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SECTION A

1.

$$(*)\begin{cases} 3x - y + z = 1 & \cdots\cdots\cdots\cdots (1) \\ 2x - 4y - 5z = 1 & \cdots\cdots\cdots\cdots (2) \\ 4x + 2y + 7z = c & \cdots\cdots\cdots\cdots (3) \end{cases}$$

$3\times(2),\ 3\times(3)$:

$$\begin{cases} 3x - y + z = 1 & \cdots\cdots (1) \\ 6x - 12y - 15z = 3 & \cdots\cdots (4) \\ 12x + 6y - 21z = 3c & \cdots\cdots (5) \end{cases}$$

$(4) - 2\times(1),\ (5) - 4\times(1)$:

$$\begin{cases} 3x-y+z=1 & \cdots\cdots (1) \\ -10y - 17z = 1 & \cdots\cdots (6) \\ 10y + 17z = 3c - 4 & \cdots\cdots (7) \end{cases}$$

$(7) + (6)$:

$$\begin{cases} 3x-y+z = 1 & \cdots\cdots (1) \\ -10y-17z = 1 & \cdots\cdots (6) \\ 0 = 3c - 3 & \cdots\cdots (8) \end{cases}$$

If the system (*) is consistent, $3c-3 = 0$
$c = 1$

Put $z = t$, from (6) and (1)

$$y = \frac{-17t - 1}{10}$$

and $$x = \frac{-9t + 3}{10}$$

i.e.

$$\begin{cases} x = \dfrac{-9t + 3}{10} \\ y = \dfrac{-17t - 1}{10} \\ z = t \end{cases} \quad \text{where } t \in R$$

2. $$\frac{1}{x(x+1)(x+2)} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

$$\therefore\ 1 \equiv A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$$
$$\equiv (A + B + C)x^2 + (3A + 2B + C)x + 2A$$

Hence
$$\begin{cases} A + B + C = 0 \\ 3A + 2B + C = 0 \\ 2A = 1 \end{cases}$$

On solving $A = \frac{1}{2}$

$B = -1$

$C = \frac{1}{2}$

$$\therefore\ \frac{1}{x(x+1)(x+2)} \equiv \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$$

(b) $$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^{n}\left[\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}\right]$$

$$= \sum_{k=1}^{n} \frac{1}{2k} - \sum_{k=1}^{n} \frac{1}{k+1} + \sum_{k=1}^{n} \frac{1}{2(k+2)}$$

$$= \sum_{k=0}^{n-1} \frac{1}{2(k+1)} - \sum_{k=1}^{n} \frac{1}{k+1} + \sum_{k=2}^{n+1} \frac{1}{2(k+1)}$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \sum_{k=2}^{n-1} \frac{1}{2(k+1)}\right) - \left(\frac{1}{2} + \sum_{k=2}^{n-1} \frac{1}{k+1} + \frac{1}{n+1}\right)$$

$$+ \left(\sum_{k=2}^{n-1} \frac{1}{2(k+1)} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)}\right)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} - \frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$+ \sum_{k=2}^{n-1}\left(\frac{1}{2(k+1)} - \frac{1}{k+1} + \frac{1}{2(k+1)}\right)$$

$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$\therefore\ \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}$$

3. (a)
$$\begin{cases} \alpha + \beta + \gamma = -A & \cdots\cdots\cdots\cdots (1) \\ \alpha\beta + \beta\gamma + \gamma\alpha = B & \cdots\cdots\cdots\cdots (2) \\ \alpha\beta\gamma = -C & \cdots\cdots\cdots\cdots (3) \end{cases}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= A^2 - 2B \quad \text{(by (1) and (2))}$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$$
$$= B^2 - 2AC \quad \text{(by (1), (2) and (3))}$$

(b) The required equation is

$$x^3 - (\alpha^2 + \beta^2 + \gamma^2)x^2 + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)x - \alpha^2\beta^2\gamma^2 = 0$$

$$x^3 - [0^2 - 2(-3)]x^2 + [(-3)^2 - 2(0)(1)]x - (1)^2 = 0$$

$$x^3 - 6x^2 + 9x - 1 = 0$$

4. (a) $C^n_k = \frac{n!}{k!(n-k)!}$

$$= \frac{(n+1)(n!)}{(k+1)[k!(n-k)!]}\cdot\frac{(k+1)}{(n+1)}$$

$$= \frac{(n+1)!}{(k+1)![(n+1)-(k+1)]!}\cdot\frac{(k+1)}{(n+1)}$$

$$= \frac{k+1}{n+1}C^{n+1}_{k+1}$$

(b) Consider $(1+x)^{n+1} = \sum_{k=0}^{n+1} C^{n+1}_k x^k$

Set $x = -1$, $0 = \sum_{k=0}^{n+1}(-1)^k C^{n+1}_k$

(c) $\sum_{k=0}^{n} \frac{(-1)^k}{k+1} C^n_k = \sum_{k=0}^{n} \frac{(-1)^k}{n+1} C^{n+1}_{k+1}$ (by (a))

$$= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{n+1} C^{n+1}_k = \sum_{k=0}^{n+1} \frac{(-1)^{k-1}}{n+1} C^{n+1}_k - \frac{(-1)^{-1}}{n+1}$$

$$= \frac{-1}{n+1}\left(\sum_{k=0}^{n+1}(-1)^k C^{n+1}_k - 1\right)$$

$$= \frac{-1}{n+1}(0 - 1) \quad \text{(by (b))}$$

$$= \frac{1}{n+1}$$

5. (a) Let $z = \cos\theta + i\sin\theta$

$$\begin{cases} \frac{1}{z} = \cos\theta - i\sin\theta \\ z^5 = \cos 5\theta + i\sin 5\theta \\ \frac{1}{z^5} = \cos 5\theta - i\sin 5\theta \end{cases}$$

$$\cos 5\theta = \frac{1}{2}(z^5 + \frac{1}{z^5})$$

$$= \frac{1}{2}[(\cos\theta + i\sin\theta)^5 + (\cos\theta - i\sin\theta)^5]$$

$$= \frac{1}{2}[2\cos^5\theta - 20\cos^3\theta\sin^2\theta + 10\cos\theta\sin^4\theta]$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(b) Consider $\cos 5\theta = 0$ ...... (*)

$\therefore$ $5\theta = (2n+1)\frac{\pi}{2}$, where $n = 0, \pm1, \pm2, \ldots$

$$\theta = (2n+1)\frac{\pi}{10}$$

Hence, the solutions of the equation

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$$

for θ between 0 and 2π are $\frac{\pi}{10}$, $\frac{3\pi}{10}$, $\frac{\pi}{2}$, $\frac{7\pi}{10}$; $\frac{9\pi}{10}$.

Since $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$, the solutions of the equation $16\cos^4\theta - 20\cos^2\theta + 5 = 0$ for θ between 0 and 2π are $\frac{\pi}{10}$, $\frac{3\pi}{10}$, $\frac{7\pi}{10}$, $\frac{9\pi}{10}$.

$\therefore$ $\cos\frac{\pi}{10}$, $\cos\frac{3\pi}{10}$, $\cos\frac{7\pi}{10}$, $\cos\frac{9\pi}{10}$ are roots of the equation $16t^4 - 20t^2 + 5 = 0$

Hence $\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10} = \frac{5}{16}$

$$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}(-\cos\frac{3\pi}{10})(-\cos\frac{\pi}{10}) = \frac{5}{16}$$

$$\cos^2\frac{\pi}{10}\cos^2\frac{3\pi}{10} = \frac{5}{16}$$

6. $|x-1| - |x+2| > 2$

$$|x-1| > 2 + |x+2|$$

$$(x-1)^2 > [2 + |x+2|]^2$$

$$(x-1)^2 > 4 + 4|x+2| + (x+2)^2$$

$$-6x-7 > 4|x+2|$$

case (i) $x \leq -2$

$$-6x-7 > -4(x+2)$$

$$1 > 2x$$

$$\frac{1}{2} > x$$

In this case, the solution is $x \leq -2$.

case (ii) $x > -2$

$$-6x-7 > 4(x+2)$$

$$-15 > 10x$$

$$-\frac{3}{2} > x$$

$\therefore$ In this case, the solution is $-2 < x < \frac{-3}{2}$.

Hence, the solution of the given inequality is

$$\{x \in \mathbb{R} : x < \frac{-3}{2}\}$$

7. Induce on n

(i) when $n = 0$, $\frac{1}{\sqrt{5}}(\alpha^0 - \beta^0) = 0$

when $n = 1$, $\frac{1}{\sqrt{5}}(\alpha - \beta) = \frac{1}{\sqrt{5}}\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$$= \frac{1}{\sqrt{5}}\sqrt{(-1)^2 - 4(-1)}$$

$$= 1$$

$\therefore$ It is true for $n=0$ and $n=1$.

(ii) Assume $a_k = \frac{1}{\sqrt{5}}(\alpha^k - \beta^k)$ and $a_{k+1} = \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1})$

when $n=k+2$, $\frac{1}{\sqrt{5}}(\alpha^{k+2} - \beta^{k+2})$

$$= \frac{1}{\sqrt{5}}[(\alpha^{k+1} - \beta^{k+1})(\alpha+\beta) - \alpha\beta(\alpha^k - \beta^k)]$$

$$= \frac{1}{\sqrt{5}}[\alpha^{k+1} - \beta^{k+1}](\alpha+\beta) - \frac{1}{\sqrt{5}}(\alpha^k - \beta^k)(\alpha\beta)$$

$$= a_{k+1}(-1) - a_k(-1)$$

$$= -a_{k+1} + a_k$$

$$= a_{k+2}$$

$\therefore$ It is true for $n=k+2$

Hence, from the Principle of Mathematical Induction, it is true for all non-negative integers n.

As α, β are roots of the equation $x^2 + x - 1 = 0$ and $\alpha > 0$, $\beta < 0$,

$$\alpha = \frac{-1 + \sqrt{5}}{2} \text{ and } \beta = \frac{-1 - \sqrt{5}}{2}$$

Consider $\frac{\alpha}{\beta} = \frac{-1 + \sqrt{5}}{-1 - \sqrt{5}}$

$$= \frac{(-1 + \sqrt{5})^2}{-4}$$

$$= \frac{6 - 2\sqrt{5}}{-4}$$

$\therefore$ $|\frac{\alpha}{\beta}| < 1$.

Hence $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})}{\frac{1}{\sqrt{5}}(\alpha^n - \beta^n)}$$

$$= \lim_{n\to\infty} \frac{(\frac{\alpha}{\beta})^{n+1} - 1}{(\frac{\alpha}{\beta})^n \cdot \frac{1}{\beta} - \frac{1}{\beta}}$$

$$= \frac{-1}{\frac{-1}{\beta}} \qquad (\because \lim_{n\to\infty}(\frac{\alpha}{\beta})^n = 0)$$

$$= \beta$$

SECTION B

8. (a) (i) $(X + Y)^2 = X^2 + XY + YX + Y^2$

$$= X^2 + 2XY + Y^2 \qquad (\because XY = YX)$$

(ii) Induce on n

(1) when $n=3$, $(X + Y)^3 = (X + Y)^2(X + Y)$

$$= (X^2 + 2XY + Y^2)(X + Y)$$

$$= X^3 + X^2Y + 2XYX + 2XY^2 + Y^2X + Y^3$$

$$= X^3 + X^2Y + 2XXY + 2XY^2 + YXY + Y^3 \qquad (\because XY = YX)$$

$$= X^3 + 3X^2Y + 2XY^2 + XY^2 + Y^3 \qquad (\because XY = YX)$$

$$= X^3 + 3X^2Y + 3XY^2 + Y^3$$

$$= \sum_{r=0}^{3} C^3_r X^{3-r} Y^r$$

(2) Assume $(X+Y)^k = \sum_{r=0}^{k} C^k_r X^{k-r} Y^r$ where k is an integer greater than 2

$$(X+Y)^{k+1} = (X+Y)(X+Y)^k$$

$$= (X+Y) \sum_{r=0}^{k} C^k_r X^{k-r} Y^r$$

$$= \sum_{r=0}^{k} C^k_r X^{k+1-r} Y^r + \sum_{r=0}^{k} C^k_r X^{k-r} Y^{r+1}$$

$$(\because XY = YX, \therefore YX^{k-r}Y^r = X^{k-r}Y^{r+1})$$

$$= \sum_{r=0}^{k} C^k_r X^{k+1-r} Y^r + \sum_{r=0}^{k+1} C^k_{r-1} X^{k-(r-1)} Y^r$$

$$= C^k_0 X^{k+1} + \sum_{r=1}^{k} (C^k_r + C^k_{r-1}) X^{k+1-r} Y^r + C^k_k Y^{k+1}$$

$$= C^{k+1}_0 X^{k+1} + \sum_{r=1}^{k} C^{k+1}_r X^{k+1-r} Y^r + C^{k+1}_{k+1} Y^{k+1}$$

$$(\because C^k_0 = C^{k+1}_0 = 1,\quad C^k_k = C^{k+1}_{k+1} = 1,\quad C^{k+1}_r = C^k_r + C^k_{r-1})$$

$$= \sum_{r=0}^{k+1} C^{k+1}_r X^{k+1-r} Y^r$$

$\therefore$ From (1) and (2), by the Principle of Mathematical Induction, it's true for n = 3, 4, 5, ...

(b) $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$= I + Y$

$$Y^2 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^{100} = \sum_{r=0}^{100} C^{100}_r I^{100-r} Y^r$$

$$= \sum_{r=0}^{2} C^{100}_r Y^r \qquad (\because Y^r = 0 \text{ for } r = 3, 4, 5 \ldots)$$

$$= \begin{pmatrix} 1 & 200 & 30100 \\ 0 & 1 & 300 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) (i) $(X+Y)^2 = X^2 + 2XY + Y^2$

$$X^2 + XY + YX + Y^2 = X^2 + 2XY + Y^2$$

$$YX = XY$$

(ii) Let $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$

$$X + Y = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

$$(X+Y)^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$(X+Y)^3 = \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{1}{4} & 0 \end{pmatrix}$$

$$X^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$Y^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

$$Y^3 = \begin{pmatrix} \frac{-1}{8} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore X^3 + 3X^2Y + 3XY^2 + Y^3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{-1}{8} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{1}{4} & 0 \end{pmatrix}$$

$$\neq (X + Y)^3$$

$$XY = \begin{pmatrix} \frac{-1}{2} & 0 \\ \frac{-1}{2} & 0 \end{pmatrix}$$

$$YX = \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore \; XY \neq YX$

Hence, $(X + Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$ does not imply $XY = YX$.

(a) $\forall \underline{u} \in S$, $\underline{u} = \alpha_1\underline{w}_1 + \alpha_2\underline{w}_2$, where $\alpha_1, \alpha_2 \in R$

$(\underline{u}\cdot\underline{w}_1)\underline{w}_1 + (\underline{u}\cdot\underline{w}_2)\underline{w}_2$

$= [(\alpha_1\underline{w}_1 + \alpha_2\underline{w}_2)\cdot\underline{w}_1]\underline{w}_1 + [(\alpha_1\underline{w}_1 + \alpha_2\underline{w}_2)\cdot\underline{w}_2]\underline{w}_2$

$= (\alpha_1\underline{w}_1\cdot\underline{w}_1 + \alpha_2\underline{w}_2\cdot\underline{w}_1)\underline{w}_1 + (\alpha_1\underline{w}_1\cdot\underline{w}_2 + \alpha_2\underline{w}_2\cdot\underline{w}_2)\underline{w}_2$

$= \alpha_1\underline{w}_1 + \alpha_2\underline{w}_2$ $\quad (\because \underline{w}_1\cdot\underline{w}_2 = 0,\ \underline{w}_1\cdot\underline{w}_1 = \underline{w}_2\cdot\underline{w}_2 = 1)$

$= \underline{u}$

(b) (i) $(\underline{v} - \underline{w})\cdot\underline{u} = \underline{v}\cdot\underline{u} - \underline{w}\cdot\underline{u}$

$= \underline{v}\cdot[(\underline{u}\cdot\underline{w}_1)\underline{w}_1 + (\underline{u}\cdot\underline{w}_2)\underline{w}_2] - [(\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2]\cdot\underline{u}$

$= (\underline{u}\cdot\underline{w}_1)(\underline{v}\cdot\underline{w}_1) + (\underline{u}\cdot\underline{w}_2)(\underline{v}\cdot\underline{w}_2) - (\underline{v}\cdot\underline{w}_1)(\underline{w}_1\cdot\underline{u}) - (\underline{v}\cdot\underline{w}_2)(\underline{w}_2\cdot\underline{u})$

$= (\underline{u}\cdot\underline{w}_1)(\underline{v}\cdot\underline{w}_1) + (\underline{u}\cdot\underline{w}_2)(\underline{v}\cdot\underline{w}_2) - (\underline{u}\cdot\underline{w}_1)(\underline{v}\cdot\underline{w}_1) - (\underline{u}\cdot\underline{w}_2)(\underline{v}\cdot\underline{w}_2)$

$= 0$

(ii) $\underline{w}\cdot\underline{w} = [(\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2]\cdot[(\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2]$

$= (\underline{v}\cdot\underline{w}_1)^2\,\underline{w}_1\cdot\underline{w}_1 + 2(\underline{v}\cdot\underline{w}_2)(\underline{v}\cdot\underline{w}_1)(\underline{w}_2\cdot\underline{w}_1) + (\underline{v}\cdot\underline{w}_2)^2(\underline{w}_2\cdot\underline{w}_2)$

$= (\underline{v}\cdot\underline{w}_1)^2 + (\underline{v}\cdot\underline{w}_2)^2$ $\quad (\because \underline{w}_1\cdot\underline{w}_2 = 0,\ \underline{w}_1\cdot\underline{w}_1 = \underline{w}_2\cdot\underline{w}_2 = 1)$

$= \underline{v}\cdot[(\underline{v}\cdot\underline{w}_1)\underline{w}_1] + \underline{v}\cdot[(\underline{v}\cdot\underline{w}_2)\underline{w}_2]$

$= \underline{v}\cdot[(\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2]$

$= \underline{v}\cdot\underline{w}$ ..... (*)

$\underline{v}\cdot\underline{v} - \underline{w}\cdot\underline{w} = \underline{v}\cdot\underline{v} - \underline{w}\cdot\underline{v} + \underline{w}\cdot\underline{v} - \underline{w}\cdot\underline{w}$

$= \underline{v}\cdot\underline{v} - \underline{w}\cdot\underline{v} + \underline{w}\cdot\underline{w} - \underline{w}\cdot\underline{v}$ (by (*))

$= (\underline{v} - \underline{w})\cdot(\underline{v} - \underline{w})$

$= |\underline{v} - \underline{w}|^2$

≥ 0

$\therefore \; \underline{v}\cdot\underline{v} \geq \underline{w}\cdot\underline{w}$

$\underline{v}\cdot\underline{v} = \underline{w}\cdot\underline{w}$

$\Leftrightarrow \; |\underline{v} - \underline{w}| = 0$

$\Leftrightarrow \; \underline{v} - \underline{w} = 0$

$\Leftrightarrow \; \underline{v} = \underline{w}$ .... (*)

"If" part:

$\underline{v} \in S$

$\Rightarrow \; \underline{v} = (\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2$ (from (a))

$\Rightarrow \; \underline{v} = \underline{w}$ (by definition of $\underline{w}$)

$\Rightarrow \; \underline{v}\cdot\underline{v} = \underline{w}\cdot\underline{w}$ (by (*))

"Only if" part:

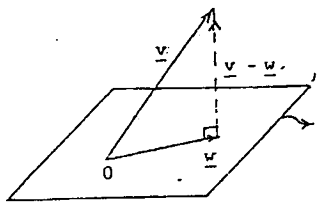
$\underline{w}\cdot\underline{w} = \underline{v}\cdot\underline{v}$

$\Rightarrow \; \underline{v} = \underline{w}$. (by (*))

$\Rightarrow \underline{v} = (\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2 \quad (\because \underline{w} = (\underline{v}\cdot\underline{w}_1)\underline{w}_1 + (\underline{v}\cdot\underline{w}_2)\underline{w}_2)$

$\Rightarrow \underline{v} \in S$

S is the set of points lying on a plane containing the vectors $\underline{w}_1$, $\underline{w}_2$ and passing through the origin.



10. (a) $f(z) = r(\cos\theta + i\sin\theta) + \dfrac{1}{r(\cos\theta + i\sin\theta)}$

$= r(\cos\theta + i\sin\theta) + \dfrac{1}{r}(\cos\theta - i\sin\theta)$

$= (r + \dfrac{1}{r})\cos\theta + (i\sin\theta)(r - \dfrac{1}{r})$

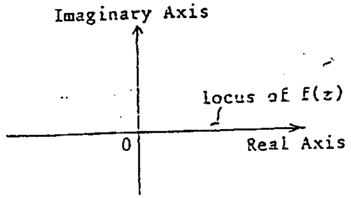
By comparing the real parts and imaginary parts,

$u = (r + \dfrac{1}{r})\cos\theta$ and $v = (r - \dfrac{1}{r})\sin\theta$

(b) (i) $|z| = 1 \Rightarrow r = 1$

$\therefore u = 2\cos\theta$ and $v = 0$

Hence, the locus of $f(z)$ is the real axis.



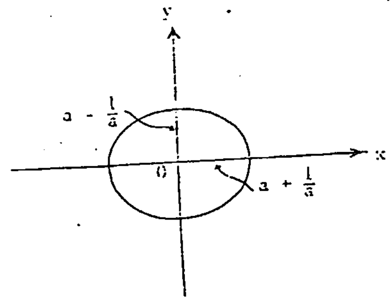
(ii) $|z| = a \Rightarrow r = a$

$\therefore \begin{cases} u = (a + \dfrac{1}{a})\cos\theta \\ v = (a - \dfrac{1}{a})\sin\theta \end{cases}$

$\dfrac{u^2}{(a + \frac{1}{a})^2} + \dfrac{v^2}{(a - \frac{1}{a})^2} = 1$

Hence, the locus of $f(z)$ is an ellipse centred at origin and with lengths of semi-axes

$a + \dfrac{1}{a}$ and $a - \dfrac{1}{a}$.



(c) $f(3) = 3 + \dfrac{1}{3} = \dfrac{10}{3}$

$f(\dfrac{1}{3}) = \dfrac{1}{3} + \dfrac{1}{\frac{1}{3}} = \dfrac{10}{3}$

$\therefore$ f is not injective.

$\forall w \in C, \ \exists \dfrac{w + \sqrt{w^2 - 4}}{2} \in C$

Suppose $\dfrac{w + \sqrt{w^2 - 4}}{2} = 0$

$w = -\sqrt{w^2 - 4}$

$w^2 = w^2 - 4$

$0 = -4$, which is false.

Hence, $\frac{w + \sqrt{w^2 - 4}}{2} \neq 0$

$\therefore \frac{w + \sqrt{w^2 - 4}}{2} \in C\setminus\{0\}$

$$f\left(\frac{w + \sqrt{w^2 - 4}}{2}\right) = \frac{w + \sqrt{w^2 - 4}}{2} + \frac{1}{\frac{w + \sqrt{w^2 - 4}}{2}}$$

$$= \frac{w + \sqrt{w^2 - 4}}{2} + \frac{2(w - \sqrt{w^2 - 4})}{w^2 - (w^2 - 4)}$$

$$= \frac{w + \sqrt{w^2 - 4}}{2} + \frac{w - \sqrt{w^2 - 4}}{2}$$

$$= w$$

$\therefore$ f is surjective.

(d) $\forall z_1, z_2 \in E$, $f_E(z_1) = f_E(z_2)$

$\Rightarrow z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2}$

$\Rightarrow (z_1 - z_2) + \frac{z_2 - z_1}{z_1 z_2} = 0$

$\Rightarrow (z_1 - z_2)(1 - \frac{1}{z_1 z_2}) = 0$

$\Rightarrow z_1 = z_2$ and $z_1 z_2 = 1$

$\Rightarrow z_1 = z_2$ ($\because |z_1| < 1$, $|z_2| < 1$, $\therefore |z_1 z_2| < 1$ and hence $z_1 z_2 \neq 1$)

$\Rightarrow$ f is injective

For $-2 \in C$, if $f_E(z) = -2$ for some $z \in C\setminus\{0\}$

($\because$ f is surjective

$\therefore$ z exists)

$z + \frac{1}{z} = -2$

$z^2 + 2z + 1 = 0$

$(z+1)^2 = 0$

$z = -1$

$\because |z| = 1 \nless 1$,

$\therefore$ The pre-image of -2 under f_E does not exist.

i.e. f_E is not surjective

11. (a) (i) $\because u_1 \geq v_1 > 0$ and $u_n = \frac{u_{n-1} + v_{n-1}}{2}$, $v_n = \frac{2u_{n-1}v_{n-1}}{u_{n-1} + v_{n-1}}$

$\therefore u_n > 0$ and $v_n > 0$, $\forall n = 0, 1, 2, \ldots$

Consider $u_n - v_n = \frac{u_{n-1} + v_{n-1}}{2} - \frac{2u_{n-1}v_{n-1}}{u_{n-1} + v_{n-1}}$

$$= \frac{(u_{n-1} - v_{n-1})^2}{2(u_{n-1} + v_{n-1})}$$

≥ 0 ($\because u_{n-1}, v_{n-1} > 0$)

$\therefore u_n \geq v_n$.

(ii) $u_n - u_{n-1} = \frac{u_{n-1} + v_{n-1}}{2} - u_{n-1}$

$$= \frac{v_{n-1} - u_{n-1}}{2}$$

≤ 0 (by (a)(i))

$\therefore u_n \leq u_{n-1}$.

Consider $u_n v_n = \left(\frac{u_{n-1} + v_{n-1}}{2}\right)\left(\frac{2u_{n-1}v_{n-1}}{u_{n-1} + v_{n-1}}\right)$

$$= u_{n-1}v_{n-1}$$

$$\frac{u_n}{u_{n-1}} = \frac{v_{n-1}}{v_n}$$

$\because \frac{u_n}{u_{n-1}} \leq 1$

$\therefore v_{n-1} \leq v_n$

Hence, $\{u_n\}$ is monotonic decreasing while $\{v_n\}$ is monotonic increasing.

(iii) From (i) & (ii), $u_0 \ge u_1 \ge u_2 \ge \cdots \ge u_n \ge v_n \ge v_{n-1}$

$\ge \cdots \ge v_2 \ge v_1 \ge v_0$

∴ $\{u_n\}$ is monotonic decreasing and is bounded from below by v_0 while $\{v_n\}$ is monotonic increasing and is bounded from above by u_0.

Hence, $\lim\limits_{n\to\infty} u_n$ and $\lim\limits_{n\to\infty} v_n$ exist.

(b) (i) $u_n - v_n = \dfrac{u_{n-1}+v_{n-1}}{2} - \dfrac{2u_{n-1}v_{n-1}}{u_{n-1}+v_{n-1}}$

$= \dfrac{1}{2}\dfrac{(u_{n-1}-v_{n-1})^2}{u_{n-1}+v_{n-1}} \le \dfrac{u_{n-1}-v_{n-1}}{2}$

$\le \dfrac{1}{2^2}(u_{n-2}-v_{n-2})$

$\cdots \qquad \left(\because \dfrac{u_{n-1}-v_{n-1}}{u_{n-1}+v_{n-1}} = 1 - \dfrac{2v_{n-1}}{u_{n-1}+v_{n-1}} \le 1\right)$

$\le \dfrac{1}{2^n}(u_0 - v_0)$

(ii) From (a)(i), $u_n \ge v_n$

∴ From (b) (i), $0 \le u_n - v_n \le \dfrac{1}{2^n}(u_0 - v_0)$

As $\lim\limits_{n\to\infty} \dfrac{1}{2^n}(u_0 - v_0) = 0$, we have $\lim\limits_{n\to\infty}(u_n - v_n) = 0$

From (a) (iii), $\lim\limits_{n\to\infty} v_n$ exists.

Hence, $\lim\limits_{n\to\infty}(u_n - v_n) + \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} v_n$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n$

(iii) Consider $u_n v_n = u_{n-1}v_{n-1}$

$= u_{n-2}v_{n-2}$

$= \cdots$

$\cdots$

$= u_0 v_0$

∴ $\lim\limits_{n\to\infty} u_n v_n = u_0 v_0$

From (b) (ii), $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n$

∴ $\lim\limits_{n\to\infty} u_n v_n = (\lim\limits_{n\to\infty} u_n)^2$

i.e. $\lim\limits_{n\to\infty} u_n = \sqrt{u_0 v_0}$ (Positive root is adopted since $u_n \ge v_0 > 0$)

12. (a) (i) $f(x) = x^p - px$

$f'(x) = px^{p-1} - p$

$= p(x^{p-1} - 1)$

$\forall x \in (0,1)$, $f'(x) < 0$ $(\because p > 1)$

$\forall x \in (1,\infty)$, $f'(x) > 0$ $(\because p > 1)$

i.e. f is strictly decreasing on $(0,1)$ and strictly increasing on $(1,\infty)$.

∴ When $x=1$, $f(x)$ attains its absolute minimum value and the absolute minimum value is $(1-p)$.

(ii) From (a)(i), $f(x) \ge f(1)$, $\forall x > 0$

$x^p - px \ge 1-p$

$x^p - 1 \ge p(x-1)$

(b) (i) For $x = \alpha\gamma$

From (a) (ii) $(\alpha\gamma)^p - 1 \geq p(\alpha\gamma - 1)$

$$\alpha^{p-1}\gamma^p - \frac{1}{\alpha} \geq p(\gamma - \frac{1}{\alpha}) \quad \ldots\ldots (1) \quad (\because \alpha > 0)$$

For $x = \beta\delta$.

from (a) (ii) $(\beta\delta)^p - 1 \geq p(\beta\delta - 1)$

$$\beta^{p-1}\delta^p - \frac{1}{\beta} \geq p(\delta - \frac{1}{\beta}) \quad \ldots (2) \quad (\because \beta > 0)$$

(1)+(2): $\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p - (\frac{1}{\alpha} + \frac{1}{\beta}) \geq p(\gamma+\delta) - p(\frac{1}{\alpha} + \frac{1}{\beta})$

$$\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p - 1 \geq p - p$$

$$\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p \geq 1$$

From (a), Equality holds

$\Leftrightarrow x = 1$

$\Leftrightarrow \alpha\gamma = \beta\delta = 1$

(ii) Put $\alpha = \frac{a+b}{a} > 0$, $\beta = \frac{a+b}{b} > 0$, $\gamma = \frac{c}{c+d} > 0$, $\delta = \frac{d}{c+d} > 0$

$\therefore \frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\gamma + \delta = 1$

From (b) (i),

$$(\frac{a+b}{a})^{p-1}(\frac{c}{c+d})^p + (\frac{a+b}{b})^{p-1}(\frac{d}{c+d})^p \geq 1$$

$$(\frac{a+b}{a})^{p-1}c^p + (\frac{a+b}{b})^{p-1}d^p \geq (c+d)^p \quad (\because c+d > 0)$$

(c) Put $a = (\sum_{j=1}^{n} a_j^p)^{\frac{1}{p}} > 0$, $b = (\sum_{j=1}^{n} b_j^p)^{\frac{1}{p}} > 0$,

$c = a_i > 0$, $d = b_i > 0$

From (b) (ii) $(\frac{a+b}{a})^{p-1}a_i^p + (\frac{a+b}{b})^{p-1}b_i^p \geq (a_i + b_i)^p$

$$\therefore \sum_{i=1}^{n} \{(\frac{a+b}{a})^{p-1}a_i^p + (\frac{a+b}{b})^{p-1}b_i^p\} \geq \sum_{i=1}^{n}(a_i + b_i)^p$$

$$(\frac{a+b}{a})^{p-1}\sum_{i=1}^{n} a_i^p + (\frac{a+b}{b})^{p-1}\sum_{i=1}^{n} b_i^p \geq \sum_{i=1}^{n}(a_i + b_i)^p$$

$$(\frac{a+b}{a})^{p-1}a^p + (\frac{a+b}{b})^{p-1}b^p \geq \sum_{i=1}^{n}(a_i + b_i)^p$$

$$(a+b)^{p-1}(a+b) \geq \sum_{i=1}^{n}(a_i + b_i)^p$$

$$(a+b)^p \geq \sum_{i=1}^{n}(a_i + b_i)^p$$

$$a+b \geq (\sum_{i=1}^{n}(a_i + b_i)^p)^{\frac{1}{p}}$$

i.e. $(\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} b_i^p)^{\frac{1}{p}} \geq (\sum_{i=1}^{n}(a_i + b_i)^p)^{\frac{1}{p}}$

Equality holds $\Leftrightarrow \alpha\gamma = \beta\delta = 1$ (from (b))

$\Leftrightarrow (\frac{a+b}{a})(\frac{c}{c+d}) = (\frac{a+b}{b})(\frac{d}{c+d}) = 1$

$\Leftrightarrow (\frac{a+b}{a})(\frac{a_i}{a_i+b_i}) = (\frac{a+b}{b})(\frac{b_i}{a_i+b_i}) = 1, \forall i$

$\Leftrightarrow \frac{a_i}{a} = \frac{b_i}{b}, \forall i$

$\Leftrightarrow \frac{a_i}{b_i} = \frac{a}{b}, \forall i$

$\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a}{b}$

13. (a) (i)

$$M_\theta M_\phi = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

$$= M_{\theta+\phi}$$

(ii) From (i) $M_\theta M_{(-\theta)} = M_0$

$= I$

$\therefore (M_\theta)^{-1} = M_{(-\theta)}$

(b) (i) (1) $\forall (x,y) \in R^2$,

$$M_0\begin{pmatrix}x\\y\end{pmatrix} = I\begin{pmatrix}x\\y\end{pmatrix}$$

$$= \begin{pmatrix}x\\y\end{pmatrix}$$

$\therefore$ $(x,y) \sim (x,y)$

i.e. $\sim$ is reflexive.

(2) $\forall (x_1,y_1), (x_2,y_2) \in R^2$,

$(x_1,y_1) \sim (x_2,y_2) \Rightarrow M_\theta\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$ for some $\theta \in R$

$\Rightarrow \begin{pmatrix}x_1\\y_1\end{pmatrix} = M_{(-\theta)}\begin{pmatrix}x_2\\y_2\end{pmatrix}$ (by (a)(iii))

$\Rightarrow (x_2,y_2) \sim (x_1,y_1)$ ($\because (-\theta) \in R$)

$\Rightarrow$ $\sim$ is symmetric

(3) $\forall (x_1,y_1), (x_2,y_2), (x_3,y_3) \in R^2$

$(x_1,y_1) \sim (x_2,y_2)$ and $(x_2,y_2) \sim (x_3,y_3)$

$\Rightarrow M_\theta\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$ and $M_\phi\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}x_3\\y_3\end{pmatrix}$ for some $\theta, \phi \in R$

$\Rightarrow M_\phi M_\theta\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_3\\y_3\end{pmatrix}$

$\Rightarrow M_{\phi+\theta}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_3\\y_3\end{pmatrix}$

$\Rightarrow (x_1,y_1) \sim (x_3,y_3)$ ($\because \phi + \theta \in R$)

$\Rightarrow$ $\sim$ is transitive

$\therefore$ From (1), (2) & (3), $\sim$ is an equivalence relation.

(ii) $(x,y) \in S$

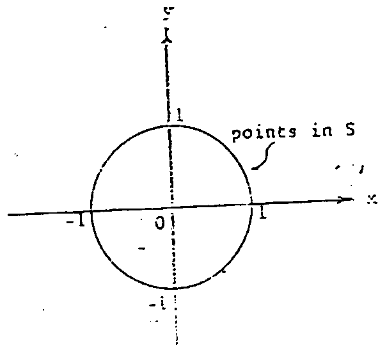
$\Rightarrow (x,y) \sim (1,0)$

$\Rightarrow \exists\theta \in R$ s.t. $\begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\Rightarrow \begin{cases}x\cos\theta + y\sin\theta = 1\\-x\sin\theta + y\cos\theta = 0\end{cases}$

$\Rightarrow (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 = 1$

$\Rightarrow x^2 + y^2 = 1$



(c) (i) $\forall [x_1,y_1], [x_2,y_2] \in R^2/\sim$

$[x_1,y_1] = [x_2,y_2] \Rightarrow (x_1,y_1) \sim (x_2,y_2)$

$\Rightarrow \begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$;

where $\theta \in R$

$\Rightarrow \begin{cases}x_1\cos\theta + y_1\sin\theta = x_2\\-x_1\sin\theta + y_1\cos\theta = y_2\end{cases}$

$\Rightarrow (x_1\cos\theta + y_1\sin\theta)^2 + (-x_1\sin\theta + y_1\cos\theta)^2$

$= x_2^2 + y_2^2$

$\Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$

$\Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$

$\Rightarrow f([x_1,y_1]) = f([x_2,y_2])$

$\Rightarrow$ f is well-defined.

(ii) (1) $\forall [x_1,y_1],\ [x_2,y_2] \in R^2$

$f([x_1,y_1]) = f([x_2,y_2])$

$\Rightarrow \sqrt{x_1^2+y_1^2} = \sqrt{x_2^2+y_2^2}$

$\Rightarrow x_1^2+y_1^2 = x_2^2+y_2^2$

$\Rightarrow \exists\theta \in R$ such that $\begin{cases} x_1\cos\theta + y_1\sin\theta = x_2 \\ -x_1\sin\theta + y_1\cos\theta = y_2 \end{cases}$

$\Rightarrow \exists\theta \in R$ such that $M_\theta\binom{x_1}{y_1} = \binom{x_2}{y_2}$

$\Rightarrow (x_1,y_1) \sim (x_2,y_2)$

$\Rightarrow [x_1,y_1] = [x_2,y_2]$

$\Rightarrow f$ is injective

(2) $\forall t \in R_+,\ \exists (t,0) \in R^2$

i.e. $t \in R_+ \Rightarrow (t,0) \in R^2$

$\Rightarrow [t,0] \in R^2/\sim$

As $f([t,0]) = \sqrt{t^2+0}$

$= t \quad (\because t > 0)$

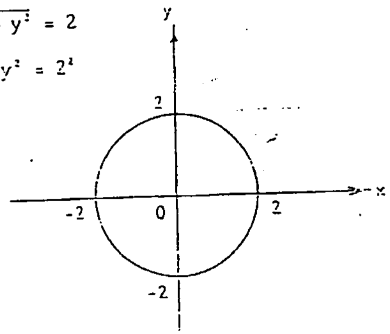
f is surjective.

Hence, f is bijective.

(iii) $(x,y) \in T \Rightarrow f([x,y]) = 2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$



90 II

PURE MATHEMATICS

1990 PAPER II

SECTION A

1. Let $f(x) = \frac{\ln x}{x}$

$f'(x) = \frac{1}{x^2}[1 - \ln x]$

$\forall x \in (e,\infty)$, $f'(x) < 0$

$\therefore f(x)$ is strictly decreasing on (e,∞).

Hence, if $b > a \geq e$, $f(a) > f(b)$

$\frac{\ln a}{a} > \frac{\ln b}{b}$

$b\ln a > a\ln b \quad (\because a,b > 0)$

$\ln a^b > \ln b^a$

$\exp(\ln a^b) > \exp(\ln b^a) \quad (\because$ exponential function is strictly increasing)

$a^b > b^a$

2. $\cot kx - \cot(k+1)x = \frac{\cos kx\sin(k+1)x - \cos(k+1)x\sin x}{\sin kx\sin(k+1)x}$

$= \frac{\sin[(k+1)x-kx]}{\sin kx\sin(k+1)x} = \frac{\sin x}{\sin kx\sin(k+1)x}$

$\therefore \frac{1}{\sin x\sin 2x} + \frac{1}{\sin 2x\sin 3x} + \dots + \frac{1}{\sin nx\sin(n+1)x}$

$= \frac{1}{\sin x}\left[\frac{\sin x}{\sin x\sin 2x} + \frac{\sin x}{\sin 2x\sin 3x} + \dots + \frac{\sin x}{\sin nx\sin(n+1)x}\right]$

$= \frac{1}{\sin x}[\cot x - \cot 2x + \cot 2x - \cot 3x + \dots + \cot nx - \cot(n+1)x]$

$= \frac{1}{\sin x}[\cot x - \cot(n+1)x]$

$$= \frac{1}{\sin x}\left[\frac{\cos x\sin(n+1)x - \sin x\cos(n+1)x}{\sin x\sin(n+1)x}\right]$$

$$= \frac{\sin nx}{\sin^2 x\sin(n+1)x}$$

3. $\int_0^a f(x)g(x)dx = \int_0^a f(x)[K-g(a-x)]dx$

$$= K\int_0^a f(x)dx - \int_0^a f(x)g(a-x)dx$$

Let $I = \int_0^a f(x)g(a-x)dx$

Put $y = a-x$

$\begin{cases} x = a \\ y = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = a \end{cases}$

$\therefore\ I = \int_a^0 f(a-y)g(y)(-dy)$

$$= \int_0^a f(a-y)g(y)dy$$

$$= \int_0^a f(a-x)g(x)dx$$

$$= \int_0^a f(x)g(x)dx \qquad (\because f(x) = f(a-x))$$

Hence, $\int_0^a f(x)g(x)dx = K\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$

$$2\int_0^a f(x)g(x)dx = K\int_0^a f(x)dx$$

$$\int_0^a f(x)g(x)dx = \frac{1}{2}K\int_0^a f(x)dx$$

Let $f(x) = \sin x\cos^4 x$ and $g(x) = x$

$\therefore\ f(\pi-x) = \sin(\pi-x)\cos^4(\pi-x)$ and $g(x) + g(\pi-x) = x + (\pi-x)$

$= \sin x(-\cos x)^4 \qquad = \pi$

$= \sin x\cos^4 x$

$= f(x)$

Hence, $\int_0^\pi x\sin x\cos^4 x dx = \frac{\pi}{2}\int_0^\pi \sin x\cos^4 x dx$

$$= \frac{-\pi}{2}\int_0^\pi \cos^4 x\, d\cos x$$

$$= \frac{-\pi}{2}\left[\frac{1}{5}\cos^5 x\right]_0^\pi$$

$$= \frac{\pi}{5}$$

4. (a) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\tan x}\right) = \lim_{x\to 0}\left(\frac{\tan x - x}{x\tan x}\right)$

$= \lim_{x\to 0}\left[\frac{\sec^2 x - 1}{\tan x + x\sec^2 x}\right]$ (by L'Hospital's Rule)

$= \lim_{x\to 0}\left[\frac{2\sec^2 x\tan x}{\sec^2 x + \sec^2 x + 2x\sec^2 x\tan x}\right]$ (by L'Hospital's Rule)

$$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{2\sec^2 x(1 + x\tan x)}$$

$$= \lim_{x\to 0}\frac{\tan x}{1 + x\tan x}$$

$$= 0$$

(b) $\int\frac{dx}{\sqrt{x^2 + 4x + 2}} = \int\frac{dx}{\sqrt{(x+2)^2 - 2}}$

$$= \int\frac{d(x+2)}{\sqrt{(x+2)^2 - 2}}$$

$$= \ln\left|\frac{x+2 + \sqrt{x^2 + 4x + 2}}{\sqrt{2}}\right| + c$$

where c is an arbitrary constant.

5. (a) $\frac{d}{dx}\int_0^{x^n} f(t)dt = \left(\frac{d}{dx^n}\int_0^{x^n} f(t)dt\right)\frac{dx^n}{dx}$

$$= f(x^n)(nx^{n-1})$$

(b) $F(x) = \int_{x^3}^{x^2} e^{-t^2}dx = \int_0^{x^2} e^{-t^2}dx - \int_0^{x^3} e^{-t^2}dx$

$$F'(x) = 2xe^{-(x^2)^2} - 3x^2e^{-(x^3)^2}$$

$$= 2xe^{-x^4} - 3x^2e^{-x^6}$$

$$F'(1) = 2e^{-1} - 3e^{-1} = -e^{-1}$$

6. Let the equation of the plane be

$(\pi) : \ell(x-1) + m(y-1) + n(z-3) = 0$, where ℓ, m, n are constants.

$\because$ $(1,-1,2)$ lies on (L) and hence on (π).

$$-2m - n = 0$$

$$2m + n = 0$$

Direction ratios of (L) is 3:2:2

$$\therefore\ 3\ell + 2m + 2n = 0$$

As $\ell^2 + m^2 + n^2 \neq 0$, the system of equations

$$\begin{cases} \ell(x-1) + m(y-1) + n(z-3) = 0 \\ 2m - n = 0 \\ 3\ell - 2m - 2n = 0 \end{cases}$$

has non-zero solution.

$$\therefore \begin{vmatrix} (x-1) & (y-1) & (z-3) \\ 0 & 2 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

i.e. $2x+3y-6z+13 = 0$

$\therefore$ The equation of the required plane is

$$2x + 3y - 6z + 13 = 0$$

7. (a) $\int \ln(1+x^2)dx = x\ln(1+x^2) - \int x d\ln(1+x^2)$ (by parts)

$$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}dx$$

$$= x\ln(1+x^2) - \int (2 - \frac{2}{1+x^2})dx$$

$= x\ln(1+x^2) - 2x + 2\tan^{-1}x + c$, where c is an arbitrary constant.

(b) $\ln u_n = \ln(\frac{1}{n^4} \prod_{k=1}^{2n} (n^2 + k^2)^{\frac{1}{n}})$

$$= \ln(\frac{1}{n^4}) + \sum_{k=1}^{2n} \ln(n^2 + k^2)^{\frac{1}{n}}$$

$$= -4\ln n + \sum_{k=1}^{2n} \frac{1}{n}\ln(n^2 + k^2)$$

$$= \frac{1}{n}\{-4n\ln n + \sum_{k=1}^{2n} \ln(n^2 + k^2)\}$$

$$= \frac{1}{n}\{\sum_{k=1}^{2n} [\ln(n^2 + k^2) - 2\ln n]\}$$

$$= \frac{1}{n}\sum_{k=1}^{2n} \ln(\frac{n^2 + k^2}{n^2})$$

$$= \frac{1}{n}\sum_{k=1}^{2n} \ln[1 + \frac{k^2}{n^2}]$$

$$\therefore \lim_{n\to\infty} \ln u_n = \int_0^2 \ln(1+x^2)dx$$

$$= [x\ln(1+x^2) - 2x + 2\tan^{-1}x]_0^2$$

$$= 2\ln 5 - 4 + 2\tan^{-1}2$$

$\ln(\lim_{n\to\infty} u_n) = 2\ln 5 - 4 + 2\tan^{-1}2$ ($\because$ $\ln x$ is continuous)

$$\lim_{n\to\infty} u_n = e^{2\ln 5 - 4 + 2\tan^{-1}2}$$

$$= \frac{25e^{2\tan^{-1}2}}{e^4}$$

SECTION B

8. (a) (i) $I_0 = \int_0^1 \frac{x}{(1+x)^2}dx$

$= \int_0^1 [\frac{1}{1+x} - \frac{1}{(1+x)^2}]dx$

$= [\ln(1+x) + \frac{1}{1+x}]_0^1$

$= \ln 2 - \frac{1}{2}$

(ii) $\forall x \in [0,1]$, $0 \leq \frac{x^{n+1}}{(1+x)^2} \leq \frac{x^n}{(1+x)^2}$

$\therefore \quad 0 \leq \int_0^1 \frac{x^{n+1}}{(1+x)^2}dx \leq \int_0^1 \frac{x^n}{(1+x)^2}dx$

$0 \leq I_n \leq I_{n-1}$

$\therefore$ $\{I_n\}$ is a monotonic decreasing sequence, and

bounded from below by 0.

Hence, $\lim_{n\to\infty} I_n$ exists and let it be ℓ .

Consider $I_n + 2I_{n-1} + I_{n-2} = \int_0^1 \frac{x^{n+1}}{(1+x)^2}dx + 2\int_0^1 \frac{x^n}{(1+x)^2}dx$

$+\int_0^1 \frac{x^{n-1}}{(1+x)^2}dx$

$= \int_0^1 \frac{x^{n-1}(x^2 + 2x + 1)}{(1+x)^2}dx$

$= \int_0^1 x^{n-1}dx$

$= \frac{1}{n}$

As $\lim_{n\to\infty} I_n = \lim_{n\to\infty} I_{n-1} = \lim_{n\to\infty} I_{n-2}$,

$4\ell = \lim_{n\to\infty} \frac{1}{n}$

$\ell = \lim_{n\to\infty} \frac{1}{4n}$

$= 0$

(b) (i) $\int_0^1 x(\frac{1-(-x)^m}{1+x})(\frac{1-(-x)^n}{1-x})dx$

$= \int_0^1 x\sum_{i=1}^{m}(-x)^{i-1}\sum_{j=1}^{n}(-x)^{j-1}dx$ (from the formula in summing a G.P.)

$= \int_0^1 \sum_{i=1}^{m}\{\sum_{j=1}^{n}(-1)^{i+j-2}x^{i+j-1}\}dx$

$= \sum_{i=1}^{m}\{\sum_{j=1}^{n}(-1)^{i+j-2}\int_0^1 x^{i+j-1}dx\}$

$= \sum_{i=1}^{m}\{\sum_{j=1}^{n}(-1)^{i+j-2}[\frac{x^{i+j}}{i+j}]_0^1\}$

$= \sum_{i=1}^{m}\{\sum_{j=1}^{n}(-1)^{i+j-2}\cdot\frac{1}{i+j}\}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}$ $\quad (\because (-1)^{i+j} = (-1)^{i+j-2}(-1)^2 = (-1)^{i+j-2})$

(ii) Consider $\int_0^1 x(\frac{1-(-x)^n}{1+x})(\frac{1-(-x)^n}{1+x})dx$

$= \int_0^1 \frac{x[1 - 2(-x)^n + x^{2n}]}{(1+x)^2}dx$

$= I_0 - 2(-1)^n I_n + I_{2n}$

$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j} = \lim_{n\to\infty}\int_0^1 x(\frac{1-(-x)^n}{1+x})(\frac{1-(-x)^n}{1+x})dx$

$= \lim_{n\to\infty}(I_0 - 2(-1)^n I_n + I_{2n})$

$= \lim_{n\to\infty} I_0 \quad (\because \lim_{n\to\infty} I_n = \lim_{n\to\infty} I_{2n} = 0)$

$= I_0$

$= \ln 2 - \frac{1}{2}$ (from (a) (i))

9. (a) Equation of the line joining the points P and Q is

$$y - \frac{c}{t_1} = \frac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2}(x - ct_1)$$

$$= \frac{-1}{t_1t_2}(x - ct_1)$$

i.e. $x + t_1t_2y - c(t_1 + t_2) = 0$

Set $t_1 \to t_2$, the equation of tangent at Q is

$$x + t_2^2y - 2ct_2 = 0$$

Set $t_2 \to t_1$, the equation of tangent at P is

$$x + t_1^2y - 2ct_1 = 0$$

(b) (i) Solving $\begin{cases} x + t_1^2y - 2ct_1 = 0 & \ldots\ldots (1) \\ x + t_2^2y - 2ct_2 = 0 & \ldots\ldots (2) \end{cases}$

(1) − (2) : $(t_1^2 - t_2^2)y - 2c(t_1 - t_2) = 0$

$$y = \frac{2c}{t_1 + t_2} \qquad (\because t_1 \neq t_2)$$

Put into (1) : $x + \frac{2ct_1^2}{t_1+t_2} - 2ct_1 = 0$

$$x = \frac{2ct_1t_2}{t_1 + t_2}$$

$$\therefore R = \left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$$

(ii) Let R = (x,y)

$$\therefore \begin{cases} x = \frac{2ct_1t_2}{t_1 + t_2} \\ y = \frac{2c}{t_1 + t_2} \end{cases}$$

$$\begin{cases} x = \frac{2ck}{t_1 + t_2}, \text{ where } k \text{ is a constant} \\ y = \frac{2c}{t_1 + t_2} \end{cases}$$

i.e. $x = ky$

$$x - ky = 0 \qquad \ldots\ldots (*)$$

Mid-point of PQ $= \left[\frac{1}{2}(ct_1 + ct_2), \frac{1}{2}\left(\frac{c}{t_1} + \frac{c}{t_2}\right)\right]$

$$= \left[\frac{c}{2}(t_1 + t_2), \frac{c}{2}\left(\frac{t_1 + t_2}{t_1t_2}\right)\right]$$

$$= \left[\frac{c}{2}(t_1 + t_2), \frac{c}{2k}(t_1 + t_2)\right]$$

As $\frac{c}{2}(t_1 + t_2) - k\left[\frac{c}{2k}(t_1 + t_2)\right] = 0$, mid-point of PQ lies on (*).

Hence, the locus of R is a straight line passing through the mid-point of PQ.

(iii) Let $A(\frac{c}{2}\cos\theta, c\sin\theta)$ be a point on the ellipse

$$4x^2 + y^2 = c^2$$

Equation of tangent at A to $4x^2 + y^2 = c^2$ is

$$4\left(\frac{c}{2}\cos\theta\right)x + yc\sin\theta = c^2$$

$$2x\cos\theta + y\sin\theta = c \qquad \ldots\ldots\ldots (**)$$

As PQ is a tangent to the ellipse $4x^2 + y^2 = c^2$, by comparing with (**), ∃θ such that

$$\frac{1}{2\cos\theta} = \frac{t_1t_2}{\sin\theta} = \frac{c(t_1 + t_2)}{c}$$

$$\therefore \begin{cases} \cos\theta = \frac{1}{2(t_1 + t_2)} \\ \sin\theta = \frac{t_1t_2}{t_1 + t_2} \end{cases}$$

If R = (x,y), then $\begin{cases} \cos\theta = \frac{y}{4c} \\ \sin\theta = \frac{x}{2c} \end{cases}$

$$\therefore \left(\frac{x}{2c}\right)^2 + \left(\frac{y}{4c}\right)^2 = 1$$

$$\frac{x^2}{4c^2} + \frac{y^2}{16c^2} = 1$$

$\therefore$ The locus of R lies on an ellipse with centre at the origin and with equation

$$\frac{x^2}{4c^2} + \frac{y^2}{16c^2} = 1$$

10. (a) $\begin{cases} x = r\cos\theta = e^{\theta}\cos\theta \\ y = r\sin\theta = e^{\theta}\sin\theta \end{cases}$

$\begin{cases} \dfrac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta \\ \dfrac{dy}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta \end{cases}$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{e^{\theta}\sin\theta + e^{\theta}\cos\theta}{e^{\theta}\cos\theta - e^{\theta}\sin\theta}$$

$$= \frac{\tan\theta + 1}{1 - \tan\theta}$$

$$= \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{4}\tan\theta}$$

$$= \tan(\theta + \frac{\pi}{4})$$

(b) From (a), slope of the tangent at P is $\tan(\theta + \frac{\pi}{4})$.

Hence, the inclination of the tangent at P to the initial line is $\theta + \frac{\pi}{4}$.

As θ is the inclination of OP to the initial line, the tangent at P always makes an angle $\frac{\pi}{4}$ with the line OP.

(c) If the tangent at Q is perpendicular to the x-axis,

$$\theta + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

when $\theta = \frac{\pi}{4}$, $r = e^{\frac{\pi}{4}}$

$\therefore$ The polar coordinates of Q is $(e^{\frac{\pi}{4}}, \frac{\pi}{4})$

and the rectangular coordinates of Q is $(e^{\frac{\pi}{4}}\cos\frac{\pi}{4}, e^{\frac{\pi}{4}}\sin\frac{\pi}{4})$

$$= (\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}}, \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}})$$

(d) (i) Area of the shaded region

$$= \text{area of } \triangle OSQ - \int_0^{\frac{\pi}{4}} \frac{1}{2}r^2 d\theta$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2} \cdot e^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2}e^{2\theta} d\theta$$

$$= \frac{1}{4}e^{\frac{\pi}{2}} - [\frac{1}{4}e^{2\theta}]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}$$

(ii) Length of arc PQ $= \int_0^{\frac{\pi}{4}} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$

$$= \int_0^{\frac{\pi}{4}} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2}e^{\theta} d\theta$$

$$= [\sqrt{2}e^{\theta}]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}(e^{\frac{\pi}{4}} - 1)$$

11. (a) Since $a_{n+1} = \sin(a_n)$ and $0 < a_1 < 1 < \frac{\pi}{2}$, it is obvious that $0 < a_n < 1$, for all $n = 1, 2, \ldots$

Further, $a_{n+1} = \sin(a_n) < a_n$ $\quad (\because 0 < a_n < 1)$

i.e. $\{a_n\}$ is a monotonic decreasing sequence.

As $\{a_n\}$ is monotonic decreasing and is bounded from below by 0, $\lim_{n\to\infty} a_n$ exists and let it be ℓ.

$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and sine function is continuous,

$\therefore \ell = \sin \ell$

Consider $\quad f(x) = x - \sin x$

$\qquad f'(x) = 1 - \cos x$

$\qquad\qquad > 0 \quad , \quad \forall x \in (0, 1)$

i.e. $f(x)$ is strictly increasing on $(0,1)$.

Also, $f(0) = 0$

Hence, $f(x) > 0, \quad \forall x \in (0,1)$

$\therefore$ The only real root of the equation $f(x) = 0$ in the interval $[0,1]$ is 0.

Therefore, $\ell = 0$.

(b) (i) $\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$

$= \lim_{x\to 0} \frac{2x - 2\sin x\cos x}{2x\sin^2 x + 2x^2\cos x\sin x}$ (by L'Hospital's Rule)

$= \lim_{x\to 0} \frac{2 - 2\cos^2 x + 2\sin^2 x}{2\sin^2 x + 4x\sin x\cos x + 4x\sin x\cos x + 2x^2\cos^2 x - 2x^2\sin^2 x}$

(by L'Hospital's Rule)

$= \lim_{x\to 0} \frac{4\sin^2 x}{2\sin^2 x + 8x\sin x\cos x + 2x^2\cos^2 x - 2x^2\sin^2 x}$

$= \lim_{x\to 0} \frac{4}{2 + \frac{8x}{\sin x}\cos x + 2(\frac{x}{\sin x})^2\cos^2 x - 2x^2}$

$= \frac{4}{2 + 8 + 2}$

$= \frac{1}{3}$

(ii) $\lim_{n\to\infty}(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}) = \lim_{n\to\infty}(\frac{1}{\sin^2 a_n} - \frac{1}{a_n^2})$

$= \lim_{a_n\to 0}(\frac{1}{\sin^2 a_n} - \frac{1}{a_n^2})$ (by (a))

$= \lim_{a_n\to 0}\frac{a_n^2 - \sin^2 a_n}{a_n^2 \sin^2 a_n}$

$= \frac{1}{3}$ (from (b)(i))

(c) Put $x_n = \frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}$

From (b)(ii), $\lim_{n\to\infty} x_n = \frac{1}{3}$

$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2})$

$= \frac{1}{n}(\frac{1}{a_{n+1}^2} - \frac{1}{a_1^2})$

$= \frac{1}{na_{n+1}^2} - \frac{1}{na_1^2}$

$\therefore \frac{1}{na_{n+1}^2} = \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{na_1^2}$

As $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} x_i$ and $\lim_{n\to\infty}\frac{1}{na_1^2}$ exist,

$\lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{na_1^2})$ exists.

Hence $\lim_{n\to\infty}(\frac{1}{na_{n+1}^2})$ exists.

$\lim_{n\to\infty}\frac{1}{na_{n+1}^2} = \frac{1}{3} \neq 0 \quad (\because \lim_{n\to\infty}\frac{1}{na_1^2} = 0)$

$\therefore \lim_{n\to\infty} na_{n+1}^2$ exists and equal to 3.

Consider $\lim_{n\to\infty} na_{n+1}^2 = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)(n+1)a_{n+1}^2$

$\because \lim_{n\to\infty} \frac{n}{n+1} = 1$ and $\lim_{n\to\infty} na_{n+1}^2 = 3$

$\therefore \lim_{n\to\infty} (n+1)a_{n+1}^2$ exists and equal to 3.

Hence, $\lim_{n\to\infty} na_n^2$ exists and is also equal to 3.

12. (a) $f(x) = (2x-1)x^{\frac{2}{3}}$

$$f'(x) = 2x^{\frac{2}{3}} + \frac{2}{3}x^{-\frac{1}{3}}(2x-1)$$
$$= 2x^{\frac{2}{3}} + \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}}$$
$$= \frac{10}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}}$$
$$= \frac{10x-2}{3x^{\frac{1}{3}}}$$
$$= \frac{2(5x-1)}{3x^{\frac{1}{3}}}$$
$$f''(x) = \frac{20}{9}x^{-\frac{1}{3}} + \frac{2}{9}x^{-\frac{4}{3}}$$
$$= \frac{2(10x+1)}{9x^{\frac{4}{3}}}$$

(b) $$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0} \frac{(2h-1)h^{\frac{2}{3}}}{h}$$
$$= \lim_{h\to 0}\left(2h^{\frac{2}{3}} - \frac{1}{h^{\frac{1}{3}}}\right)$$

As $\lim_{h\to 0} \frac{1}{h^{\frac{1}{3}}}$ does not exist and $\lim_{h\to 0} 2h^{\frac{2}{3}} = 0$, $f'(0)$ does not exist.

(c) (i) $f'(x) = 0$ for $x = \frac{1}{5}$

(ii) $f'(x) > 0$, $\frac{2(5x-1)}{x^{\frac{1}{3}}} > 0$

$\therefore x < 0$ or $x > \frac{1}{5}$

(iii) $f'(x) < 0$, $\frac{2(5x-1)}{x^{\frac{1}{3}}} < 0$

$\therefore 0 < x < \frac{1}{5}$

(iv) $f''(x) = 0$ when $x = \frac{-1}{10}$

(v) $f''(x) > 0$, $\frac{2(10x+1)}{9x^{\frac{4}{3}}} > 0$

$\therefore x > \frac{-1}{10}$ and $x \neq 0$

(vi) $f''(x) < 0$, $\frac{2(10x+1)}{9x^{\frac{4}{3}}} < 0$

$x < \frac{-1}{10}$

(d) When $x = \frac{1}{5}$, $y = \frac{-3}{5}\left(\frac{1}{5}\right)^{\frac{2}{3}}$

From (c) (i) & (v), $f'(\frac{1}{5}) = 0$ and $f''(\frac{1}{5}) > 0$.

$\therefore$ $\left(\frac{1}{5}, \frac{-3}{5}\left(\frac{1}{5}\right)^{\frac{2}{3}}\right)$ is a minimum point.

When $x = 0$, $y = 0$

As x is slightly less than 0, $f'(x) > 0$ (from (c) (ii))

As x is slightly greater than 0, $f'(x) < 0$ (from (c) (iii))

$\therefore$ (0,0) is a maximum point.

When $x = \frac{-1}{10}$, $y = \frac{-5}{6}\left(\frac{-1}{10}\right)^{\frac{2}{3}}$

As x is slightly less than $\frac{-1}{10}$, $f''(x) < 0$ (from (c)(v))

As x is slightly greater than $\frac{-1}{10}$, $f''(x) > 0$ (from (c)(v))

$\therefore$ $\left(\frac{-1}{10}, \frac{-5}{6}\left(\frac{-1}{10}\right)^{\frac{2}{3}}\right)$ is an inflexion point.

(e) $f(x) \to \infty$ only when $x \to \infty$

$\therefore$ There is no vertical asymptote.

Suppose there exists a slant asymptote $y = mx+c$

$$m = \lim_{x\to\infty} \frac{f(x)}{x}$$

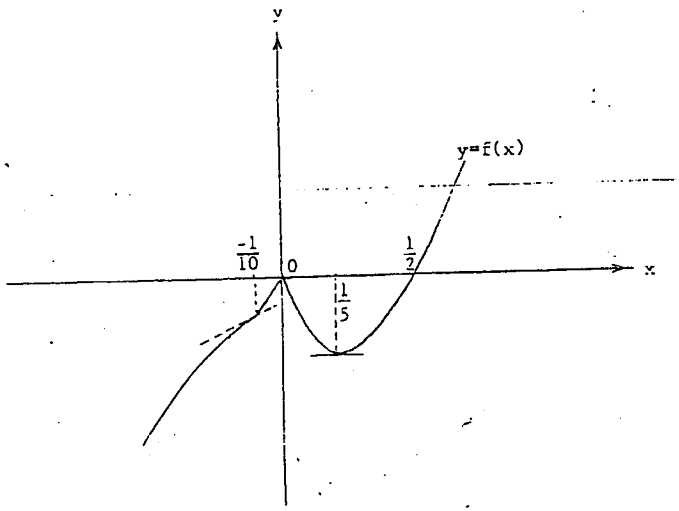
$$= \lim_{x\to\infty} (2x^{\frac{2}{3}} - \frac{1}{x^{\frac{1}{3}}})$$

As $\lim_{x\to\infty} 2x^{\frac{2}{3}} = \infty$ and $\lim_{x\to\infty} \frac{1}{x^{\frac{1}{3}}} = 0$; m does not exist.

$\therefore$ There is no slant asymptote.

There is no asymptote.

(f)



13. (a) $f(x) = \frac{1}{\sqrt{1+x^2}}$

$$f'(x) = \frac{-x}{(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{-x}{1+x^2} \cdot \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{-x\,f(x)}{1+x^2}$$

$(1+x^2)f'(x) + xf(x) = 0$ .... (*)

Differentiate both sides of (*) with respect to x by n times,

$$(1+x^2)f^{(n+1)}(x) + C_1^n(2x)f^{(n)}(x) + C_2^n(2)f^{(n-1)}(x)$$
$$+ xf^{(n)}(x) + C_1^n f^{(n-1)}(x) = 0$$

$$(1+x^2)f^{(n+1)}(x) + (2n+1)xf^{(n)}(x) + n^2f^{(n-1)}(x) = 0$$

(b) (i) $P_{n+1}(x) = (1+x^2)^{n+\frac{1}{2}}f^{(n+1)}(x)$

$$P_n'(x) = (1+x^2)^{n+\frac{1}{2}}f^{(n+1)}(x) + (n+\frac{1}{2})(2x)(1+x^2)^{n-\frac{1}{2}}f^{(n)}(x)$$

$$= (1+x^2)^{n+\frac{1}{2}}f^{(n+1)}(x) + (2n+1)x(1+x^2)^{n-\frac{1}{2}}f^{(n)}(x)$$

$\therefore$ $(1+x^2)P_n'(x) - (2n+1)xP_n(x)$

$$= (1+x^2)^{n+\frac{3}{2}}f^{(n+1)}(x) + (2n+1)x(1+x^2)^{n+\frac{1}{2}}f^{(n)}(x)$$
$$- (2n+1)x(1+x^2)^{n+\frac{1}{2}}f^{(n)}(x)$$

$$= (1+x^2)^{n+\frac{3}{2}}f^{(n+1)}(x)$$

$$= P_{n+1}(x)$$

Induce on n,

(1) when n = 0

$$P_0(x) = (1+x^2)^{\frac{1}{2}}f^{(0)}(x)$$

$$= 1$$

$= (-1)^0 0! x^0$

It is true for $n = 0$.

(2) Assume $P_k(x) = (-1)^k k! x^k + a_{k-1}x^{k-1} + \dots + a_1 x + a_0$

$\therefore P_k'(x) = (-1)^k k! k x^{k-1} + a_{k-1}(k-1)x^{k-2} + \dots + a_1$

$$P_{k+1}(x) = (1+x^2)[(-1)^k k! k x^{k-1} + a_{k-1}(k-1)x^{k-2} + \dots + a_1] - (2k+1)x[(-1)^k k! x^k + a_{k-1}x^{k-1} + \dots + a_1 x + a_0]$$

$$= x^{k+1}[(-1)^k k! k - (-1)^k k!(2k+1)] + x^k[a_{k-1}(k-1) - (2k+1)a_{k-1}] + \dots$$

$$= x^{k+1}[(-1)^k k!(-(k+1))] + \dots$$

$$= (-1)^{k+1}(k+1)! x^{k+1} + \dots$$

$\therefore$ It is also true for $n = k+1$.

Hence, $P_n(x)$ is a polynomial of degree n with leading coefficient $(-1)^n n!$

(ii) $P_{n+1}(x) + (2n+1)xP_n(x) + n^2(1+x^2)P_{n-1}(x)$

$$= (1+x^2)^{n+\frac{1}{2}} f^{(n+1)}(x) + (2n+1)x(1+x^2)^{n+\frac{1}{2}} f^{(n)}(x) + n^2(1+x^2)(1+x^2)^{n-\frac{1}{2}} f^{(n-1)}(x)$$

$$= (1+x^2)^{n+\frac{1}{2}}[(1+x^2)f^{(n+1)}(x) + (2n+1)xf^{(n)}(x) + n^2 f^{(n-1)}(x)]$$

$= 0$ (by (a))

When $x = 0$, we have

$$P_{n+1}(0) + n^2 P_{n-1}(0) = 0$$

$$P_{n+1}(0) = -n^2 P_{n-1}(0)$$

$$\therefore P_n(0) = -(n-1)^2 P_{n-2}(0)$$

When n is even, $P_n(0) = (-1)(n-1)^2 P_{n-2}(0)$

$$= (-1)^2(n-1)^2(n-3)^2 P_{n-4}(0)$$

$$= \dots$$

$$= (-1)^{\frac{n}{2}}(n-1)^2(n-3)^2 \dots 1^2 P_0(0)$$

$$= (-1)^{\frac{n}{2}}(n-1)^2(n-3)^2 \dots 1^2 \quad (\because P_0(0) = 1)$$

When n is odd, $P_n(0) = (-1)^{\frac{n-1}{2}}(n-1)^2(n-3)^2 \dots 2^2 P_1(0)$

$$= (-1)^{\frac{n-1}{2}}(n-1)^2(n-3)^2 \dots 2^2 \cdot (1+0^2)^{\frac{3}{2}} f'(0)$$

$$= 0 \quad (\because f'(0) = 0)$$

(iii) From (b) (i), $P_n'(x) = \frac{1}{1+x^2}[P_{n+1}(x) + (2n+1)xP_n(x)]$

$$= \frac{1}{1+x^2}[-n^2(1+x^2)P_{n-1}(x)] \quad \text{(by (b)(ii))}$$

$$= -n^2 P_{n-1}(x)$$

$$P_n^{(r)}(x) = \frac{d^{r-1}}{dx^{r-1}}[-n^2 P_{n-1}(x)]$$

$$= \frac{d^{r-2}}{dx^{r-2}}[-n^2 P'_{n-1}(x)]$$

$$= \frac{d^{r-2}}{dx^{r-2}}[(-n^2)[-(n-1)^2]P_{n-2}(x)]$$

$$= \frac{d^{r-3}}{dx^{r-3}}[(-n^2)[-(n-1)^2]P'_{n-2}(x)]$$

$$= \frac{d^{r-3}}{dx^{r-3}}[(-n^2)[-(n-1)^2][-(n-2)^2]P_{n-3}(x)]$$

$$= \dots$$

$$= (-n^2)[-(n-1)^2][-(n-2)^2]\dots[-(n-(r-1))^2]P_{n-r}(x)$$

$$= (-1)^r[n(n-1)(n-2)\dots(n-r+1)]^2 P_{n-r}(x)$$

(iv) From MacLaurin's Expansion,

$$P_n(x) = \sum_{r=0}^{n} \frac{P_n^{(r)}(0)}{r!} x^r$$

case (i) when n is even, let n = 2m

$$\therefore \quad P_n(x) = \sum_{r=0}^{m} \frac{P_{2m}^{(2r)}(0)}{(2r)!} x^{2r} + \sum_{r=1}^{m} \frac{P_{2m}^{(2r-1)}(0)}{(2r-1)!} x^{2r-1}$$

From (b)(iii), $P_{2m}^{(2r-1)}(0) = (-1)^{2r-1}[2m(2m-1)\dots(2m-2r+2)]^2 P_{2m-2r+1}(0)$

As 2m-2r+1 is odd, from (b)(ii), $P_{2m-2r+1}(0) = 0$

Hence $P_n(x) = \sum_{r=0}^{m} \frac{P_{2m}^{(2r)}(0)}{(2r)!} x^{2r}$

As $x^{2r} = (-x)^{2r}$, $P_n(x)$ is an even function.

case (ii), when n is odd, let n = 2m + 1

$$\therefore \quad P_n(x) = \sum_{r=0}^{m} \frac{P_{2m+1}^{(2r)}(0)}{(2r)!} x^{2r} + \sum_{r=0}^{m} \frac{P_{2m+1}^{(2r+1)}(0)}{(2r+1)!} x^{2r+1}$$

Consider $P_{2m+1}^{(2r)}(0) = (-1)^{2r}[(2m+1)(2m)\dots(2m-2r)]^2 P_{2m+1-2r}(0)$

As 2m + 1 - 2r is odd, from (b)(ii) $P_{2m-2r+1}(0) = 0$

$$\therefore \quad P_n(x) = \sum_{r=0}^{m} \frac{P_{2m+1}^{(2r+1)}(0)}{(2r+1)!} x^{2r+1}$$

Since $P_{2m+1}^{(2r+1)}(0) = (-1)^{2r+1}[(2m+1)(2m)\dots(2m-2r+1)]^2 P_{2m-2r}(0)$

$= (-1)^{2r+1}[(2m+1)(2m)\dots(2m-2r+1)]^2 (-1)^{m-r}(2m-2r-1)^2 \dots 1^2$

$\neq 0$

and $(-x)^{2r+1} = (-1)^{2r+1}x^{2r+1}$

$= -x^{2r+1}$

$P_n(x)$ is an odd function.