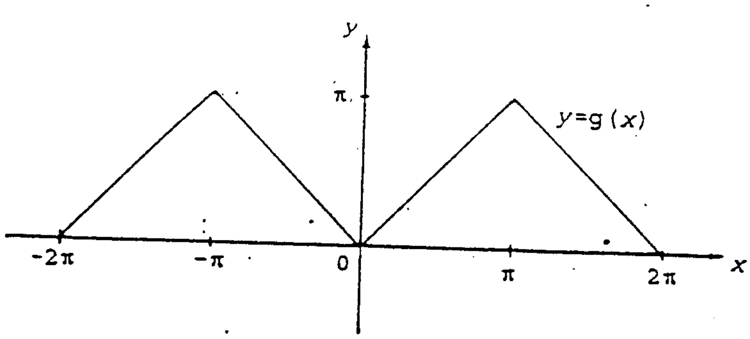
Solution	Marks
1. (a) The statement is clearly true for $n=1$. Assume it is true for $n=k$, then	1
$A^{k+1} = \begin{pmatrix} a^k & \frac{a^k-b^k}{a-b} \\ 0 & b^k \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$	1A
$= \begin{pmatrix} a^{k+1} & a^k + \frac{a^k-b^k}{a-b}\cdot b \\ 0 & b^{k+1} \end{pmatrix}$	
$= \begin{pmatrix} a^{k+1} & \frac{a^{k+1}-b^{k+1}}{a-b} \\ 0 & b^{k+1} \end{pmatrix}$	1A
The statement is also true for $n=k+1$. By the principle of mathematical induction, $A^n = \begin{pmatrix} a^n & \frac{a^n-b^n}{a-b} \\ 0 & b^n \end{pmatrix}$ for all positive integers n.	
(b) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = 2^{95}\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{3}{2} \end{pmatrix}^{95}$	1M + 1A
$= 2^{95}\begin{pmatrix} \left(\frac{1}{2}\right)^{95} & \frac{\left(\frac{1}{2}\right)^{95}-\left(\frac{3}{2}\right)^{95}}{\frac{1}{2}-\frac{3}{2}} \\ 0 & \left(\frac{3}{2}\right)^{95} \end{pmatrix}$	
$= \begin{pmatrix} 1 & 3^{95}-1 \\ 0 & 3^{95} \end{pmatrix}$	1A
	(6)

Solution	Marks
(a) $(1+x)^n = 1+C_1^n x+C_2^n x^2+C_3^n x^3+\dots+C_n^n x^n$	1A
Differentiating both sides w.r.t. x, we have	
$n(1+x)^{n-1} = C_1^n+2C_2^n x+3C_3^n x^2+\dots+nC_n^n x^{n-1}$ $\quad\dots(*)$	1A
Putting $x=1$ into (*), we get	
$C_1^n+2C_2^n+3C_3^n+\dots+nC_n^n = 2^{n-1}n$	1
(b) Putting $x=-1$ into (*), we get	
$C_1^n+2(-1)C_2^n+3C_3^n+\dots+n(-1)^{n-1}C_n^n = 0$	1A
$\therefore \dfrac{n!}{(n-1)!}+\dfrac{-2(n!)}{2!(n-2)!}+\dfrac{3(n!)}{3!(n-3)!}+\dots+\dfrac{(-1)^{n-1}n(n!)}{n!} = 0$	
$\therefore \dfrac{1}{(n-1)!}+\dfrac{-2}{2!(n-2)!}+\dfrac{3}{3!(n-3)!}+\dots+\dfrac{(-1)^{n-1}n}{n!} = 0$	1A
Alternatively,	
$\dfrac{1}{(n-1)!}+\dfrac{-2}{2!(n-2)!}+\dfrac{3}{3!(n-3)!}+\dots+\dfrac{(-1)^{n-1}n}{n!}$	
$= \dfrac{1}{n!}\left[C_1^n-2C_2^n+3C_3^n+\dots+n(-1)^{n-1}C_n^n\right]$	1A
$= \dfrac{1}{n!}n(1-1)^{n-1} = 0$	1A
	(5)

Solution	Marks
3. (a) Comparing coefficients on both sides, we have	
$\begin{cases} a_1 = 2p \\ a_2 = p^2+2q-\alpha^2 \\ a_3 = 2pq-2\alpha\beta \\ a_4 = q^2-\beta^2 \end{cases}$	1A (For any 3 being co[rrect])
Eliminating p, we have	
$\begin{cases} \alpha^2 = \dfrac{a_1^2}{4}+2q-a_2 \\ \alpha\beta = \dfrac{1}{2}(a_1q-a_3) \\ \beta^2 = q^2-a_4 \end{cases}$	1
(b) By (a), $\begin{cases} \alpha^2 = 2(q+8) \\ \alpha\beta = 2(q-6) \\ \beta^2 = q^2+9 \end{cases}$	1A
$\therefore 2(q^2+9)(q+8) = 4(q-6)^2$	
$q(q^2+6q+33) = 0$ (or an equation useful for getting q, α or β)	1A
Since q is real, therefore $q=0$.	
Hence $\begin{cases} p=2 \\ q=0 \\ \alpha=4 \\ \beta=-3 \end{cases}$ or $\begin{cases} p=2 \\ q=0 \\ \alpha=-4 \\ \beta=3 \end{cases}$	1A
(c) By (b), $x^4+4x^3-12x^2+24x-9 = (x^2+2x)^2-(4x-3)^2$.	
$\therefore (x^2+2x+4x-3)(x^2+2x-4x+3) = 0$	
$(x^2+6x-3)(x^2-2x+3) = 0$	1A
$x = -3\pm2\sqrt{3}$ or $1\pm\sqrt{2}i$	1A
	(7)
4. (a) For every $x\in\mathbb{R}$,	
(i) $g(-x) = f[\cos(-x)] = f(\cos x) = g(x)$	
(ii) $g(x+2k\pi) = f[\cos(x+2k\pi)] = f(\cos x) = g(x)$ for any integer k.	1
$\therefore g(x)$ is even and periodic.	1
(b) For $x\in[0,\pi]$, $g(x) = \arccos(\cos x)$ $= x$	1A
	1A + 1A (1A for any 2 out of the 4 segments)
	(5)

Solution	Marks
5. (a) Let $f(x) = \ln x - x + 1$ for $x>0$,	
then $f'(x) = \frac{1}{x} - 1 \begin{cases} >0 & \text{for } 0<x<1 \\ <0 & \text{for } x>1 \end{cases}$	1M + 1A (1A for differentiatio
Hence $f(x)$ is strictly increasing for $0<x<1$ and strictly decreasing for $x>1$.	
$\because$ $f(x)$ is continuous,	
$\therefore$ $f(x) \le f(1)$ for $x>0$ and the equality holds iff $x=1$.	1
i.e. $\ln x \le x-1$ for $x>0$ and the equality holds iff $x=1$.	1 (for eithe
(b) For $r>1$, put $x = \frac{r}{r-1}$, then $x>1$.	
By (a), $\ln\frac{r}{r-1} < \frac{r}{r-1} - 1 = \frac{1}{r-1}$	1
$\sum_{k=2}^{n} [\ln r - \ln(r-1)] < \sum_{k=2}^{n} \frac{1}{k-1}$	1M
$\ln n < \sum_{k=1}^{n-1} \frac{1}{k}$	1
	(7)
6. For $n=1$,	
$\because$ $1 \le a_1^2 \le 1+1+(-1) = 1$	
$\therefore$ $a_1 = 1$ ($\because$ a_1 is non-negative)	1
Assume $a_1 = a_2 = \ldots = a_n = 1$.	1 (can be absorbed bel
Then $\sum_{k=1}^{n+1} a_k^2 = n + a_{n+1}^2$	1
Hence $n+1 \le n + a_{n+1}^2 \le n+1+1+(-1)^{n+1}$	
$1 \le a_{n+1}^2 \le 2 + (-1)^{n+1} < 4$	1A
$\because$ a_{n+1} is a non-negative integer,	
$\therefore$ $a_{n+1} = 1$	
By the principle of mathematical induction, $a_{n+1} = 1$ for all $n \ge 1$.	
<u>Alternatively</u>,	
For any positive integer m,	
$\because$ $2m-1 \le \sum_{k=1}^{2m-1} a_k^2 \le (2m-1)+1+(-1)^{2m-1} = 2m-1$	
$\therefore$ $\sum_{k=1}^{2m-1} a_k^2 = 2m-1$	1
(i) For $m=1$, $a_1^2 = 1 \Rightarrow a_1 = 1$.	1
(ii) For any positive integer m,	
$\sum_{k=1}^{2m+1} a_k^2 - \sum_{k=1}^{2m-1} a_k^2 = a_{2m+1}^2 + a_{2m}^2 = 2$	1A
$\Rightarrow a_{2m+1} = a_{2m} = 1$	1
	(4)

Solution	Marks
7. (a) $\because \lvert z\rvert = \lvert z-a\rvert$	
$\therefore z\bar{z} = (z-a)\overline{(z-a)}$	1A
$z\bar{a}+a\bar{z} = a\bar{a}$	1A
$\frac{z}{a}+\frac{\bar{z}}{\bar{a}} = 1$	
$\text{Re}\left(\frac{z}{a}\right) = \frac{1}{2}$	1
(b) By (a), $\frac{z}{a} = \frac{1}{2}+yi$ for some $y\in\mathbb{R}$.	1A
$\because \lvert z\rvert=\lvert a\rvert$	
$\therefore \left\lvert\frac{z}{a}\right\rvert=1$	
$\left(\frac{1}{2}\right)^2+y^2=1$	
$y=\pm\frac{\sqrt{3}}{2}$	1A
$\Rightarrow z=\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)a$	1A
<u>Alternatively</u>,	
$\because \lvert z\rvert=\lvert z-a\rvert=\lvert a\rvert \Rightarrow$ The points representing 0, z, a form an equilateral triangle in the Argand diagram	1A
$\therefore z=ae^{\pm\frac{\pi}{3}i}$	1A
$=\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)a$	1A
	(6)

<u>Note</u>

If a is mistaken as a real number, no mark will be awarded for part (a). However, no mark will be deducted from part (b) in this case.

Solution	Marks
8. (a) (i) $A = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(s_1\ s_2)$	
$= \begin{pmatrix} u_1 s_1 & u_1 s_2 \\ u_2 s_1 & u_2 s_2 \end{pmatrix}$	1A
$\Rightarrow \det A = u_1 u_2 s_1 s_2 - u_1 u_2 s_1 s_2 = 0$	1
(ii) If $\det A = 0$, then the vectors $(a_1\ a_2)$, $(b_1\ b_2)$ are linearly dependent.	1M
W.l.g let $(a_1\ a_2) = k(b_1\ b_2)$ for some $k \in \mathbb{R}$, then	
$A = \begin{pmatrix} kb_1 & kb_2 \\ b_1 & b_2 \end{pmatrix}$	1A
$= \begin{pmatrix} k \\ 1 \end{pmatrix}(b_1\ b_2)$ [illegible]	
Taking $B = \begin{pmatrix} k \\ 1 \end{pmatrix} \in M_{21}$ and $C = (b_1\ b_2) \in M_{12}$, we have $A = BC$.	1A
(b) (i) $D = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}\begin{pmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ u_3 & v_3 & 0 \end{pmatrix}\begin{pmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \\ 0 & 0 & 0 \end{pmatrix}$.	1M+1A+1A
$\Rightarrow \det D = \begin{vmatrix} u_1 & v_1 & 0 \\ u_2 & v_2 & 0 \\ u_3 & v_3 & 0 \end{vmatrix}\begin{vmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \\ 0 & 0 & 0 \end{vmatrix}$	1A
$= 0$	1
<u>Alternatively</u>,	
$D = \begin{pmatrix} u_1 s_1 + v_1 t_1 & u_1 s_2 + v_1 t_2 & u_1 s_3 + v_1 t_3 \\ u_2 s_1 + v_2 t_1 & u_2 s_2 + v_2 t_2 & u_2 s_3 + v_2 t_3 \\ u_3 s_1 + v_3 t_1 & u_3 s_2 + v_3 t_2 & u_3 s_3 + v_3 t_3 \end{pmatrix}$	1A
$\Rightarrow \det D = \begin{vmatrix} u_1 s_1 & u_1 s_2 + v_1 t_2 & u_1 s_3 + v_1 t_3 \\ u_2 s_1 & u_2 s_2 + v_2 t_2 & u_2 s_3 + v_2 t_3 \\ u_3 s_1 & u_3 s_2 + v_3 t_2 & u_3 s_3 + v_3 t_3 \end{vmatrix} + \begin{vmatrix} v_1 t_1 & u_1 s_2 + v_1 t_2 & u_1 s_3 + v_1 t_3 \\ v_2 t_1 & u_2 s_2 + v_2 t_2 & u_2 s_3 + v_2 t_3 \\ v_3 t_1 & u_3 s_2 + v_3 t_2 & u_3 s_3 + v_3 t_3 \end{vmatrix}$	1M + 1A
$= \begin{vmatrix} u_1 s_1 & u_1 s_2 & u_1 s_3 + v_1 t_3 \\ u_2 s_1 & u_2 s_2 & u_2 s_3 + v_2 t_3 \\ u_3 s_1 & u_3 s_2 & u_3 s_3 + v_3 t_3 \end{vmatrix} + \begin{vmatrix} u_1 s_1 & v_1 t_2 & u_1 s_3 + v_1 t_3 \\ u_2 s_1 & v_2 t_2 & u_2 s_3 + v_2 t_3 \\ u_3 s_1 & v_3 t_2 & u_3 s_3 + v_3 t_3 \end{vmatrix} +$	
$\begin{vmatrix} v_1 t_1 & u_1 s_2 & u_1 s_3 + v_1 t_3 \\ v_2 t_1 & u_2 s_2 & u_2 s_3 + v_2 t_3 \\ v_3 t_1 & u_3 s_2 & u_3 s_3 + v_3 t_3 \end{vmatrix} + \begin{vmatrix} v_1 t_1 & v_1 t_2 & u_1 s_3 + v_1 t_3 \\ v_2 t_1 & v_2 t_2 & u_2 s_3 + v_2 t_3 \\ v_3 t_1 & v_3 t_2 & u_3 s_3 + v_3 t_3 \end{vmatrix}$	1A
$= 0$	1

Solution	Marks
(ii) If $c_i = \alpha a_i + \beta b_i$ $(i=1,2,3)$, then	
$$D = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 & \alpha a_3 + \beta b_3 \end{pmatrix}$$	1A
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$	1A
Taking $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{pmatrix} \in M_{32}$ and $T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \in M_{23}$,	
we have $D = ST$.	1
(iii) If $\det A = 0$, then the vectors $(a_1\ a_2\ a_3)$, $(b_1\ b_2\ b_3)$ and $(c_1\ c_2\ c_3)$ are linearly dependent.	1M
W.l.g. let $c_i = \alpha a_i + \beta b_i$ $(i=1,2,3)$.	
By (b)(ii), $D=ST$ for some $S \in M_{32}$ and $T \in M_{23}$.	
Taking $P=S$ and $Q=T$, the result follows.	1

Solution	Marks
9. (a) For (S), $\Delta = \begin{vmatrix} 2 & 2 & -1 \\ h & -3 & -1 \\ -3 & h & 1 \end{vmatrix} = -(h^2-9)$.	1A
(S) has a unique solution if and only if $\Delta \neq 0$, i.e. $h^2 \neq 9$.	1
In this case, S.S. $= \left\{ \left(-\frac{k}{h+3}, \frac{k}{h+3}, -k \right) \right\}$.	1A
(b) (i) If $h=3$, the augmented matrix of (S) is $\left(\begin{array}{ccc\|c} 2 & 2 & -1 & k \\ 3 & -3 & -1 & 0 \\ -3 & 3 & 1 & 0 \end{array} \right) \to \left(\begin{array}{ccc\|c} 12 & 0 & -5 & 3k \\ 1 & -5 & 0 & -k \end{array} \right)$	1M
or $\left(\begin{array}{ccc\|c} 1 & -5 & 0 & -k \\ 0 & -12 & 1 & -3k \end{array} \right)$ or $\left(\begin{array}{ccc\|c} 0 & 12 & -1 & 3k \\ 12 & 0 & -5 & 3k \end{array} \right)$	
It is consistent for all values of k.	1A
S.S. $= \{(x,y,z): x=t, y=\frac{k+t}{5}, z=\frac{12t-3k}{5}, t\in\mathbb{R}\}$	2A
or $\{(x,y,z): x=5t-k, y=t, z=12t-3k, t\in\mathbb{R}\}$	
or $\{(x,y,z): x=\frac{3k+5t}{12}, y=\frac{3k+t}{12}, z=t, t\in\mathbb{R}\}$	
(ii) If $h=-3$, the augmented matrix of (S) is $\left(\begin{array}{ccc\|c} 2 & 2 & -1 & k \\ -3 & -3 & -1 & 0 \\ -3 & -3 & 1 & 0 \end{array} \right) \to \left(\begin{array}{ccc\|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \end{array} \right)$	1M
It is consistent for $k = 0$ only.	1A
S.S. $= \{(x,y,z): x=t, y=-t, z=0, t\in\mathbb{R}\}$	1A
or $\{(x,y,z): x=-t, y=t, z=0, t\in\mathbb{R}\}$	

Solution	Marks
(c) Putting $k=\frac{2}{3}$, (S) becomes the first 3 eqtns of (T).	
(i) If $h^2\neq 9$ and $k=\frac{2}{3}$, then (S) has a unique solution.	1M
(T) is consistent if the solution	
$x=-\frac{2}{3(h+3)}$, $y=\frac{2}{3(h+3)}$, $z=-\frac{2}{3}$ satisfies the 4th eqtn.:	
$-5\left(-\frac{2}{3(h+3)}\right)-2\left(\frac{2}{3(h+3)}\right)+6\left(-\frac{2}{3}\right)=h$	1 for h
$\Rightarrow \quad h^2+7h+10=0$	
$\Rightarrow \quad h=-2$ or -5.	
When $h=-2$, the solution of (T) is $(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3})$.	2 for solution
When $h=-5$, the solution of (T) is $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$.	
(ii) If $h=3$ and $k=\frac{2}{3}$, (S) has infinitely many solutions.	
(T) is consistent if $\left(t, \frac{2+3t}{15}, \frac{36t-6}{15}\right)$, $t\in\mathbb{R}$,	1M
satisfies the 4th eqtn.:	
$-5t-2\left(\frac{2+3t}{15}\right)+6\left(\frac{36t-6}{15}\right)=3$	
$\Rightarrow \quad 135t=85$	
$\Rightarrow \quad t=\frac{17}{27}$	
When $h=3$, the solution of (T) is $(\frac{17}{27}, \frac{7}{27}, \frac{10}{9})$.	1A

Solution	Marks
10. (a) (i) $(x-\alpha)(x-\beta)(x-\gamma) = x^3+px^2+qx+r$ for all x.	
Differentiating w.r.t. x on both sides, we have	
$(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)+(x-\alpha)(x-\beta) = 3x^2+2px+q$ ......(1)	1M + 1A
Hence	
$\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma} = \frac{(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)+(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}$	
$= \frac{3x^2+2px+q}{x^3+px^2+qx+r}$	1
Alternatively,	
$\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma} = \frac{(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)+(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}$	
$= \frac{3x^2-2(\alpha+\beta+\gamma)x+(\alpha\beta+\beta\gamma+\gamma\alpha)}{x^3+px^2+qx+r}$	1M + 1A
$= \frac{3x^2+2px+q}{x^3+px^2+qx+r}$	1
(ii) Sub. $x=\alpha$ into (1), we have $3\alpha^2+2p\alpha+q = (\alpha-\beta)(\alpha-\gamma)$.	1
Alternatively,	
Form (i), $1+\frac{x-\alpha}{x-\beta}+\frac{x-\alpha}{x-\gamma} = \frac{3x^2+2px+q}{(x-\beta)(x-\gamma)}$	
Sub. $x=\alpha$, we have $3\alpha^2+2p\alpha+q = (\alpha-\beta)(\alpha-\gamma)$.	1
(b) (i) Let $(3x^2+2px+q)f(x) = (x^3+px^2+qx+r)Q(x) + Ax^2+Bx+C$	1M
$= (x-\alpha)(x-\beta)(x-\gamma)Q(x) + Ax^2+Bx+C$.	
Then $(3\alpha^2+2p\alpha+q)f(\alpha) = A\alpha^2+B\alpha+C$ ......(2)	1A
Let $\frac{Ax^2+Bx+C}{x^3+px^2+qx+r} = \frac{k_1}{x-\alpha}+\frac{k_2}{x-\beta}+\frac{k_3}{x-\gamma}$.	1M
Then $Ax^2+Bx+C = k_1(x-\beta)(x-\gamma)+k_2(x-\gamma)(x-\alpha)+k_3(x-\alpha)(x-\beta)$.	1A
Putting $x=\alpha$, we have	
$A\alpha^2+B\alpha+C = k_1(\alpha-\beta)(\alpha-\gamma)$	1A
$(3\alpha^2+2p\alpha+q)f(\alpha) = k_1(\alpha-\beta)(\alpha-\gamma)$ by (2)	
$k_1 = f(\alpha)$ by (a)(ii)	1A
Similarly, $k_2 = f(\beta)$ and	
$k_3 = f(\gamma)$.	1M
Hence $\frac{f(\alpha)}{x-\alpha}+\frac{f(\beta)}{x-\beta}+\frac{f(\gamma)}{x-\gamma} = \frac{Ax^2+Bx+C}{x^3+px^2+qx+r}$.	
(ii) From (b)(i),	
$Ax^2+Bx+C = f(\alpha)(x-\beta)(x-\gamma)+f(\beta)(x-\gamma)(x-\alpha)+f(\gamma)(x-\alpha)(x-\beta)$.	1A
Equating coefficients of x^2, x and the constant terms, we have	
$A = f(\alpha)+f(\beta)+f(\gamma)$.	1A
$B = -[(\beta+\gamma)f(\alpha)+(\gamma+\alpha)f(\beta)+(\alpha+\beta)f(\gamma)]$	1A
$C = \beta\gamma f(\alpha)+\gamma\alpha f(\beta)+\alpha\beta f(\gamma)$	1A

Solution	Marks
11. (a) $\arg\left(\dfrac{z_C-z_M}{z_P-z_M}\right) = -\dfrac{\pi}{2}$	1M + 1A
$\left\lvert\dfrac{z_C-z_M}{z_P-z_M}\right\rvert = \cot\dfrac{\alpha}{2}$	1A
$\therefore \dfrac{z_C-z_M}{z_P-z_M} = -i\cot\dfrac{\alpha}{2}$	1A
$z_C-z_M = i(z_M-z_P)\cot\dfrac{\alpha}{2}$	1
<u>Note</u>: Award 1 mark for drawing appropriate figure.	
(b) $\because z_M = \dfrac{1}{2}(z_P+z_Q)$	1A
$\therefore z_C = \dfrac{1}{2}(z_P+z_Q)+i\left[\dfrac{1}{2}(z_P+z_Q)-z_P\right]\cot\dfrac{\alpha}{2}$	
$= \dfrac{1}{2}(z_P+z_Q)-\dfrac{1}{2}i(z_P-z_Q)\cot\dfrac{\alpha}{2}$	1A
$r = \dfrac{1}{2}\lvert z_P-z_Q\rvert\operatorname{cosec}\dfrac{\alpha}{2}$	1M + 1A
(c) (i) Any circle in the complex plane with centre α and radius r has equation $\lvert z-\alpha\rvert = r$ $(z-\alpha)(\overline{z}-\overline{\alpha}) = r^2$ $z\overline{z}-\overline{\alpha}z-\alpha\overline{z}+\alpha\overline{\alpha}-r^2 = 0$ which is in the form of $z\overline{z}+az+b\overline{z}+c=0$ where $a, b \in \mathbf{C}$ and $c \in \mathbf{R}$.	1
(ii) By (b), the radius of $\mathscr{C} = \dfrac{1}{2}\lvert 1+i-(-i)\rvert\operatorname{cosec}\dfrac{\pi}{6}$	
$= \sqrt{5}$	1A
There are two possible positions for the centre of $\mathscr{C}$:	
(1) Taking $z_P = 1+i$ and $z_Q = -i$ in (b), $z_C = \dfrac{1}{2}(1+i-i)-\dfrac{1}{2}i(1+i+i)\cot\dfrac{\pi}{6}$	
$= \dfrac{1}{2}+\sqrt{3}-\dfrac{\sqrt{3}}{2}i$	1A
(2) Taking $z_P = -i$ and $z_Q = 1+i$ in (b), $z_C = \dfrac{1}{2}(-i+1+i)-\dfrac{1}{2}i(-i-1-i)\cot\dfrac{\pi}{6}$	
$= \dfrac{1}{2}-\sqrt{3}+\dfrac{\sqrt{3}}{2}i$	1A
<u>**Alternatively,**</u> $z_C = [1+i-(-i)]e^{\pm i\frac{\pi}{3}}+(-i)$ $= (1+2i)\left(\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}\right)-i$ $= \dfrac{1}{2}-\sqrt{3}+\dfrac{\sqrt{3}}{2}i$ or $\dfrac{1}{2}+\sqrt{3}-\dfrac{\sqrt{3}}{2}i$	1A + 1A

Solution	Marks
In case (1),	
$a = -\frac{1}{2}-\sqrt{3}-\frac{\sqrt{3}}{2}i$	
$b = -\frac{1}{2}-\sqrt{3}+\frac{\sqrt{3}}{2}i$	
$c = -1+\sqrt{3}$.	1A
In case (2),	
$a = -\frac{1}{2}+\sqrt{3}-\frac{\sqrt{3}}{2}i$	
$b = -\frac{1}{2}+\sqrt{3}-\frac{\sqrt{3}}{2}i$	
$c = -1-\sqrt{3}$.	1A

Solution	Marks
12. (a) If $\lim_{n\to\infty} a_n$ exists, then	
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}} + \frac{1}{p} \lim_{n\to\infty} a_{n-1}$	
$= \frac{1}{p} \lim_{n\to\infty} a_n$	1A
Since $p \neq 1$, we have $\lim_{n\to\infty} a_n = 0$.	1
	($p \neq 1$ is necessary)
(b) (i) Suppose on the contrary that $\lim_{n\to\infty} a_n$ exists.	
By (a), $\lim_{n\to\infty} a_n = 0$.	1
$2 = a_0 < a_1 < a_2 < \ldots \Rightarrow 2 = a_0 \leq \lim_{n\to\infty} a_n = 0$	
which is a contradiction.	1
Hence $\lim_{n\to\infty} a_n$ does not exist.	
(ii) Given $a_{k-1} \geq a_k$ for some $k \geq 1$.	
Assume $a_{m-1} \geq a_m$ for some $m \geq k$, then	
$a_m - a_{m+1} = \left(\frac{1}{m^{\frac{1}{p}}} - \frac{1}{(m+1)^{\frac{1}{p}}}\right) + \frac{1}{p}(a_{m-1} - a_m) \geq 0$	
By the principle of mathematical induction, $a_{n-1} \geq a_n$ for all $n \geq k$.	1
Thus $\{a_{n+k-1}\}$ is monotonic decreasing and bounded below by 0, hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+k-1}$ exists.	
By (a), $\lim_{n\to\infty} a_n = 0$.	1
(c) (i) If $0<p<1$, $a_n = \frac{1}{n^{\frac{1}{p}}} + \frac{1}{p} a_{n-1} > \frac{1}{p} a_{n-1} > a_{n-1}$ for all $n \geq 1$.	1A
By (b)(i), $\lim_{n\to\infty} a_n$ does not exist.	1
(ii) If $p \geq 2$, then $a_1 = \frac{1}{1^{\frac{1}{p}}} + \frac{1}{p} a_0 = 1 + \frac{2}{p} \leq 2 = a_0$.	1A
By (b)(ii), $\lim_{n\to\infty} a_n = 0$.	1

Solution	Marks
(d) (i) If $1<p<2$, then $0<p-1<1 \Rightarrow a_0=2<\frac{2}{p-1}$	1A
Assume $a_n<\frac{2}{p-1}$ for some $n\geq 0$, then	
$$\begin{aligned} a_{n+1} &= \frac{1}{(n+1)^{\frac{1}{p}}}+\frac{1}{p}a_n \\ &< \frac{1}{n+1}+\frac{1}{p}a_n \\ &\leq 1+\frac{1}{p}a_n \\ &< 1+\frac{1}{p}\left(\frac{2}{p-1}\right) \\ &= \frac{p^2-p+2}{p(p-1)} \\ &= \frac{(p-1)(p-2)+2p}{p(p-1)} \\ &< \frac{2p}{p(p-1)} \\ &= \frac{2}{p-1} \end{aligned}$$	1
(ii) By (d)(i), $\{a_n\}$ is bounded above.	
Suppose $\{a_n\}$ is strictly increasing, then $\lim_{n\to\infty} a_n$ exists	1
which contradicts the result of (b)(i).	1
Thus $\{a_n\}$ is not strictly increasing.	
By (b)(ii), $\lim_{n\to\infty} a_n=0$.	1

Solution	Marks
13. (a) Without loss of generality, assume $a \ge b$.	1M
Then $a-b \ge 0$, $\frac{a}{b} \ge 1$ and hence	1
$\frac{a^a b^b}{a^b b^a} = \left(\frac{a}{b}\right)^{a-b} \ge 1$	
$\therefore a^a b^b \ge a^b b^a$.	1
If the equality holds, then	
$\left(\frac{a}{b}\right)^{a-b} = 1 \Rightarrow (a-b)\log\left(\frac{a}{b}\right) = 0$	
$\Rightarrow \frac{a}{b} = 1$ or $a-b=0$	
$\Rightarrow a=b$.	1
(b) $\left(\frac{a+b}{2}\right)^{a+b} \ge (\sqrt{ab})^{a+b}$	1M
$= \sqrt{a^a b^b a^b b^a}$	
$\ge \sqrt{a^b b^a a^b b^a}$	
$= a^b b^a$ .... (*)	1
If the equality holds, then	
$\frac{a+b}{2} = \sqrt{ab}$ and $a^a b^b = a^b b^a \Rightarrow a=b$.	1
(c) Let $f(x) = \ln[x^x(1-x)^{1-x}]$	
$= x\ln x + (1-x)\ln(1-x)$ for $0<x<1$	1A
Then $f'(x) = \ln x - \ln(1-x)$	1A
$= \ln\frac{x}{1-x}$	
$\begin{cases} <0 & \text{if } 0<x<\frac{1}{2} \\ =0 & \text{if } x=\frac{1}{2} \\ >0 & \text{if } \frac{1}{2}<x<1 \end{cases}$	
$\therefore f(x) \ge f(\frac{1}{2})$ for $0<x<1$	
and if the equality holds, $x=\frac{1}{2}$.	1
Hence $\ln[x^x(1-x)^{1-x}] \ge \ln[(\frac{1}{2})^{\frac{1}{2}}(1-\frac{1}{2})^{1-\frac{1}{2}}] = \ln\frac{1}{2}$	
$x^x(1-x)^{1-x} \ge \frac{1}{2}$ for $0<x<1$	
and if the equality holds, $x=\frac{1}{2}$.	1

Solution	Marks
Put $x=\frac{a}{a+b}$, then $0<x<1$ and $1-x=\frac{b}{a+b}$	1M + 1A
Hence $\left(\frac{a}{a+b}\right)^{\frac{a}{a+b}}\left(\frac{b}{a+b}\right)^{\frac{b}{a+b}} \ge \frac{1}{2}$	
$\left(\frac{a}{a+b}\right)^{a}\left(\frac{b}{a+b}\right)^{b} \ge \left(\frac{1}{2}\right)^{a+b}$	
$a^a b^b \ge \left(\frac{a+b}{2}\right)^{a+b}$	1
If the equality holds,	
then $\frac{a}{a+b}=\frac{1}{2} \Rightarrow a=b$.	1

Solution	Marks
1. (a) Let $y = \left(\frac{a^x+b^x+1}{3}\right)^{\frac{1}{x}}$, then $\ln y = \frac{1}{x}\ln\left(\frac{a^x+b^x+1}{3}\right)$	1A
$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{3}{a^x+b^x+1}\left[\frac{a^x\ln a+b^x\ln b}{3}\right]$	1M
$= \frac{1}{3}\ln ab$	
$\therefore \lim_{x\to 0} y = (ab)^{\frac{1}{3}}$	1A
(b) $\lim_{n\to\infty}\left(\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\dots+\frac{n^2}{n^3+n^3}\right)$	
$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{(\frac{1}{n})^2}{1+(\frac{1}{n})^3}+\frac{(\frac{2}{n})^2}{1+(\frac{2}{n})^3}+\dots+\frac{(\frac{n}{n})^2}{1+(\frac{n}{n})^3}\right]$	1A
$= \int_0^1\frac{x^2}{1+x^3}dx$	1M
$= \frac{1}{3}\left[\ln(1+x^3)\right]_0^1$	
$= \frac{1}{3}\ln 2$	1A
	(6)
2. (a) For $0<\theta<\frac{\pi}{2}$, let $x=\sin^2\theta$, then $dx=2\sin\theta\cos\theta\,d\theta$.	
$\int\frac{f(x)}{\sqrt{x(1-x)}}dx = \int\frac{f(\sin^2\theta)\cdot 2\sin\theta\cos\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}}d\theta = 2\int f(\sin^2\theta)\,d\theta$.	1
(b) By (a),	
$\int\frac{dx}{\sqrt{x(1-x)}} = 2\int d\theta$	
$= 2\theta + c$	
$= 2\sin^{-1}\sqrt{x} + c$	1A
$\int\sqrt{\frac{x}{1-x}}\,dx = \int\frac{x}{\sqrt{x(1-x)}}dx$	
$= 2\int\sin^2\theta\,d\theta$	1A
$= \int(1-\cos 2\theta)\,d\theta$	
$= \theta - \frac{1}{2}\sin 2\theta + c$	1A
$= \sin^{-1}\sqrt{x} - \sqrt{x(1-x)} + c$	1A
	(5)

Solution	Marks
3. (a) $y^2 = 4ax$	
$2y\frac{dy}{dx} = 4a$	
$\left.\frac{dy}{dx}\right\|_P = \left.\frac{2a}{y}\right\|_P = \frac{1}{t}$	1 (for attempting to find slope of tangent)
Normal at P is $\frac{y-2a}{x-at^2} = -t$	
$y + tx = 2at + at^3$	1
(b) Suppose the normals at P_i $(i=1,2,3)$ are concurrent and intersect at (x_0, y_0), then	
$y_0 + t_i x_0 = 2at_i + at_i^3$ for $i=1,2,3$	1
i.e. t_1, t_2, t_3 are the roots of $at^3 + (2a - x_0)t - y_0 = 0$	1
$\therefore$ Sum of the roots $= t_1 + t_2 + t_3 = 0$	1
	(5)
4. (a) For $x \ge 0$, $F'(x) = \frac{\sin x}{x+1}$	1A
$\begin{cases} > 0 & \text{for } x \in (0,\pi) \\ < 0 & \text{for } x \in (\pi, 2\pi) \end{cases}$	1M
$\therefore$ $F(x) \le F(\pi)$ $\forall x \in [0, 2\pi]$	
i.e. $x_0 = \pi$	1A
(b) $\because F(0) = 0$ and	1A
$F(2\pi) = \int_0^{2\pi} \frac{\sin t}{t+1} dt$	
$= \int_0^{\pi} \frac{\sin t}{t+1} dt + \int_{\pi}^{2\pi} \frac{\sin t}{t+1} dt$	1
$= \int_0^{\pi} \frac{\sin t}{t+1} dt + \int_0^{\pi} \frac{\sin(\pi + t)}{\pi + t + 1} dt$	
$= \int_0^{\pi} \frac{\sin t}{t+1} dt - \int_0^{\pi} \frac{\sin t}{\pi + t + 1} dt$	
≥ 0	1
$\because$ F is strictly increasing on $(0, \pi]$, $\therefore F(x) > F(0) = 0$ for $x \in (0, \pi]$.	
$\because$ F is strictly decreasing on $[\pi, 2\pi)$, $\therefore F(x) > F(2\pi) \ge 0$ for $x \in [\pi, 2\pi)$.	
Hence $F(x) > 0$ for $x \in (0, 2\pi)$.	1
	(7)

5. (a) $\begin{cases} r = -2\cos\theta \\ r = 2+2\cos\theta \end{cases}$

$-2\cos\theta = 2+2\cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ 1A

The intersecting points other than the pole are

$(1, \frac{2\pi}{3})$ and $(1, \frac{4\pi}{3})$. 1A

(b) Area of the shaded region

$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-2\cos\theta)^2 d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (2+2\cos\theta)^2 d\theta$ 1M + 1A

$= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\cos^2\theta \, d\theta + \int_{\frac{2\pi}{3}}^{\pi} (2+4\cos\theta+2\cos^2\theta) \, d\theta$

$= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (1+\cos 2\theta) \, d\theta + \int_{\frac{2\pi}{3}}^{\pi} (2+4\cos\theta+1+\cos 2\theta) \, d\theta$

$= \left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} + \left[3\theta+4\sin\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{2\pi}{3}}^{\pi}$ 1 (Working out any one)

$= (\frac{2\pi}{3}-\frac{\pi}{2}-\frac{\sqrt{3}}{4}) + (3\pi-2\pi-2\sqrt{3}+\frac{\sqrt{3}}{4})$

$= \frac{7\pi}{6} - 2\sqrt{3}$ 1A

(6)

Solution	
6. (a) $y = \left(\frac{x+1}{x-1}\right)^r$	
$\ln y = r[\ln(x+1) - \ln(x-1)]$	1M
$\frac{1}{y}\frac{dy}{dx} = r\left[\frac{1}{x+1} - \frac{1}{x-1}\right]$	
$\frac{dy}{dx} = \frac{-2ry}{x^2-1}$	1
Alternatively, $\frac{dy}{dx} = r\left(\frac{x+1}{x-1}\right)^{r-1} \cdot \frac{(x-1)-(x+1)}{(x-1)^2}$ $= ry \cdot \left(\frac{x-1}{x+1}\right) \cdot \frac{-2}{(x-1)^2}$ $= \frac{-2ry}{x^2-1}$	1M 1
(b) $(x^2-1)\frac{dy}{dx} = -2ry$	
By Leibniz's formula,	
$\sum_{k=0}^{n} C_k^n (x^2-1)^{(k)} (y')^{(n-k)} = -2ry^{(n)}$	
$C_0^n (x^2-1) y^{(n+1)} + C_1^n (2x) y^{(n)} + C_2^n (2) y^{(n-1)} = -2ry^{(n)}$	1M + 1A
$(x^2-1) y^{(n+1)} + 2(nx+r) y^{(n)} + (n^2-n) y^{(n-1)} = 0$	1
Alternatively, From (a), $(x^2-1)y' = -2ry$ $(x^2-1)y'' + 2xy' = -2ry'$ $(x^2-1)y^{(1+1)} + 2(1 \cdot x + r)y^{(1)} + (1^2-1)y^{(1-1)} = 0$ ∴ The statement holds for $n=1$. Assume $(x^2-1)y^{(n+1)} + 2(nx+r)y^{(n)} + (n^2-n)y^{(n-1)} = 0$, then $(x^2-1)y^{(n+2)} + 2xy^{(n+1)} + 2(nx+r)y^{(n+1)} + 2ny^{(n)} + (n^2-n)y^{(n)} = 0$ $(x^2-1)y^{((n+1)+1)} + 2[(n+1)x+r]y^{(n+1)} + [(n+1)^2-(n+1)]y^{((n+1)-1)} = 0$. By the principle of M.I., the statement holds for $n \geq 1$.	 1A 1M 1A
	(5)

Solution	Marks
7. (a) $\frac{d}{dx}\ln[1+f(x)] = \frac{f'(x)}{1+f(x)}$	1A
(b) $\because f(x) = x^3+\int_0^x 3t^2 f(t)\,dt$	
$\therefore f'(x) = 3x^2+3x^2 f(x)$	1A
$= 3x^2[1+f(x)]$	
$\frac{f'(x)}{1+f(x)} = 3x^2$ [$1+f(x)>0$ as $f(x)>-1$]	1M
$\int\frac{f'(x)}{1+f(x)}dx = \int 3x^2 dx$	
$\ln[1+f(x)] = x^3+c$	1
$\because f(0)=0$	
$\therefore \ln(1-0)=0+c \Rightarrow c=0$	1A
Hence $1+f(x) = e^{x^3}$	
$f(x) = e^{x^3}-1 \quad \forall x\in R$	1A
Alternatively,	
$1+f(x) = e^{x^3+c}$	
$f(x) = Ae^{x^3}-1$ where $A = e^c$	1A
Putting $x=0$, we have	
$A-1 = 0+\int_0^0 3t^2 f(t)\,dt = 0$	
$\therefore A = 1$	
$\Rightarrow f(x) = e^{x^3}-1 \quad \forall x\in R$	1A
	(6)

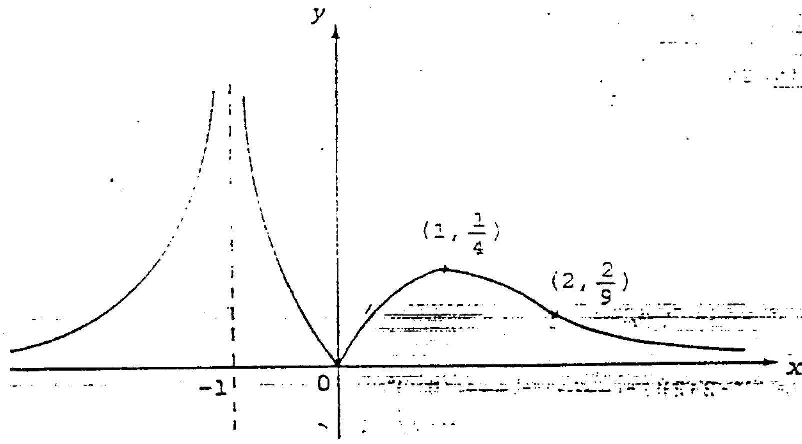
Solution	Marks
8. (a) $I_0 = \int_0^1 \frac{1-x}{1+x^3}\,dx$	
$= \int_0^1 [\frac{2}{3(1+x)} - \frac{2x-1}{3(1-x+x^2)}]\,dx$	1M + 1A
$= \left[\frac{2}{3}\ln\|1+x\| - \frac{1}{3}\ln\|1-x+x^2\|\right]_0^1$	1A
$= \frac{2}{3}\ln 2$	1A
(b) $\|I_k\| = \left\|\int_0^1 \frac{(-1)^k(1-x)x^{3k}}{1+x^3}\,dx\right\|$	
$\le \int_0^1 \left\|\frac{(-1)^k(1-x)x^{3k}}{1+x^3}\right\|\,dx$	1
$= \int_0^1 \frac{(1-x)x^{3k}}{1+x^3}\,dx$ $\quad (\because \frac{(1-x)x^{3k}}{1+x^3} \ge 0\ \forall x\in[0,1])$	
$\le \int_0^1 x^{3k}\,dx$ $\quad (\because \frac{1-x}{1+x^3} \le 1\ \forall x\in[0,1])$	1
$= \frac{1}{3k+1}$	
$\therefore -\frac{1}{3k+1} \le I_k \le \frac{1}{3k+1}$.	1
(c) $I_{k+1}-I_k = \int_0^1 [\frac{(-1)^{k+1}(1-x)x^{3(k+1)}}{1+x^3} - \frac{(-1)^k(1-x)x^{3k}}{1+x^3}]\,dx$	1M
$= (-1)^{k+1}\int_0^1 \frac{(1-x)x^{3k}(x^3+1)}{1+x^3}\,dx$	
$= (-1)^{k+1}\int_0^1 (x^{3k}-x^{3k+1})\,dx$	1A
$= (-1)^{k+1}\left[\frac{x^{3k+1}}{3k+1} - \frac{x^{3k+2}}{3k+2}\right]_0^1$	
$= \frac{(-1)^{k+1}}{(3k+1)(3k+2)}$	1A
(d) By (c), $\sum_{k=0}^{n}(I_{k+1}-I_k) = \sum_{k=0}^{n}\frac{(-1)^{k+1}}{(3k+1)(3k+2)}$	1M
$I_{n+1}-I_0 = -b_n$	1A
$\therefore I_{n+1} = \frac{2}{3}\ln 2 - b_n$ .....(*) by (a)	1A
By (b), $-\frac{1}{3k+1} \le I_k \le \frac{1}{3k+1}$ and $\lim_{k\to\infty}\frac{1}{3k+1} = 0$	
$\therefore \lim_{k\to\infty} I_k = 0$ by squeezing principle.	1
As $\lim_{n\to\infty} I_{n+1} = 0$, we have $\lim_{n\to\infty} b_n = \frac{2}{3}\ln 2$ by letting $n\to\infty$ in (*)	1

Solution	Marks
9. (a) (i) For $x>0$, $f(x)=\frac{x}{(x+1)^2}$,	
$f'(x)=-\frac{x-1}{(x+1)^3}$ and $f''(x)=\frac{2x-4}{(x+1)^4}$	1A
(ii) For $x<0$ and $x\neq-1$, $f(x)=-\frac{x}{(x+1)^2}$,	
$f'(x)=\frac{x-1}{(x+1)^3}$ and $f''(x)=-\frac{2x-4}{(x+1)^4}$	1A
(iii) $\therefore \frac{f(x)-f(0)}{x-0}=\begin{cases}\frac{1}{(x+1)^2} & \text{for } x>0\\ -\frac{1}{(x+1)^2} & \text{for } x<0 \text{ and } x\neq-1\end{cases}$	1A
$\therefore \lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=1$ and $\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=-1$,	
$\therefore f'(0)$ does not exist.	1
(b) (i) For $x>0$, $f'(x)<0 \Rightarrow x>1$.	
For $x<0$, $f'(x)<0 \Rightarrow -1<x<0$.	
$\therefore f'(x)<0$ on $(-1,0)\cup(1,\infty)$.	1A
(ii) For $x>0$, $f'(x)>0 \Rightarrow 0<x<1$.	
For $x<0$, $f'(x)>0 \Rightarrow x<-1$.	
$\therefore f'(x)<0$ on $(-\infty,-1)\cup(0,1)$.	1A
(iii) For $x>0$, $f''(x)<0 \Rightarrow 0<x<2$.	
For $x<0$, $f''(x)<0 \Rightarrow$ no solution.	
$\therefore f''(x)<0$ on $(0,2)$.	1A
(iv) For $x>0$, $f''(x)>0 \Rightarrow x>2$.	
For $x<0$, $f''(x)>0 \Rightarrow x<-1$ or $-1<x<0$.	
$\therefore f''(x)>0$ on $(-\infty,-1)\cup(-1,0)\cup(2,\infty)$.	1A

(c)

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,2)$	2	$(2,\infty)$
$f(x)$	↑	Undefined	↓	0	↑	$\frac{1}{4}$	↓	$\frac{2}{9}$	↓
$f'(x)$	+	Undefined	−	∄	+	0	−	−	−
$f''(x)$	+	Undefined	+	∄	−	−	−	0	+

Solution	Marks
$\therefore (1,\frac{1}{4})$ is a relative maximum point.	1A
$(0,0)$ is a relative minimum point.	1A
$(2,\frac{2}{9})$ is a point of inflexion.	1A

Solution	Marks
(d) The vertical symptote is $x=-1$.	1A
Let the oblique/horizontal asymptote be $y=mx+c$.	
$m = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{\lvert x\rvert}{x(x+1)^3} = 0$	
$c = \lim\limits_{x\to\infty} [f(x)-0] = 0$	
$\therefore$ the horizontal asymptote is $y=0$.	1A
	1A + 1

10. (a) (i) For $n\geq 2$, $a_n^2 = a_{n-1}^2+2\beta+\left(\frac{\beta}{a_{n-1}}\right)^2$

$\geq a_{n-1}^2+2\beta$ 1

(ii) The statement holds for $n=1$ since $a_1^2 = (\beta+1)^2 = \beta^2+2\beta+1$.

Assume it holds for some $k\geq 1$, then

$a_{k+1}^2 \geq a_k^2+2\beta \geq \beta^2+2k\beta+1+2\beta = \beta^2+2(k+1)\beta+1$.

By the principle of M.I., the statement holds for $n\geq 1$. 1

> <u>Alternatively</u>,
>
> $a_1^2 = (\beta+1)^2 = \beta^2+2\beta+1$.
>
> For $n\geq 2$, $a_n^2 \geq a_{n-1}^2+2\beta \geq a_{n-2}^2+2(2\beta) \geq \ldots \geq a_1^2+(n-1)(2\beta)$
>
> $= \beta^2+2\beta+1+(n-1)(2\beta) = \beta^2+2n\beta+1$ 1

(b) For $n\geq 2$, $a_n^2 = a_{n-1}^2+2\beta+\left(\frac{\beta}{a_{n-1}}\right)^2$

$= a_{n-2}^2+2(2\beta)+\left(\frac{\beta}{a_{n-1}}\right)^2+\left(\frac{\beta}{a_{n-2}}\right)^2$ 1

$= a_1^2+(n-1)(2\beta)+\sum_{k=1}^{n-1}\frac{\beta^2}{a_k^2}$

$= \beta^2+2\beta+1+(n-1)(2\beta)+\sum_{k=1}^{n-1}\frac{\beta^2}{a_k^2}$

$= \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{a_k^2}$ 1

$\leq \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{\beta^2+2k\beta+1}$ 1

> <u>Alternatively</u>,
>
> $\because a_2^2 = a_1^2+2\beta+\frac{\beta^2}{a_1^2} = \beta^2+2(2)\beta+1+\frac{\beta^2}{\beta^2+2\beta+1}$
>
> $\therefore$ The statement holds for $n=2$. 1
>
> Assume it holds for $n\geq 2$, then
>
> $a_{n+1}^2 = a_n^2+2\beta+\frac{\beta^2}{a_n^2}$
>
> $\leq \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{\beta^2+2k\beta+1}+2\beta+\frac{\beta^2}{a_n^2}$
>
> $\leq \beta^2+2(n+1)\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{\beta^2+2k\beta+1}+\frac{\beta^2}{\beta^2+2n\beta+1}$
>
> $= \beta^2+2(n+1)\beta+1+\sum_{k=1}^{n}\frac{\beta^2}{\beta^2+2k\beta+1}$. 1
>
> $\therefore$ By the principle of M.I., the statement holds for $n\geq 2$. 1

Solution	Marks
(c) For $k\ge 1$,	
$\dfrac{1}{\beta^2+2k\beta+1} \le \dfrac{1}{\beta^2+2\beta x+1} \quad \forall x\in[k-1,k]$	1
$\therefore \int_{k-1}^{k}\dfrac{dx}{\beta^2+2k\beta+1} \le \int_{k-1}^{k}\dfrac{dx}{\beta^2+2\beta x+1}$	
$\Rightarrow \dfrac{1}{\beta^2+2k\beta+1} \le \int_{k-1}^{k}\dfrac{dx}{\beta^2+2\beta x+1}$	1
(d) By (a)(ii) and (c),	
$\beta^2+2n\beta+1 \le a_n^2 \le \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\dfrac{\beta^2}{\beta^2+2k\beta+1}$.	1
$\because \sum_{k=1}^{n-1}\dfrac{\beta^2}{\beta^2+2k\beta+1} \le \sum_{k=1}^{n-1}\int_{k-1}^{k}\dfrac{\beta^2}{\beta^2+2\beta x+1}dx$	
$= \int_0^{n-1}\dfrac{\beta^2}{\beta^2+2\beta x+1}dx$	1A
$= \dfrac{\beta^2}{2\beta}[\ln\|\beta^2+2\beta x+1\|]_0^{n-1}$	
$= \dfrac{\beta}{2}\ln\left[\dfrac{\beta^2+2(n-1)\beta+1}{\beta^2+1}\right]$	1A
$\dfrac{\beta^2+1}{n}+2\beta \le \dfrac{a_n^2}{n} \le \dfrac{\beta^2+1}{n}+2\beta+\dfrac{\beta}{2n}\ln\left[\dfrac{\beta^2+2(n-1)\beta+1}{\beta^2+1}\right]$	
As $\lim_{n\to\infty}\dfrac{\beta}{2n}\ln\left[\dfrac{\beta^2+2(n-1)\beta+1}{\beta^2+1}\right]$	
$= \lim_{n\to\infty}\dfrac{\beta}{2}\cdot\dfrac{\beta^2+1}{\beta^2+2(n-1)\beta+1}\cdot\dfrac{2\beta}{\beta^2+1}$	1M
$= 0$	1
By squeezing principle, $\lim_{n\to\infty}\dfrac{a_n^2}{n}$ exists and	
$\lim_{n\to\infty}\dfrac{a_n^2}{n} = 2\beta$	1A
Suppose $\lim_{n\to\infty}\dfrac{a_n^2}{\sqrt{n}} = \ell$ exists, then	
$\lim_{n\to\infty}\dfrac{a_n^2}{n} = \lim_{n\to\infty}\dfrac{a_n^2}{\sqrt{n}}\cdot\lim_{n\to\infty}\dfrac{1}{\sqrt{n}}$	1
$= \ell\cdot 0$	
$= 0$	
$\ne 2\beta \quad (\because \beta>0)$	
$\therefore \lim_{n\to\infty}\dfrac{a_n^2}{\sqrt{n}}$ does not exist.	1

II-2 (a) Let $t=\tan\frac{\theta}{2}$, then $dt=\frac{1}{2}\sec^2\frac{\theta}{2}d\theta$ or $d\theta=\frac{2}{1+t^2}dt$. 1M + 1A

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{2\sin\theta+\cos\theta+2} = \int_0^1\frac{2dt}{t^2+4t+3}$$ 1A

$$= \int_0^1\left(\frac{1}{t+1}-\frac{1}{t+3}\right)dt$$ 1A

$$= \left[\ln\left|\frac{t+1}{t+3}\right|\right]_0^1$$

$$= \ln\frac{3}{2}$$ 1A

(b) $pg(\theta)+qg'(\theta)+r$

$= p(A\sin\theta-B\cos\theta+C)+q\frac{d}{d\theta}(A\sin\theta-B\cos\theta+C)+r$

$= (Ap-Bq)\sin\theta+(Bp+Aq)\cos\theta+(Cp+r)$ 1A

There exist real numbers p, q, r such that

$a\sin\theta+b\cos\theta+c = pg(\theta)+qg'(\theta)+r$

if $\begin{cases}Ap-Bq=a\\Bp+Aq=b\\Cp+r=c\end{cases}$ is consistent. 1M

Since $\Delta = \begin{vmatrix}A & -B & 0\\B & A & 0\\C & 0 & 1\end{vmatrix}$

$= A^2+B^2$ 1A

$\neq 0$ as A, B are not both zero. 1

Therefore the system of linear equations is consistent and hence p, q, r exist. 1

Alternatively,
Comparing coefficients of

$a\sin\theta+b\cos\theta+c = pg(\theta)+qg'(\theta)+r$,

we have (*): $\begin{cases}Ap-Bq=a\\Bp+Aq=b\\Cp+r=c\end{cases}$ 1M

Solving (*) and using the fact that $A^2+B^2\neq0$,

we have $p=\frac{Aa+Bb}{A^2+B^2}$, $q=\frac{Ab-Ba}{A^2+B^2}$ and $r=c-C\left(\frac{Aa+Bb}{A^2+B^2}\right)$. 1A+1A+1A

$\therefore$ p, q, r exist.

(c) Let $7\sin\theta - 4\cos\theta + 3 = p(2\sin\theta + \cos\theta + 2) + q\dfrac{d}{d\theta}(2\sin\theta + \cos\theta + 2) + r$,

then $\begin{cases} 2p - q = 7 \\ p + 2q = -4 \\ 2p + r = 3 \end{cases}$

$\Rightarrow \quad p = 2, \; q = -3, \; r = -1.$ 1A

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \frac{7\sin\theta - 4\cos\theta + 3}{2\sin\theta + \cos\theta + 2}\,d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{2(2\sin\theta + \cos\theta + 2) - 3\dfrac{d}{d\theta}(2\sin\theta + \cos\theta + 2) - 1}{2\sin\theta + \cos\theta + 2}\,d\theta$$ 1A

$$= \int_0^{\frac{\pi}{2}} 2\,d\theta - 3[\ln|2\sin\theta + \cos\theta + 2|]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta + \cos\theta + 2}$$ 1A + 1A

$$= \pi - 3\ln\frac{4}{3} - \ln\frac{3}{2}$$

$$= \pi - 5\ln 2 + 2\ln 3$$ 1A

12. (a) Equations of L_1 and L_2 are

(A) $\begin{cases} x = 2t+5 \\ y = 2t+1 \\ z = -t \end{cases}$ and (B) $\begin{cases} x = 2s+4 \\ y = 5s-8 \\ x = 2s+1 \end{cases}$ respectively.

Putting (A) into (B), we have

(C) $\begin{cases} 2t-2s = -1 & \ldots(1) \\ 2t-5s = -9 & \ldots(2) \\ t+2s = -1 & \ldots(3) \end{cases}$ (1M)

Solving (1) and (2), $s=\frac{8}{3}$ and solving (2) and (3), $s=\frac{7}{9}$.

[OR $\left(\begin{array}{cc|c} 2 & -2 & -1 \\ 2 & -5 & -9 \\ 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -2 & -1 \\ 0 & 1 & \frac{8}{3} \\ 0 & 1 & -\frac{1}{6} \end{array}\right)$]

$\therefore$ (C) is inconsistent $\Rightarrow$ L_1 and L_2 do not intersect. (1)

Alternatively,

From L_1, $x-y=4$ and $x+2z=5$.

From L_2, $5x-2y=36$ and $x-z=3$.

$\begin{cases} x-y=4 \\ 5x-2y=36 \end{cases} \Rightarrow x=\frac{28}{3}$ (1M)

$\begin{cases} x+2z=5 \\ x-z=3 \end{cases} \Rightarrow x=\frac{11}{3}$

The equations are not consistent $\Rightarrow$ L_1 and L_2 do not intersect. (1)

(b) Let the direction ratios of L be $[a:b:c]$, then

$\begin{cases} 2a+2b-c = 0 \\ 2a+5b+2c = 0 \end{cases}$ (1M)

$\Rightarrow$ $a:b:c = 3:-2:2$ (1A)

Let $A = (2t+5,\ 2t+1,\ -t)$ and $B = (2s+4,\ 5s-8,\ 2s+1)$, then

$\frac{2s-2t-1}{3} = \frac{5s-2t-9}{-2} = \frac{2s+t+1}{2}$ (1M)

$\Rightarrow$ $t = -1$ and $s = 1$ (1A)

Therefore $A = (3,\ -1,\ 1)$
$B = (6,\ -3,\ 3)$ (1A)

and eqtn. of L is $\frac{x-3}{3} = \frac{y+1}{-2} = \frac{z-1}{2}$ (1A)

or $\frac{x-6}{3} = \frac{y+3}{-2} = \frac{z-3}{2}$.

(c) (i) Equation of π is

$2(x-3)+2(y+1)-(z-1)=0$ — 1M

or $2x+2y-z-3=0$. — 1A

(ii) Sub. $B=(6,-3,3)$ into the equation of π,

L.H.S. $=2(6)+2(-3)-(3)-3=0$.

$\therefore B$ lies on π. — 1

(ii) Let the projection of L_2 on π be $\frac{x-6}{p}=\frac{y+3}{q}=\frac{z-3}{r}$, then

$$\begin{cases} 2p+2q-r=0 \\ 3p-2q+2r=0 \end{cases}$$ — 1M + 1A

$p:q:r=-2:7:10$ — 1A

Hence the equation required is

$\frac{x-6}{-2}=\frac{y-3}{7}=\frac{z-3}{10}$. — 1A

13\. (a) (i) $\int_a^b f(x)g(x)\,dx = \int_a^b f(x)w'(x)\,dx$ | 1 (for applying integration by pa

$= [f(x)w(x)]_a^b - \int_a^b w(x)f'(x)\,dx$

$= f(b)\int_a^b g(x)\,dx - \int_a^b w(x)f'(x)\,dx$ | 1

(ii) By (a)(i) and Theorem (*),

$\int_a^b f(x)g(x)\,dx = f(b)\int_a^b g(x)\,dx - w(c)\int_a^b f'(x)\,dx$

for some $c\in[a,b]$ | 1A

$= f(b)\int_a^b g(x)\,dx - \left(\int_a^c g(x)\,dx\right)(f(b)-f(a))$ | 1A

$= f(b)\int_c^b g(x)\,dx + f(a)\int_a^c g(x)\,dx$ | 1

(b) Putting $f(x) = -\frac{1}{F'(x)}$ and $g(x) = -F'(x)\cos F(x)$,

then $f(x)$ and $g(x)$ are continuously differentiable and

$f'(x) = \frac{F''(x)}{(F'(x))^2} \geq 0$ for $a\leq x\leq b$. By (a), | 1

$\left|\int_a^b \cos F(x)\,dx\right| = \left|\frac{-1}{F'(b)}\int_c^b(-F'(x)\cos F(x))\,dx\right.$

$\left.+\frac{-1}{F'(a)}\int_a^c(-F'(x)\cos F(x))\,dx\right|$ | 1A

$= \left|\frac{1}{F'(b)}[\sin F(x)]_c^b + \frac{1}{F'(a)}[\sin F(x)]_a^c\right|$ | 1M

$\leq \left|\frac{1}{F'(b)}\right||\sin F(b)| + \left|\frac{1}{F'(b)}\right||\sin F(c)|$

$+\left|\frac{1}{F'(a)}\right||\sin F(c)| + \left|\frac{1}{F'(a)}\right||\sin F(b)|$ | 1A

$\leq \frac{4}{m}$ | 1

(c) (i) For $0 \le x \le 1$ and $n \ge 1$, we have

$x^{n+1} \le x^n \Rightarrow \cos(x^n) \le \cos(x^{n+1})$

$$\Rightarrow \int_0^1 \cos(x^n)\,dx \le \int_0^1 \cos(x^{n+1})\,dx$$ 1

$\because \int_0^1 \cos(x^n)\,dx \le \int_0^1 dx = 1$

$\therefore \left\{\int_0^1 \cos(x^n)\,dx\right\}$ is monotonic increasing and bounded above.

Hence $\lim_{n\to\infty} \int_0^1 \cos(x^n)\,dx$ exists. 1

(ii) Let $F(x) = x^n$ for $x \in [1, 2\pi]$. 1A

When $n \ge 2$, $F'(x) = nx^{n-1} \ge n > 0$ and $F''(x) = n(n-1)x^{n-2} \ge 0$.

$\therefore \left|\int_1^{2\pi} \cos(x^n)\,dx\right| \le \frac{4}{n}$ by (b)

$\Rightarrow \lim_{n\to\infty} \left|\int_1^{2\pi} \cos(x^n)\,dx\right| = 0$

$\Rightarrow \lim_{n\to\infty} \int_1^{2\pi} \cos(x^n)\,dx = 0$

1M (for either)

Hence $\lim_{n\to\infty} \int_0^{2\pi} \cos(x^n)\,dx$

$= \lim_{n\to\infty} \left(\int_0^1 \cos(x^n)\,dx + \int_1^{2\pi} \cos(x^n)\,dx\right)$

$= \lim_{n\to\infty} \int_0^1 \cos(x^n)\,dx$ exists. 1