4. $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two sequences of real numbers. Define $s_k = a_1 + a_2 + \ldots + a_k$ for $k = 1, 2, \ldots, n$.

(a) Prove that $\sum_{k=1}^{n} a_k b_k = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_{n-1}(b_{n-1} - b_n) + s_n b_n$.

(b) If $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$
and there are constants m and M such that

$$m \leq s_k \leq M \quad \text{for } k = 1, 2, \ldots, n,$$

prove that $mb_1 \leq \sum_{k=1}^{n} a_k b_k \leq Mb_1$.

(5 marks)

5. Let $\{a_n\}$ be a sequence of positive numbers such that

$$a_1 + a_2 + \ldots + a_n = \left(\frac{1 + a_n}{2}\right)^2$$

for $n = 1, 2, 3, \ldots$.

Prove by induction that $a_n = 2n - 1$ for $n = 1, 2, 3, \ldots$.

(5 marks)

6. Let $\operatorname{Arg} z$ denote the principal value of the argument of the complex number z ($-\pi < \operatorname{Arg} z \leq \pi$).

(a) If $z \neq 0$ and $z + \bar{z} = 0$, show that $\operatorname{Arg} z = \pm \frac{\pi}{2}$.

(b) If $z_1, z_2 \neq 0$ and $|z_1 + z_2| = |z_1 - z_2|$, show that $\frac{z_1}{z_2} + \overline{\left(\frac{z_1}{z_2}\right)} = 0$

and hence find all possible values of $\operatorname{Arg} \frac{z_1}{z_2}$.

(5 marks)

7. (a) Let m and n be positive integers. Using the identity

$$(1+x)^n + (1+x)^{n+1} + \ldots + (1+x)^{n+m} = \frac{(1+x)^{n+m+1} - (1+x)^n}{x},$$

where $x \neq 0$, show that

$$C_n^n + C_n^{n+1} + \ldots + C_n^{n+m} = C_{n+1}^{n+m+1}.$$

(b) Using (a), or otherwise, show that

$$\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3) = 24\left(C_5^{m+5} - 1\right).$$

Hence evaluate $\sum_{r=0}^{k} r(r-1)(r-2)(r-3)$ for $k \geq 4$.

(7 marks)

SECTION B (60 marks)
Answer any **FOUR** questions from this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

8. Let $M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$, where a, b and c are non-negative real numbers.

(a) Show that $\det(M) = \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$ and $0 \le \det(M) \le (a+b+c)^3$.

(4 marks)

(b) Let $M^n = \begin{pmatrix} a_n & b_n & c_n \\ c_n & a_n & b_n \\ b_n & c_n & a_n \end{pmatrix}$ for any positive integer n, show that a_n, b_n and c_n are non-negative real numbers satisfying $a_n + b_n + c_n = (a+b+c)^n$.

(4 marks)

(c) If $a+b+c = 1$ and at least two of a, b and c are non-zero, show that

(i) $\lim_{n\to\infty} \det(M^n) = 0$,

(ii) $\lim_{n\to\infty} (a_n - b_n) = 0$ and $\lim_{n\to\infty} (a_n - c_n) = 0$,

(iii) $\lim_{n\to\infty} a_n = \frac{1}{3}$.

(7 marks)

9. (a) Consider

$$\text{(I)} \quad \begin{cases} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{cases}$$

and

$$\text{(II)} \quad \begin{cases} a_{11}x + a_{12}y + a_{13} = 0 \\ a_{21}x + a_{22}y + a_{23} = 0 \\ a_{31}x + a_{32}y + a_{33} = 0 \end{cases}.$$

(i) Show that if (I) has a unique solution, then (II) has no solution.

(ii) Show that (u, v) is a solution of (II) if and only if (ut, vt, t) are solutions of (I) for all $t \in \mathbf{R}$.

(iii) If (II) has no solution and (I) has nontrivial solutions, what can you say about the solutions of (I) ?

(5 marks)

(b) Consider

$$\text{(III)} \quad \begin{cases} -(3+k)x + y - z = 0 \\ -7x + (5-k)y - z = 0 \\ -6x + 6y + (k-2)z = 0 \end{cases}$$

and

$$\text{(IV)} \quad \begin{cases} -(3+k)x + y - 1 = 0 \\ -7x + (5-k)y - 1 = 0 \\ -6x + 6y + (k-2) = 0 \end{cases}.$$

(i) Find the values of k for which (III) has non-trivial solutions.

(ii) Find the values of k for which (IV) is consistent. Solve (IV) for each of these values of k.

(iii) Solve (III) for each k such that (III) has non-trivial solutions.

(10 marks)

10. Let **a** , **b** and **c** be vectors in $\mathbf{R}^3$.

(a) Show that **a** , **b** and **c** are linearly dependent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 ,$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$.

(5 marks)

(b) Suppose **a** , **b** and **c** are linearly independent. Show that for any vector **x** in $\mathbf{R}^3$, there are unique $x_1, x_2, x_3 \in \mathbf{R}$ such that

$$\mathbf{x} = x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} .$$

(4 marks)

(c) Let $S = \{\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} : \alpha, \beta, \gamma \in \mathbf{R}\}$.

Under what conditions on **a** , **b** and **c** will S represent

(i) a point ?
(ii) a line ?
(iii) a plane ?
(iv) the whole space ?

[Note: You are not required to give reasons.]

(6 marks)

11. A function $f : \mathbf{C} \to \mathbf{C}$ is said to be *real linear* if

$$f(\alpha z_1 + \beta z_2) = \alpha f(z_1) + \beta f(z_2)$$

for all $\alpha, \beta \in \mathbf{R}$ and $z_1, z_2 \in \mathbf{C}$.

(a) Suppose f is a real linear function. Show that

(i) if $z = 0$ whenever $f(z) = 0$, then f is injective;

(ii) if $f(i) = i f(1)$ and $f(i) \neq 0$, then f is bijective.

(4 marks)

(b) Suppose $\lambda, \mu \in \mathbf{C}$ and

$$g(z) = \lambda z + \mu \bar{z} \quad \text{for all } z \in \mathbf{C} .$$

Show that

(i) g is real linear;

(ii) g is injective if and only if $|\lambda| \neq |\mu|$.

(8 marks)

(c) If f is a real linear function, find $a, b \in \mathbf{C}$ such that

$$f(z) = az + b\bar{z} \quad \text{for all } z \in \mathbf{C} .$$

(3 marks)

12. Let $p(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, where $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Suppose $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_2 = \cos\theta_2 + i\sin\theta_2$ are two roots of $p(x) = 0$, where $0 < \theta_1 < \theta_2 < \pi$.

(a) Show that

(i) $p(x) = (x^2 - 2x\cos\theta_1 + 1)(x^2 - 2x\cos\theta_2 + 1)$,

(ii) $p'(x) = 2p(x)\left(\dfrac{x - \cos\theta_1}{x^2 - 2x\cos\theta_1 + 1} + \dfrac{x - \cos\theta_2}{x^2 - 2x\cos\theta_2 + 1}\right)$.

(5 marks)

(b) Suppose $p(w) = 0$, by considering $p(x) - p(w)$, show that

$$\frac{p(x)}{x - w} = x^3 + (w + a_1)x^2 + (w^2 + a_1w + a_2)x + (w^3 + a_1w^2 + a_2w + a_3).$$

(3 marks)

(c) Let $s_n = z_1^n + \overline{z_1}^n + z_2^n + \overline{z_2}^n$, using (a)(ii) and (b), show that

$$p'(x) = 4x^3 + (s_1 + 4a_1)x^2 + (s_2 + a_1s_1 + 4a_2)x + (s_3 + s_2a_1 + s_1a_2 + 4a_3).$$

[Hint: $\dfrac{2(x - \cos\theta_r)}{x^2 - 2x\cos\theta_r + 1} = \dfrac{1}{x - z_r} + \dfrac{1}{x - \overline{z_r}}$, $r = 1, 2$.]

Hence show that

$$s_n + a_1s_{n-1} + \dots + a_{n-1}s_1 + na_n = 0 \quad \text{for } n = 1, 2, 3, 4.$$

(7 marks)

13. Let $\mathbf{Z}_+$ be the set of all positive integers and $m, n \in \mathbf{Z}_+$.

Let $A(m,n) = (1 - x^m)(1 - x^{m+1})\dots(1 - x^{m+n-1})$,

$B(n) = (1 - x)(1 - x^2)\dots(1 - x^n)$.

(a) Show that $A(m+1, n+1) - A(m, n+1)$ is divisible by $(1 - x^{n+1})A(m+1, n)$.

(2 marks)

(b) Suppose $P(m,n)$ denote the statement

" $A(m,n)$ is divisible by $B(n)$. "

(i) Show that $P(1,n)$ and $P(m,1)$ are true.

(ii) Using (a), or otherwise, show that if $P(m, n+1)$ and $P(m+1, n)$ are true, then $P(m+1, n+1)$ is also true.

(iii) Let k be a fixed positive integer such that $P(m,k)$ is true for all $m \in \mathbf{Z}_+$. Show by induction that $P(m, k+1)$ is true for all $m \in \mathbf{Z}_+$.

(10 marks)

(c) Using (b), or otherwise, show that $P(m,n)$ is true for all $m, n \in \mathbf{Z}_+$.

(3 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 1994

PURE MATHEMATICS A-LEVEL PAPER II

2.00 pm-5.00 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C1) answer book.

1. Evaluate

(a) $\lim\limits_{x \to 1} \dfrac{1-\sqrt{x}}{1-\sqrt[5]{x}}$,

(b) $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$.

(4 marks)

2. Evaluate

(a) $\int \tan^3 x \, dx$,

(b) $\int \dfrac{x^2 - x + 2}{x(x-2)^2} dx$.

(6 marks)

3. Find the equations of the straight line which satisfies the following two conditions:

(i) passing through the point $(4, 2, -3)$,

(ii) parallel to the planes $x+y+z-10 = 0$ and $x+2y = 0$.

(4 marks)

4. The equation of a curve C in polar coordinates is

$$r = 1 + \sin\theta , \quad 0 \le \theta \le 2\pi .$$

(a) Sketch curve C .

(b) Find the area bounded by curve C .

(5 marks)

5. For $n = 1, 2, 3, \ldots$ and $\theta \in \mathbf{R}$, let $s_n = \sum_{k=1}^{n} 3^{k-1}\sin^3\left(\frac{\theta}{3^k}\right)$.

Using the identity $\sin^3\phi = \frac{3}{4}\sin\phi - \frac{1}{4}\sin 3\phi$, show that

$$s_n = \frac{3^n}{4}\sin\left(\frac{\theta}{3^n}\right) - \frac{1}{4}\sin\theta .$$

Hence, or otherwise, evaluate $\lim_{n\to\infty} s_n$.

(4 marks)

6. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function.

Show that $\int_0^1 x f(xt)\,dt = \int_0^x f(t)\,dt$ for all $x \in \mathbf{R}$.

If $\int_0^1 f(xt)\,dt = 0$ for all $x \in \mathbf{R}$,
show that $f(x) = 0$ for all $x \in \mathbf{R}$.

(5 marks)

7. Let $f(x) = \int_1^x \sin(\cos t)\,dt$, where $x \in [0, \frac{\pi}{2})$.

(a) Show that f is injective.

(b) If g is the inverse function of f, find $g'(0)$.

(6 marks)

8. (a) Show that for any $a, y \in \mathbf{R}$, $e^y - e^a \geq e^a(y-a)$.

(b) By taking $y = x^2$ in the inequality in (a), prove that

$$\int_0^1 e^{x^2}dx \geq e^{\frac{1}{3}} .$$

(6 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

9. Given an ellipse

$$(E): \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and a point $P(h, k)$ outside (E).

(a) If $y = mx + c$ is a tangent from P to (E), show that
$(h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0$.

(4 marks)

(b) Suppose the two tangents from P to (E) touch (E) at A and B.

(i) Find the equation of the line passing through A and B.

(ii) Find the coordinates of the mid-point of AB.

(6 marks)

(c) Show that the two tangents from P to (E) are perpendicular if and only if P lies on the circle $x^2 + y^2 = a^2 + b^2$.

(5 marks)

10. Let $f(x) = \dfrac{\sqrt[3]{x^2}}{x^2+1}$, $x \in \mathbf{R}$.

(a) (i) Evaluate $f'(x)$ for $x \neq 0$.
Prove that $f'(0)$ does not exist.

(ii) Determine those values of x for which $f'(x) > 0$ and those values of x for which $f'(x) < 0$.

(iii) Find the relative extreme points of $f(x)$.

(8 marks)

(b) (i) Evaluate $f''(x)$ for $x \neq 0$. Hence determine the points of inflexion of $f(x)$.

(ii) Find the asymptote of the graph of $f(x)$.

(4 marks)

(c) Using the above results, sketch the graph of $f(x)$.

(3 marks)

11. For any non-negative integer n , let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx .$$

(a) (i) Show that $\dfrac{1}{n+1}\left(\dfrac{\pi}{4}\right)^{n+1} \leq I_n \leq \dfrac{1}{n+1}\left(\dfrac{\pi}{4}\right)$.

[Note: You may assume without proof that $x \leq \tan x \leq \dfrac{4x}{\pi}$ for $x \in [0, \dfrac{\pi}{4}]$.]

(ii) Using (i), or otherwise, evaluate $\lim_{n\to\infty} I_n$.

(iii) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$ for $n = 2, 3, 4, \ldots$.

(8 marks)

(b) For $n = 1, 2, 3, \ldots$, let $a_n = \sum_{k=1}^{n} \dfrac{(-1)^{k+1}}{2k-1}$.

(i) Using (a)(iii), or otherwise, express a_n in terms of I_{2n} .

(ii) Evaluate $\lim_{n\to\infty} a_n$.

(7 marks)

12. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function satisfying the following conditions for all $x \in \mathbf{R}$:

A. $f(x) > 0$;

B. $f(x+1) = f(x)$;

C. $f(\frac{x}{4})f(\frac{x+1}{4}) = f(x)$.

Define $g(x) = \frac{d}{dx}\ln f(x)$ for $x \in \mathbf{R}$.

(a) Show that for all $x \in \mathbf{R}$,

(i) $f'(x+1) = f'(x)$;

(ii) $g(x+1) = g(x)$;

(iii) $\frac{1}{4}[g(\frac{x}{4}) + g(\frac{x+1}{4})] = g(x)$.

(8 marks)

(b) Let M be a constant such that $|g(x)| \leq M$ for all $x \in [0,1]$.

(i) Using (a), or otherwise, show that

$$|g(x)| \leq \frac{M}{2} \quad \text{for all } x \in \mathbf{R} .$$

Hence deduce that

$$g(x) = 0 \quad \text{for all } x \in \mathbf{R} .$$

(ii) Show that $f(x) = 1$ for all $x \in \mathbf{R}$.

(7 marks)

13. Let $L_n = \left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx\right\}^{\frac{1}{n}}$ for any positive integer n .

(a) Show that $L_n \leq \pi^{\frac{1}{n}}$.

(3 marks)

(b) For $n = 1, 2, 3, \ldots$, let $r_n = \cos\frac{1}{2n}$.

Find the values of x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\cos x \geq r_n$.

Hence show that $L_n \geq r_n\left(\frac{1}{n}\right)^{\frac{1}{n}}$.

(5 marks)

(c) Show that

(i) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$,

(ii) $\lim_{n\to\infty} L_n = 1$.

(7 marks)

14. (a) $f(x)$ is a continuously differentiable and strictly increasing function on $[0, c]$ such that $f(0) = 0$.
Let $b \in [0, f(c)]$.
Define $g(t) = tb - \int_0^t f(x)\,dx$, $t \in [0, c]$.

(i) Determine the interval on which $g(t)$ is strictly increasing and the interval on which $g(t)$ is strictly decreasing. Hence show that

$$g(t) \le g(f^{-1}(b)) \quad \text{for all } t \in [0, c] .$$

(ii) Using the substitution $y = f(x)$ and integration by parts, show that

$$\int_0^b f^{-1}(y)\,dy = g(f^{-1}(b)) .$$

(iii) If $a \in [0, c]$, prove the inequality

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \ge ab .$$

(10 marks)

(b) If a, b, p, q are positive numbers and $\frac{1}{p} + \frac{1}{q} = 1$, prove that

$$\frac{1}{p} a^p + \frac{1}{q} b^q \ge ab .$$

(5 marks)

END OF PAPER