Solution	Marks	Remarks
1. (a) $A^3 = I \Rightarrow \det(A^3) = \det I = 1$ $\quad |A^3| = |A\cdot A\cdot A|$		
$\Rightarrow (\det A)^3 = 1$ …………………… $= |A||A||A|$	1	
$\Rightarrow \det A = 1$ (as it is real) $\quad = |A|^3$	1	
	2	
(b) (i) $B^2 + B + I = 0$ $\quad \Rightarrow B^2 = -(B+I) \leftarrow$ or $\Rightarrow (B^2+B) = -I$		
$B(B^2 + B + I) = 0$ $\quad B^2\cdot B = -(B+I)B$ $\quad B^3 = I$	1	
$\boxed{B^3} = -(B^2 + B)$ $\quad B^3 = -(B^2+B)$ $\quad B^2\cdot B = B\cdot B^2 = I$		
$= \boxed{I}$ …………………………………	1	
$\therefore$ B^{-1} exists and $B^{-1} = B^2 = -(B + I)$ $\quad AB = BA = I$	1	
(ii) $I + B + B^2 + \ldots + B^{100}$ $\quad B^{-1} = A$		
$= (I+B+B^2) + B^3(I+B+B^2) + \ldots + B^{96}(I+B+B^2) + B^{99} + B^{100}$	1	
$= \overset{I}{I} + \overset{B^3}{B}$ (or $= -B^2$) as $I + B + B^2 = 0$, $\overset{B^6}{B^{3n}} = I$	1	
(iii) Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$		
$B^{-1} = -(B + I)$		
$\Rightarrow \frac{1}{\det B}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -(a+1) & -b \\ -c & -(d+1) \end{pmatrix}$		
By (a), $\det B = 1$ as $B^3 = I$.		
$\therefore \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -(a+1) & -b \\ -c & -(d+1) \end{pmatrix}$ ………………	1	
$d = -(a + 1)$ and $a = -(d + 1)$		
i.e. $a + d = -1$ …………………………………	1	
	8	
(c) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with integral entries such that $M^2+M+I=0$	1	
By (b), $M \neq I$, $M^3 = I$ and $a + d = -1$.	1	
Further $ad - bc = 1$ by (a).		
Putting $a = 0$, then $d = -1$ Putting $b = 1$, then $c = -1$	1	
On checking, $M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ satisfies $M^3 = I$.	1	
只可能未必satisfy $M^2+M+I=0$	4	

Note that $M = \begin{pmatrix} a & b \\ \frac{-1-a-a^2}{b} & -1-a \end{pmatrix}$

Solution	Marks	Remarks
2(a) (i) $|z|^2 = z\bar{z}$		
$= \frac{a + ib}{1 - c}\cdot\frac{a - ib}{1 - c}$ …………………………	1	
$= \frac{a^2 + b^2}{(1 - c)^2}$		
$= \frac{1 - c^2}{(1 - c)^2}$		
$= \frac{1 + c}{1 - c}$ …………………………	1	
(ii) $\frac{1 + c}{1 - c} = z\bar{z} \Rightarrow 1 + c = z\bar{z}(1 - c)$		
$\Rightarrow c(1 + z\bar{z}) = z\bar{z} - 1$		
$\Rightarrow c = \frac{z\bar{z} - 1}{z\bar{z} + 1}$. $(z\bar{z} + 1 \neq 0)$	1	
(iii) $z + \bar{z} = \frac{a + ib}{1 - c} + \frac{a - ib}{1 - c}$		
$= \frac{2a}{1 - c}$ …………………………	1	
$= \frac{2a}{1 - \frac{z\bar{z} - 1}{z\bar{z} + 1}}$		
$= a(z\bar{z} + 1)$		
$\therefore a = \frac{z + \bar{z}}{z\bar{z} + 1}$ …………………………	1	
$z - \bar{z} = \frac{2bi}{1 - c}$	1	
$= ib(z\bar{z} + 1)$		
$\therefore b = \frac{i(\bar{z} - z)}{z\bar{z} + 1}$ (or $\frac{z - \bar{z}}{i(z\bar{z} + 1)}$)	1	
	7	

2 (b) (i) For any $z \in \mathbb{C}$, let $a = \frac{z + \bar{z}}{z\bar{z} + 1}$, $b = \frac{-i(\bar{z} - z)}{(z\bar{z} + 1)}$, $c = \frac{z\bar{z} - 1}{z\bar{z} + 1}$ 1

Then $a, b, c \in \mathbb{R}$ and $c \neq 1$ 1

Further $a^2 + b^2 + c^2 = \frac{(z + \bar{z})^2 - (\bar{z} - z)^2 + (z\bar{z} - 1)^2}{(z\bar{z} + 1)^2}$

$= \frac{(z\bar{z})^2 + 2z\bar{z} + 1}{(z\bar{z} + 1)^2}$

$= 1$ 1

$\therefore \exists\ (a, b, c) \in S$ such that $f((a, b, c)) = z$. 1

i.e. f is surjective.

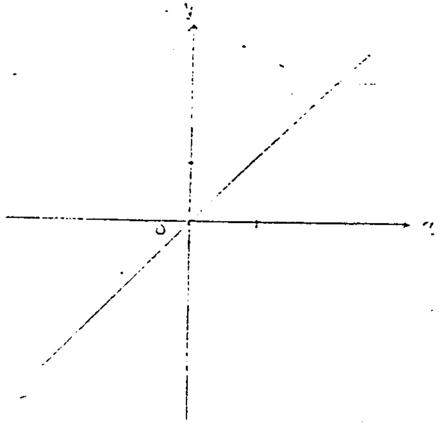
Next let $f((a_1, b_1, c_1)) = z_1$, $f((a_2, b_2, c_2)) = z_2$.

Then by (a)(ii),

$z_1 = z_2 \Rightarrow (a_1, b_1, c_1) = (a_2, b_2, c_2)$

Hence f is injective. 1

(ii) $f[A]$ is the set of all complex numbers with equal real and imaginary parts. Its graph is 1



$\frac{1}{7}$

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3 (a) (i) By L'Hospital's rule,

$\lim_{h \to 0} \frac{a^h - 1}{h} = \lim_{h \to 0} \frac{a^h \ln n}{1}$ 1

$= \ln a$ 1

(ii) From (i), $\lim_{n\to\infty} n(\sqrt[n]{a} - 1) = \lim_{n\to\infty} \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}}$

$= \ln a$ 1

$\therefore \lim_{n\to\infty} x_n = \lim_{n\to\infty} n\left(\frac{\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1}{2}\right)$

$= \frac{1}{2}\left[\lim_{n\to\infty} n(\sqrt[n]{a} - 1) + \lim_{n\to\infty} n(\sqrt[n]{b} - 1)\right]$

$= \frac{1}{2}\left[\ln a + \ln b\right]$

$= \ln\sqrt{ab}$ 1

(iii) $\lim_{n\to\infty} x_n = \ln\sqrt{ab} = 0$

$\Rightarrow ab = 1$ 1

$x_n = n\left(\frac{\sqrt[n]{a} + \frac{1}{\sqrt[n]{a}}}{2} - 1\right)$

$= \frac{n}{2\sqrt[n]{a}}(\sqrt[n]{a} - 1)^2$ 1

$x_n = 0$ for some $n \Rightarrow a = 1$

$\Rightarrow b = 1$,

contradicting the definition of a and b.

Hence $x_n \neq 0\ \forall n$. $\frac{1}{7}$

~~(iv)~~ $y_n = (1 + \frac{x_n}{n})^n$ 1

Observe that if $\lim_{n\to\infty} x_n = 0$, $x_n \neq 0\ \forall n$,

if $\lim_{n\to\infty} x_n \neq 0$, $x_n \neq 0$ for sufficiently large n. 1

As $n \to \infty$, $\frac{x_n}{n} = \frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2} - 1 \to 0$

$\therefore \lim_{n\to\infty} (1 + \frac{x_n}{n})^{\frac{n}{x_n}} = e$ 1

Now $\ln y_n = \ln\left[(1 + \frac{x_n}{n})^{\frac{n}{x_n}}\right]^{x_n}$

$= x_n \ln(1 + \frac{x_n}{n})^{\frac{n}{x_n}}$ 1

$\lim_{n\to\infty} \ln y_n = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} \ln(1 + \frac{x_n}{n})^{\frac{n}{x_n}}$ 1

$= \ln\sqrt{ab} \cdot \ln \lim_{n\to\infty} (1 + \frac{x_n}{n})^{\frac{n}{x_n}}$ (as ln is continuous) 1

$= \ln\sqrt{ab} \cdot \ln e$ $\downarrow$ $? \to e$

$= \ln\sqrt{ab}$

$\therefore \lim_{n\to\infty} y_n = \sqrt{ab}$ ....

$\frac{1}{7}$

Solution | 87 I | Marks | Remarks

4(a) Total number of ways that n different books can be put into n different boxes is n^n. 1

Of these, there are only $n!$ ways in which no box is empty.

The probability is therefore $\frac{n!}{n^n}$. 1

2

(b) There are $n(n-1)$ ways of choosing a box to hold 2 books and another box to be empty. 1

There are C_2^n ways of choosing 2 books to be put into the same box. 1

∴ the probability exactly one box is empty is

$$\frac{(n-2)!n(n-1)C_2^n}{n^n} = \frac{n!n(n-1)}{2n^n} \quad \ldots\ldots\ldots\ldots \quad 1$$

3

(c) The required probability is $\frac{(n-1)!nC_2^{n+1}}{n^{n+1}}$ 1

$$= \frac{(n+1)!}{2n^n} \quad \ldots\ldots\ldots\ldots \quad 1$$

2

(d) There are two possibilities:

(1) One box containing 3 books and the remaining boxes each containing one. 1

The probability is $\frac{(n-1)!nC_3^{n+2}}{n^{n+2}}$ 1

$$= \frac{(n+2)!}{6n^{n+1}}$$

(2) Two boxes each containing 2 books and the remaining boxes each containing one. 1

The probability is $\frac{(n-2)!nC_2^{n+2} \cdot (n-1)C_2^n}{n^{n+2}}$ 1

$$= \frac{(n+2)!(n-1)}{4n^{n+1}}$$

∴ the required probability $= \frac{(n+2)!}{6n^{n+1}} + \frac{(n+2)!(n-1)}{4n^{n+1}}$

$$= \frac{(n+2)!(3n-1)}{12n^{n+1}} \quad 1$$

5

(e) There are n ways of choosing a box to contain no book. 1

By (d), the probability is $\frac{n(n+1)!(3n-4)}{12(n-1)^n}$ ………… 1

2

Solution | 87 I | Marks | Remarks

5(a) Let $f(x) = 1 + \alpha x - (1+x)^\alpha$

$f'(x) = \alpha - \alpha(1+x)^{\alpha-1}$ ………… 1

$= \alpha[1 - (1+x)^{\alpha-1}]$

> 0 for $0 < \alpha < 1$ and $x \in (0, 1]$.

∴ f is strictly increasing there and continuous at $x = 0$.

∴ $f(x) > f(0) = 0$, i.e. $(1+x)^\alpha < 1 + \alpha x \quad \forall x \in (0, 1]$ …… 1

Let $g(x) = 1 - \alpha x - (1-x)^\alpha$

$g'(x) = -\alpha + \alpha(1-x)^{\alpha-1}$ 1

$= -\alpha(1 - (1-x)^{\alpha-1}$

> 0 for $0 < \alpha < 1$ and $x \in (0, 1]$.

∴ g is strictly increasing there and continuous at $x = 0, 1$.

∴ $g(x) > g(0) = 0$, i.e. $(1-x)^\alpha < 1 - \alpha x \quad \forall x \in (0, 1]$. 1

4

(b) For any positive integers n and k, putting $\alpha = \frac{n}{n+1}$, $x = \frac{1}{k}$, 1+1

then $0 < \alpha < 1$ and $0 < x \leqslant 1$, 1

By (a), $(1 + \frac{1}{k})^{\frac{n}{n+1}} < 1 + \frac{n}{n+1} \frac{1}{k}$ ………… 1

$(k+1)^{\frac{n}{n+1}} < (1 + \frac{n}{n+1} \cdot \frac{1}{k}) k^{\frac{n}{n+1}}$ $\leftarrow (k+1)^{\frac{n}{n+1}} - k^{\frac{n}{n+1}} < \frac{n}{n+1} \div k^{\frac{n}{n+1}}$

$\Leftarrow$?

Hence $(\frac{n+1}{n})[(k+1)^{\frac{n}{n+1}} - k^{\frac{n}{n+1}}] < \frac{1}{\frac{1}{k^{n+1}}}$ ………… 1

Also $(1 - \frac{1}{k})^{\frac{n}{n+1}} < 1 - \frac{n}{n+1} \cdot \frac{1}{k}$

$(k-1)^{\frac{n}{n+1}} < (1 - \frac{n}{n+1} \cdot \frac{1}{k}) k^{\frac{n}{n+1}}$

i.e. $\frac{1}{\frac{1}{k^{n+1}}} < (\frac{n+1}{n})[k^{\frac{n}{n+1}} - (k-1)^{\frac{n}{n+1}}]$ ………… 1

The result follows.

6

(c) Put $n = 2$ in (b), one has

$k=1 \quad \frac{3}{2}[2^{\frac{2}{3}} - 1^{\frac{2}{3}}] < \frac{1}{\sqrt[3]{1}} < \frac{3}{2}[1^{\frac{2}{3}} - 0^{\frac{2}{3}}]$ 1

$k=2 \quad \frac{3}{2}[3^{\frac{2}{3}} - 2^{\frac{2}{3}}] < \frac{1}{\sqrt[3]{2}} < \frac{3}{2}[2^{\frac{2}{3}} - 1^{\frac{2}{3}}]$

…………

$\frac{3}{2}[(10^6+1)^{\frac{2}{3}} - (10^6)^{\frac{2}{3}}] < \frac{1}{\sqrt[3]{1,000,000}} < \frac{3}{2}[(10^6)^{\frac{2}{3}} - (10^6-1)^{\frac{2}{3}}]$ 1

Adding, $\frac{3}{2}[(10^6+1)^{\frac{2}{3}} - 1] \leqslant S < \frac{3}{2}[10^4]$ 1

where S is the required sum.

∴ $14998 < S < 15000$ 1

4

Solution | 87 I Marks | Remarks

6\. (a) (i) g is continuous and for any $t_1, t_2 \in \mathbb{R}$, $e^{t_1}, e^{t_2} > 0$ and

$g(t_1 + t_2) = f(e^{t_1+t_2})$ 1

$= f(e^{t_1} \cdot e^{t_2})$

$= f(e^{t_1}) + f(e^{t_2})$

$= g(t_1) + g(t_2)$

$\therefore g(t) = g(1)t \quad \forall t \in \mathbb{R}$. 1

(ii) For any $x > 0$, let $x = e^t$.

Then $f(x) = f(e^t)$ 1

$= g(t)$

$= g(1)t$

$= g(1)\log_e x$ 1

Since f is non-constant, $g(1) = f(e) \neq 0$ and 1

$f(x) = \log_b x$, where $b = e^{\frac{1}{f(e)}} > 0$ 1

6

(b) (i) Consider $H(x) = \log_e h(x)$, $x > 0$.

For any $x > 0$, to show that $h(x) > 0$.

$h(x) = h(\sqrt{x} \cdot \sqrt{x})$ 1

$= [h(\sqrt{x})]^2$

$\geqslant 0$ 1

Also, $h(x_0) = 0$ for some $x_0 > 0 \Rightarrow h(x) = h(\frac{x}{x_0} \cdot x_0)$ 1

$= h(\frac{x}{x_0})h(x_0)$

$= 0 \quad \forall x > 0$

But h is non-constant, $\therefore h(x) > 0 \quad \forall x > 0$ and

H is well defined. 1

(ii) Now H is continuous and for any $x, y > 0$,

$H(xy) = \log_e h(xy)$

$= \log_e h(x)h(y)$

$= \log_e h(x) + \log_e h(y)$

$= H(x) + H(y)$ 1

By (a), $H(x) = \log_b x$, where $b = e^{\frac{1}{H(e)}}$ 1

$= \log_e h(e) \log_e x$ 1

$\therefore h(x) = x^c$, where $c = \log_e h(e)$ 1

8

Solution | 87 I Marks | Remarks

7 (a) For any $(m, n), (m', n'), (m'', n'') \in A$,

Reflexive: $(m, n)R(m, n)$ as $m + n = n + m$. 1

Symmetric: $(m, n)R(m', n') \Rightarrow m + n' = n + m'$

$\Rightarrow m' + n = n' + m$

$\Rightarrow (m', n')R(m, n)$ 1

Transitive: $(m, n)R(m', n')$ and $(m', n')R(m'', n'')$

$\Rightarrow m - n = m' - n'$ and $m' - n' = m'' - n''$

$\Rightarrow m - n = m'' - n''$

$\Rightarrow (m, n)R(m'', n'')$ 1

Hence R is an equivalence relation.

3

(b) (i) Suppose $[m, n] = [m', n']$.

Then $(m, n)R(m', n') \Rightarrow m - n = m' - n'$

$\Rightarrow f([m, n]) = f([m', n'])$

Hence f is well defined. 1

(ii) To show that f is surjective, for any $k \in \mathbb{Z}$,

$f([k, 0]) = k \quad$ if $k \geq 0$ 1

$f([0, -k]) = k \quad$ if $k < 0$ 1

Next, for any $[m, n], [m', n'] \in A/R$,

$f([m, n]) = f([m', n']) \Rightarrow m - n = m' - n'$

$\Rightarrow (m, n)R(m', n')$

$\Rightarrow [m, n] = [m', n']$ 1

Thus f is bijective.

4

7\. (c) (i) $(f \circ h)([m, n]) = f([am + bn, bm + an])$

$= (am + bn) - (bm + an)$

$= (a - b)(m - n)$ 1

(ii) If $a \neq b$, for any $[m, n], [m', n'] \in A/R$,

$h([m, n]) = h([m', n'])$

$\Rightarrow [am+bn, bm+an] = [am'+bn', bm'+an']$

$\Rightarrow (am+bn) - (bm+an) = (am'+bn') - (bm'+an')$

$\Rightarrow (a - b)(m - n) = (a - b)(m' - n')$

$\Rightarrow [m, n] = [m', n']$ as $a \neq b$ 1

i.e. h is injective.

If $a = b$, $[am+bn, bm+an] = [a(m+n), a(m+n)]$

$= [0, 0]$

$\therefore h([m, n]) = [0, 0] \ \forall [m, n] \in A/R$

and h is not injective. The answer follows. 1

[OR As f is bijective, h is injective iff $f \circ h$ is injective, and so on.]

(iii) If h is surjective, there exists $[m, n] \in A/R$ such that

$h([m, n]) = [1, 0]$.

$(a - b)(m - n) = 1$

$\Rightarrow a - b = \pm 1$ as a, b, m, n are all integers. 1+1

For $a - b = 1$, $h([m, n]) = [am+bn, bm+an]$

$= [(b+1)m+bn, bm+(b+1)n]$

$= [b(m+n)+m, b(m+n)+n]$ 1

$= [m, n]$

For $a - b = -1$, $h([m, n]) = [am+(a+1)n, (a+1)m+an]$

$= [a(m+n)+n, a(m+n)+m]$

$= [n, m]$ 1

In both cases, h is surjective.

7

9(a) (i) $MM^T = \begin{pmatrix} \underline{a}\cdot\underline{a} & \underline{a}\cdot\underline{b} & \underline{a}\cdot\underline{c} \\ \underline{b}\cdot\underline{a} & \underline{b}\cdot\underline{b} & \underline{b}\cdot\underline{c} \\ \underline{c}\cdot\underline{a} & \underline{c}\cdot\underline{b} & \underline{c}\cdot\underline{c} \end{pmatrix}$ 1

$= I$, by definition of $\underline{a}, \underline{b}, \underline{c}$. 1

M is invertible.

AB = BA = I

A = B⁻¹

A is called invertible non-singular.

(ii) Since $\underline{u}\cdot\underline{a} = \underline{u}\cdot\underline{b} = \underline{u}\cdot\underline{c} = 0$, we have

$$\begin{cases} a_1u_1 + a_2u_2 + a_3u_3 = 0 \\ b_1u_1 + b_2u_2 + b_3u_3 = 0 \\ c_1u_1 + c_2u_2 + c_3u_3 = 0 . \end{cases}$$

From (i), M is invertible. 1

The above system has a unique solution $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 1

i.e. $\underline{u} = \underline{0}$

(iii) Let $\underline{u} = \underline{v} - [\, (\underline{v}\cdot\underline{a})\underline{a} + (\underline{v}\cdot\underline{b})\underline{b} + (\underline{v}\cdot\underline{c})\underline{c} \,]$

$\underline{u}\cdot\underline{a} = \underline{v}\cdot\underline{a} - [\, (\underline{v}\cdot\underline{a})\underline{a} + (\underline{v}\cdot\underline{b})\underline{b} + (\underline{v}\cdot\underline{c})\underline{c} \,]\cdot\underline{a}$ 1

$= \underline{v}\cdot\underline{a} - (\underline{v}\cdot\underline{a})(\underline{a}\cdot\underline{a})$

$= 0$ 1

Similarly $\underline{u}\cdot\underline{b} = \underline{u}\cdot\underline{c} = 0$

By (ii) $\underline{u} = \underline{0}$, i.e. $\underline{v} = (\underline{v}\cdot\underline{a})\underline{a} + (\underline{v}\cdot\underline{b})\underline{b} + (\underline{v}\cdot\underline{c})\underline{c}$ 1

7

(b) $\phi(\underline{i})\cdot\phi(\underline{i}) = \underline{i}\cdot\underline{i}$

$= 1$ 1

Similarly $\phi(\underline{j})\cdot\phi(\underline{j}) = \phi(\underline{k})\cdot\phi(\underline{k}) = 1$

and $\phi(\underline{i})\cdot\phi(\underline{j}) = \phi(\underline{j})\cdot\phi(\underline{k}) = \phi(\underline{k})\cdot\phi(\underline{i}) = 0$ 1

Putting $\phi(\underline{i}) = \underline{a}$, $\phi(\underline{j}) = \underline{b}$, $\phi(\underline{k}) = \underline{c}$, by (a)(iii)

$\phi(\underline{x}) = (\phi(\underline{x})\cdot\phi(\underline{i}))\phi(\underline{i}) + (\phi(\underline{x})\cdot\phi(\underline{j}))\phi(\underline{j}) + (\phi(\underline{x})\cdot\phi(\underline{k}))\phi(\underline{k})$ 1

$= (\underline{x}\cdot\underline{i})\phi(\underline{i}) + (\underline{x}\cdot\underline{j})\phi(\underline{j}) + (\underline{x}\cdot\underline{k})\phi(\underline{k})$ 1

$= x_1\phi(\underline{i}) + x_2\phi(\underline{j}) + x_3\phi(\underline{k})$ 1

For any $\lambda, \mu \in \mathbb{R}$, $\underline{x}, \underline{y} \in \mathbb{R}^3$, let $\underline{x} = (x_1, x_2, x_3)$, $\underline{y} = (y_1, y_2, y_3)$.

$\phi(\lambda\underline{x} + \mu\underline{y}) = (\lambda x_1 + \mu y_1)\phi(\underline{i}) + (\lambda x_2 + \mu y_2)\phi(\underline{j}) + (\lambda x_3 + \mu y_3)\phi(\underline{k})$ 1

$= \lambda[x_1\phi(\underline{i}) + x_2\phi(\underline{j}) + x_3\phi(\underline{k})] + \mu[y_1\phi(\underline{i}) + y_2\phi(\underline{j}) + y_3\phi(\underline{k})]$

$= \lambda\phi(\underline{x}) + \mu\phi(\underline{y})$ 1

Hence ϕ is linear.

7

1 (a) $I_{k+2} = \int_0^{\frac{\pi}{2}} \cos^{k+2}x\,dx$

$$= \cos^{k+1}x\sin x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (k+1)\sin^2 x\cos^k x\,dx$$ 1

$$= (k+1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^k x\,dx\,.$$

$$= (k+1)I_k - (k+1)I_{k+2}$$ 1

$\therefore\ I_{k+2} = \frac{k+1}{k+2}I_k\,,\quad k = 0, 1, 2, \ldots$ 1

$I_0 = \frac{\pi}{2}\,,\quad I_1 = 1\,.$ 1

For $n = 1, 2, 3, \ldots,$

$$I_{2n} = \frac{2n-1}{2n}I_{2n-2}$$

$$= \text{etc.}$$

$$= \frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}I_0$$

$$= \frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}\cdot\frac{\pi}{2}$$ 1

Similarly,

$$I_{2n+1} = \frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3}$$ 1

$\underline{6}$

(b) For any $x \in [0, \frac{\pi}{2}]$, $0 \leqslant \cos x \leqslant 1$, we have 1

$$0 \leqslant \cos^{m+2}x \leqslant \cos^{m+1}x \leqslant \cos^m x$$ 1

$$\int_0^{\frac{\pi}{2}} \cos^{m+2}x\,dx \leqslant \int_0^{\frac{\pi}{2}} \cos^{m+1}x\,dx \leqslant \int_0^{\frac{\pi}{2}} \cos^m x\,dx$$ 1

Dividing throughout by $\int_0^{\frac{\pi}{2}} \cos^m x\,dx$ (which is positive),

$$\frac{\int_0^{\frac{\pi}{2}} \cos^{m+2}x\,dx}{\int_0^{\frac{\pi}{2}} \cos^m x\,dx} \leqslant \frac{\int_0^{\frac{\pi}{2}} \cos^{m+1}x\,dx}{\int_0^{\frac{\pi}{2}} \cos^m x\,dx} \leqslant 1 \quad \text{for all positive } m\,.$$ 1

---

1 (b) Now by (a), $I_{m+2} = \frac{m+1}{m+2}I_m$

$$\frac{m+1}{m+2} \leqslant \frac{\int_0^{\frac{\pi}{2}} \cos^{m+1}x\,dx}{\int_0^{\frac{\pi}{2}} \cos^m x\,dx} \leqslant 1\,;$$ 1

$\frac{1+\frac{1}{m}}{1+\frac{2}{m}}$ 1

As $\lim_{m\to\infty}\frac{m+1}{m+2} = 1\,,\ \lim_{m\to\infty}\frac{\int_0^{\frac{\pi}{2}} \cos^{m+1}x\,dx}{\int_0^{\frac{\pi}{2}} \cos^m x\,dx} = 1$ by Sandwitch theorem 1

$\underline{6}$

(c) $$\frac{I_{2n+1}}{I_{2n}} = \frac{\frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3}}{\frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}\cdot\frac{\pi}{2}}$$ 1

$$= \frac{(2)(2)(4)(4)(6)(6)\ldots(2n-2)(2n-2)(2n)(2n)}{(1)(3)(3)(5)(5)(7)(7)\ldots(2n-1)(2n-1)(2n+1)\cdot\frac{\pi}{2}}$$

As $\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = 1$, the required limit $= \frac{\pi}{2}$

$\underline{\frac{1}{2}}$

2(a) $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

Slope of tangent at $(at^2, 2at)$ $(t \neq 0)$ is $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$ 1

$\therefore$ equation of the normal is $y - 2at = -t(x - at^2)$

i.e. $tx + y - (at^3 + 2at) = 0$ (which also holds for $t = 0$). $\frac{1}{2}$

(b) The equation of the normal at t_1 is

$$t_1x + y - (at_1^3 + 2at_1) = 0$$ 1

Putting $x = at^2$, $y = 2at$,

$$at_1t^2 + 2at - (at_1^3 + 2at_1) = 0$$ 1

As t_1 is a real root of the equation, it has another real root

t_2 given by $t_1 + t_2 = \frac{-2a}{at_1}$

i.e. $t_2 = -\frac{2}{t_1} - t_1$ .... 1 ← If cand. shows Δ>0. award the mark in (c) when used

Further, $t_1t_2 = -(2 + t_1^2) < 0$

$\therefore$ $t_2 \neq t_1$ .... $\frac{1}{4}$

(c) (i) Let $P_n = (at_n^2 , 2at_n)$, where $t_n \neq 0$.

By (b), $x_{n+1} - x_n = a(t_{n+1}^2 - t_n^2)$ $\left(\frac{2+t_n^2}{t_n}\right)^2$

$= a[\frac{(2+t_n^2)^2}{t_n^2} - t_n^2]$ 1

$= \frac{4a}{t_n^2} - 4a$

$= \frac{4a^2}{x_n} - 4a$ 1

(ii) $x_{n+1} - x_1 = \sum_{k=1}^{n} (x_{k+1} - x_k)$ 1

$= \sum_{k=1}^{n} (\frac{4a^2}{x_k} + 4a)$

$> 4na$ as $x_k > 0$ 1

As $n \to \infty$, $x_{n+1} \to \infty$ and therefore $\lim_{n\to\infty} \frac{1}{x_n} = 0$ 1

(iii) $\lim_{n\to\infty} (x_{n+1} - x_n) = \lim_{n\to\infty} (\frac{4a^2}{x_n} + 4a)$

$= 4a$ 1

$|y_{n+1}| - |y_n| = 2\sqrt{a}(\sqrt{x_{n+1}} - \sqrt{x_n})$

$= \frac{2\sqrt{a}(x_{n+1} - x_n)}{\sqrt{x_{n+1}} + \sqrt{x_n}}$ 1

$\because$ as $n \to \infty$, $x_n \to \infty$ and $\lim_{n\to\infty}(x_{n+1} - x_n) = 4a$,

$\lim_{n\to\infty} (|y_{n+1}| - |y_n|) = 0$ .... $\frac{1}{8}$

3(a) $a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$

$\{a_n\}$ is monotonic increasing. 1

For $r \geq 2$, $r! \geq r(r - 1)$

$\frac{1}{r!} \leq \frac{1}{r-1} - \frac{1}{r}$

$\sum_{r=0}^{n} \frac{1}{r!} = 2 + \sum_{r=2}^{n} (\frac{1}{r-1} - \frac{1}{r})$ 1

$= 3 - \frac{1}{n} < 3$ 1

Hence $\{a_n\}$ is bounded above by 3 and is therefore convergent. $\frac{}{3}$

(b) $b_n = \sum_{r=0}^{n} C_r^n \frac{1}{n^r}$ 1

$= 2 + \sum_{r=2}^{n} \frac{n(n-1)\ldots(n-r+1)}{r!\,n^r}$

$= 2 + \sum_{r=2}^{n} \frac{1}{r!}(1 - \frac{1}{n}) \ldots (1 - \frac{r-1}{n})$. 1

$\leq 2 + \sum_{r=2}^{n} \frac{1}{r!}$

$= a_n$ $\frac{1}{3}$

(c) Let n be fixed and greater than 1. We shall prove the inequality by induction on r . The inequality holds for $r = 2$ as

$1 - \frac{(2)(1)}{n} \leq 1 - \frac{1}{n}$. 1

Assume that it holds for some r where $2 \leq r < n$, then

$(1 - \frac{1}{n})\ldots(1 - \frac{r-1}{n})(1 - \frac{r}{n}) \geq [1 - \frac{r(r-1)}{n}](1 - \frac{r}{n})$ $(1 - \frac{r}{n} \geq 0)$ 1

$= 1 - \frac{r(r-1)}{n} - \frac{r}{n} + \frac{r^2(r-1)}{n^2}$

$> 1 - \frac{r^2}{n}$

$> 1 - \frac{r(r+1)}{n}$ 2

$\therefore$ the inequality holds for r where $2 \leq r \leq n$.

Solution 87 II	Marks	Remarks
3(c) From (b), $b_n = 2 + \sum_{r=2}^{n} \frac{1}{r!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{r-1}{n})$		
$\geqslant 2 + \sum_{r=2}^{n} \frac{1}{r!}(1 - \frac{r(r-1)}{n})$	1	
$= \sum_{r=0}^{n} \frac{1}{r!} - \sum_{r=2}^{n} \frac{1}{(r-2)!n}$		
$\geqslant \sum_{r=0}^{n} \frac{1}{r!} - \frac{1}{n} \sum_{r=0}^{n} \frac{1}{r!}$		
$= (1 - \frac{1}{n})a_n$	1	
Now $(1 - \frac{1}{n})a_n \leqslant b_n \leqslant a_n$		
As $\lim_{n \to \infty} a_n$ exists by (a), and $\lim_{n \to \infty} (1 - \frac{1}{n})a_n = \lim_{n \to \infty} a_n$, ∴ $\{b_n\}$ converges	1	
and $\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n$.	1 8	

Solution 87 II	Marks	Remarks
4(a) $B(m, n) = \int_0^1 x^m (1 - x)^n \, dx$		
$= \frac{1}{m+1} x^{m+1} (1 - x)^n \Big\|_0^1 + \int_0^1 \frac{n}{m+1} x^{m+1} (1 - x)^{n-1} \, dx$		
$= \frac{n}{m+1} B(m+1, n-1)$ ...($m \geqslant 0$, $n \geqslant 1$)	2	
Hence $B(m, n) = \frac{n}{m+1} \cdot \frac{(n-1)}{(m+2)} B(m+2, n-2)$	1	
= etc.		
$= \frac{n(n-1) \ldots 1}{(m+1)(m+2) \ldots (m+n)} B(m+n, 0)$	1	
$= \frac{n(n-1) \ldots 1}{(m+1)(m+2) \ldots (m+n)} \cdot \frac{x^{m+n+1}}{m+n+1}\Big\|_0^1$		
$= \frac{m!\,n!}{(m+n+1)!}$ (which also holds for m=n=0.)	1 5	
(b)(i) $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx = \int_0^1 \frac{x^4(x^4 - 4x^3 + 6x^2 - 4x + 1)}{1+x^2} \, dx$	1	
$= \int_0^1 [\, x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \,] \, dx$	1	
$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$	1	
$= \frac{22}{7} - \pi$	1	
(ii) For $x \in [0, 1]$, $\frac{x^4(1-x)^4}{2} \leqslant \frac{x^4(1-x)^4}{1+x^2} \leqslant x^4(1-x)^4$.	1+1	
$\frac{1}{2} \int_0^1 x^4(1-x)^4 \, dx \leqslant \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx \leqslant \int_0^1 x^4(1-x)^4 \, dx$	1	
But $\int_0^1 x^4(1-x)^4 \, dx = \frac{4!\,4!}{9!}$		
$= \frac{1}{630}$		
∴ $\frac{1}{1260} \leqslant \frac{22}{7} - \pi \leqslant \frac{1}{630}$	1	
	1 9	

5 (a) (i) $\int_0^1 [f(x)]^2 dx = \int_0^1 \sum_{i=0}^{n} a_i x^i f(x) dx$

$$\boxed{= \sum_{i=0}^{n} a_i \int_0^1 x^i f(x) dx}$$

$$\boxed{= a_0 \int_0^1 f(x) dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

as $\int_0^1 x^i f(x) dx = 0$ for $i = 1, 2, \ldots, n$.

(ii) For $k = 1, 2, \ldots, n$,

$$\boxed{\int_0^1 x^k f(x) dx = 0}$$

$$\boxed{\int_0^1 x^k (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0) dx = 0}$$

$$\left[\frac{a_n}{k+n+1} x^{k+n+1} + \frac{a_{n-1}}{k+n} x^{k+n} + \ldots + \frac{a_0}{k+1} x^{k+1}\right]_0^1 = 0$$

$$\frac{a_n}{k+n+1} + \frac{a_{n-1}}{k+n} + \ldots + \frac{a_0}{k+1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

(b) Let $\frac{a_n}{t+n+1} + \frac{a_{n-1}}{t+n} + \ldots + \frac{a_0}{t+1} = \frac{Q(t)}{(t+n+1)(t+n) \ldots (t+1)}$,

where $Q(t)$ is a polynomial in t of degree $\leq n$.

By (a)(ii), $Q(t) = 0$ for $t = 1, 2, \ldots, n$.

Hence $Q(t) = C(t-1)(t-2) \ldots (t-n)$ for some constant C.

(c) Putting $t = 0$ in (b),

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \ldots + \frac{a_0}{1} = (-1)^n \frac{n!\,C}{(n+1)!}.$$

But $\int_0^1 f(x) dx = \left[\frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \ldots + a_0 x \right]_0^1$

$$= \frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \ldots + a_0 .$$

The answers follows.

Multiplying both sides of (b) by $(t+1)$ and putting $t = -1$,

we have $a_0 = \frac{(-1)^n (n+1)!C}{n!} = (-1)^n (n+1)C$.

By (a), $\int_0^1 [f(x)]^2 dx = a_0 \int_0^1 f(x) dx$

$$= (-1)^n (n+1)C \int_0^1 f(x) dx = (n+1)^2 \left\{\int_0^1 f(x) dx\right\}^2$$

6(a) (i) Suppose l_1 and l_2 intersect at $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$. $\exists\, t_1, t_2 \in \mathbb{R}$

such that $\begin{pmatrix}a_1\\ b_1\\ c_1\end{pmatrix} + t_1 \begin{pmatrix}p_1\\ q_1\\ r_1\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$

$= \begin{pmatrix}a_2\\ b_2\\ c_2\end{pmatrix} + t_2 \begin{pmatrix}p_2\\ q_2\\ r_2\end{pmatrix}$ ……… 1

$$\begin{pmatrix}a_1-a_2\\ b_1-b_2\\ c_1-c_2\end{pmatrix} + t_1 \begin{pmatrix}p_1\\ q_1\\ r_1\end{pmatrix} - t_2 \begin{pmatrix}p_2\\ q_2\\ r_2\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

The system $\begin{pmatrix}a_1-a_2 & p_1 & p_2\\ b_1-b_2 & q_1 & q_2\\ c_1-c_2 & r_1 & r_2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ has a non-trivial

solution $\begin{pmatrix}1\\ t_1\\ -t_2\end{pmatrix}$. ……………………………… 1

$$\begin{vmatrix}a_1-a_2 & p_1 & p_2\\ b_1-b_2 & q_1 & q_2\\ c_1-c_2 & r_1 & r_2\end{vmatrix} = 0.$$ 1

(ii) The vector $\begin{pmatrix}p_1\\ q_1\\ r_1\end{pmatrix} \times \begin{pmatrix}p_2\\ q_2\\ r_2\end{pmatrix}$ is normal to the plane containing 1

l_1 and l_2.

($\because$ l_1 and l_2 are distinct and they intersect, $\begin{pmatrix}p_1\\ q_1\\ r_1\end{pmatrix} \times \begin{pmatrix}p_2\\ q_2\\ r_2\end{pmatrix} \neq \underline{0}$)

The plane containing l_1 and l_2 is given by

$$\begin{pmatrix}x-a_1\\ y-b_1\\ z-c_1\end{pmatrix} \cdot \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\end{vmatrix} = 0$$ 2

~~$\overrightarrow{P_0P}$~~

$\overrightarrow{P_0P} \cdot \overrightarrow{m} = 0$

or $\begin{vmatrix}x-a_1 & y-b_1 & z-c_1\\ p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\end{vmatrix} = 0$ ……………………… 1

(b) (i) The required plane is parallel to the vector

$$\begin{pmatrix} q \\ r \\ p \end{pmatrix} \times \begin{pmatrix} r \\ p \\ q \end{pmatrix} = (rq - p^2)\underline{i} + (pr - q^2)\underline{j} + (pq - r^2)\underline{k}$$ 1

As L_1, L_2, L_3 are concurrent at the origin, by (a)(ii), the equation of the plane is given by 1

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ p & q & r \\ rq-p^2 & pr-q^2 & pq-r^2 \end{vmatrix} = 0$$ 1

This can be written as

$$(pq + qr + rp)[(q - r)x + (r - p)y + (p - q)z] = 0$$ 1

As p, q, r are distinct, this plane is uniquely defined if $pq + qr + rp \neq 0$ 1

Its equation is $(q - r)x + (r - p)y + (p - q)z = 0$.

(ii) If $pq + qr + rp = 0$, L_1, L_2, L_3 are mutually perpendicular.

There are infinitely many planes each of which contains L_1 and which is perpendicular to the plane containing L_2 and L_3. 1

The equation of one such plane is

$$\begin{vmatrix} x & y & z \\ p & q & r \\ q & r & p \end{vmatrix} = 0 \text{ .}$$ 1

7

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7. Let $g(x) = \frac{x^2(x + 7)}{x - 1} = x^2 + 8x + 8 + \frac{8}{x - 1}$ $(x \neq 1)$

$g'(x) = 2x + 8 - \frac{8}{(x - 1)^2}$ $\left(= \frac{2x(x^2 + 2x - 7)}{(x - 1)^2} \right)$

$g''(x) = 2 + \frac{16}{(x - 1)^3}$, $\left(= \frac{2(x + 1)(x^2 - 4x + 7)}{(x - 1)^3} \right)$ 1

As $f(x) = \begin{cases} g(x) & , x > 0 \\ 0 & , x = 0 \\ -g(x) & , x < 0 \end{cases}$

$f'(x) = \begin{cases} g'(x) & , x > 0 \\ 0 & , x = 0 \\ -g'(x) & , x < 0 \end{cases}$ 1

$f''(x) = \begin{cases} g''(x) & , x > 0 \\ -g''(x) & , x < 0 \end{cases}$ 1

3

(b) (i) Putting $f'(x) = 0$, $x = 0$ or $-1 \pm 2\sqrt{2}$

At $x = -1 + 2\sqrt{2}$, $f''(x) > 0$

$\therefore$ $(-1 + 2\sqrt{2}, 13 + 16\sqrt{2})$ is a minimum point [or (1.83, 35.63)] 1

At $x = -1 - 2\sqrt{2}$, $f''(x) < 0$

$\therefore$ $(-1 - 2\sqrt{2}, -13 + 16\sqrt{2})$ is a maximum point [or (-3.83, 9.63)] 1

[The point (0, 0) will be discussed in (ii).]

Next $x = 1$ is obviously an asymptote 1

As neither $\lim_{x \to \infty} \frac{f(x)}{x}$ nor $\lim_{x \to -\infty} \frac{f(x)}{x}$ exists, the graph of $f(x)$ has no oblique asymptotes. 1

(ii) Putting $f''(x) = 0$, $x = -1$

As $f''(x)$ changes sign at $x = -1$, $(-1, 3)$ is a point of inflexion. 1

Consider the point (0, 0).

For $x > 0$ slightly, $f''(x) < 0$. $\therefore$ $f'(x)$ is decreasing.

For $x < 0$ slightly, $f''(x) > 0$. $\therefore$ $f'(x)$ is increasing.

$\therefore$ (0, 0) is another inflexion point. 1

As $f''(x)$ exists in $\mathbb{R}$ except at $x = 0$ or 1 , these two are the only inflexion points. 1

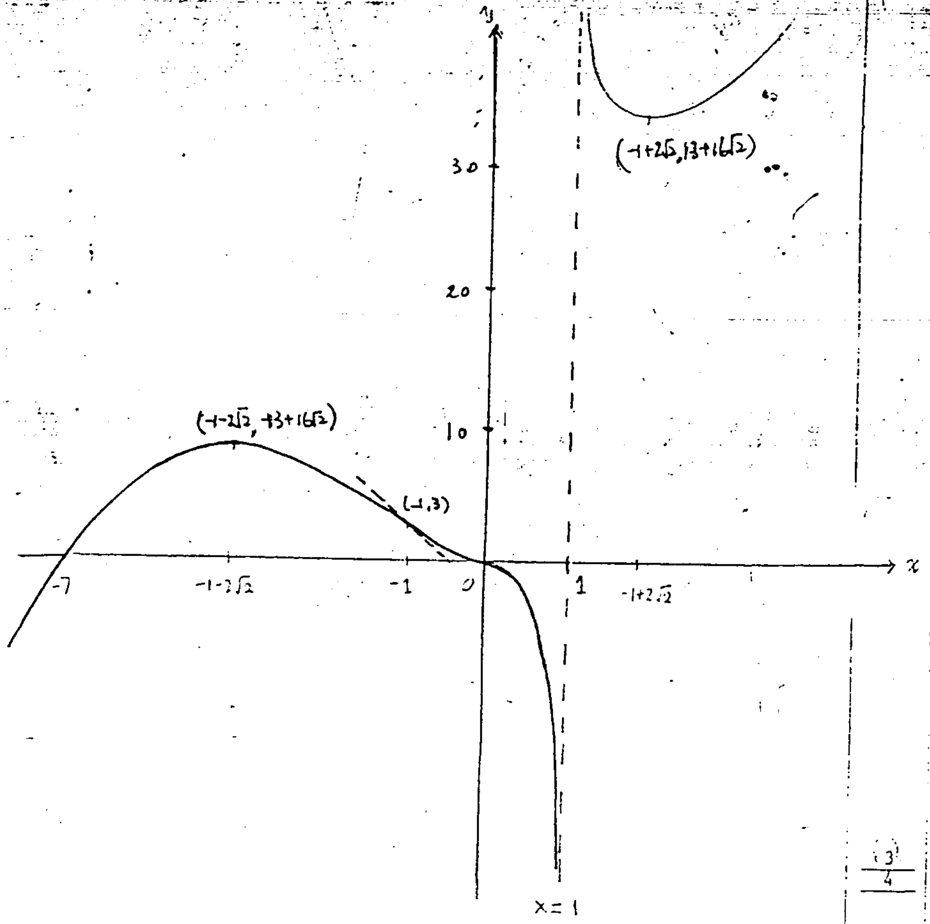
7

Solution

87 | Marks | Remarks

II

7 (c) The curve cuts the axes at (0, 0) and (−7, 0) 1



(3)
4

Solution

87 | Marks | Remark

II

8 (a) Integrating by parts,

$$\int_a^x (x-t)g''(t)dt = (x-t)g'(t)\Big|_a^x + \int_a^x g'(t)dt \qquad 2$$

$$= -(x-a)g'(a) + g(x) - g(a)$$

$$\therefore g(x) = g(a) + (x-a)g'(a) + \int_a^x (x-t)g''(t)dt \quad \forall x \in \mathbb{R} \qquad 1$$

$$\overline{3}$$

(b) (i) If $g(x) = \int_a^x f(t)dt - \frac{(x-a)}{2}[f(x) + f(a)]$, g has a continuous second derivative and $g'(x) = f(x) - \frac{(x-a)}{2}f'(x) - \frac{1}{2}[f(x)+f(a)]$

$$= -\frac{(x-a)}{2}f'(x) + \frac{1}{2}[f(x)-f(a)] \qquad 1$$

$$g''(x) = -\frac{(x-a)}{2}f''(x) \;\ldots\ldots\ldots\ldots\ldots \qquad 1$$

$$\therefore \text{ by (a), } g(x) = 0 + (x-a)(0) + \int_a^x (x-t)\left[-\frac{(t-a)}{2}f''(t)\right]dt$$

$$= -\int_a^x \frac{(x-t)(t-a)}{2}f''(t)dt \qquad 1$$

(ii) Putting x = b in (i) $|g(b)| = \left|\int_a^b \frac{(b-t)(t-a)}{2}f''(t)dt\right|$ 1

$$\leq \frac{M}{2}\left|\int_a^b (b-t)(t-a)dt\right| \qquad 1$$

$$= \frac{M}{2}\left|\left[-abt + \frac{(a+b)}{2}t^2 - \frac{t^3}{3}\right]_a^b\right|$$

$$= \frac{M}{12}(b-a)^3 \qquad 1$$

Dividing the interval [0, 1] into n equal sub-intervals by the points $a_k = \frac{k}{n}$, k = 0, 1, 2, ..., n, we have

$$\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt - \frac{1}{2}\cdot\frac{1}{n}\left[f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right]\right| \leq \frac{M}{12n^3} \qquad 1$$

$$\left|\int_0^1 f(t)dt - \sum_{k=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right]\right|$$

$$= \left|\sum_{k=0}^{n-1}\left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt - \frac{1}{2n}\left[f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right]\right]\right| \qquad 1$$

$$\leq \sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt - \frac{1}{2n}\left[f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right]\right| \qquad 1$$

$$\leq \frac{M}{12n^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 1$$

$$\overline{11}$$

87 II Marks Remarks

9(a) (i) Taking $f(x) = e^{\lambda x}$, $g(x) = e^{x}$,

$$h^{(n)}(x) = \frac{d^n}{dx^n} e^{(1+\lambda)x}$$

$$= (1+\lambda)^n e^{(1+\lambda)x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

$$f^{(k)}(x)g^{(n-k)}(x) = \left(\frac{d^k}{dx^k} e^{\lambda x}\right)\left(\frac{d^{n-k}}{dx^{n-k}} e^{x}\right)$$

$$= \lambda^k e^{\lambda x} \cdot e^{x}$$

$$= \lambda^k e^{(1+\lambda)x}, \quad k = 0, 1, \ldots, n. \quad 1$$

(ii) $(1+\lambda)^n e^{(1+\lambda)x} = \sum_{k=0}^{n} a_k \lambda^k e^{(1+\lambda)x}$ …………………… 1

$$(1+\lambda)^n = \sum_{k=0}^{n} a_k \lambda^k$$

Since this is true for any λ,

$a_k = C^n_k$, $k = 0, 1, \ldots, n$.

(b) (i) Using the above result,

$$u^{(m)}(x) = \sum_{k=0}^{m} C^m_k (x^m)^{(k)} \cdot \frac{d^{m-k}}{dx^{m-k}} e^{-x} \quad \ldots\ldots\ldots\ldots \quad 1$$

$$= \sum_{k=0}^{m} C^m_k \frac{m!}{(m-k)!} x^{m-k} \cdot (-1)^{m-k} e^{-x}$$

$$= \sum_{k=0}^{m} (-1)^k C^m_k \frac{m!}{k!} x^k e^{-x}$$

$$\therefore y(x) = \sum_{k=0}^{m} (-1)^k C^m_k \frac{m!}{k!} x^k,$$

which is a polynomial of degree m. 1

The coefficient of the term x^k is $(-1)^k C^m_k \frac{m!}{k!}$. 1

87 II Marks Remarks

9(b) (ii) $xu'(x) + (x-m)u(x) = x[mx^{m-1}e^{-x} - x^m e^{-x}] + (x-m)x^m e^{-x} = 0$ 1

Differentiating $(m+1)$ times, 1

$[xu'(x)]^{(m+1)} = xu^{(m+2)}x + (m+1)u^{(m+1)}(x)$)

$[(x-m)u(x)]^{(m+1)} = (x-m)u^{(m+1)}(x) + (m+1)u^{(m)}(x)$) 1 for either

$\therefore$ L.S. $= xu^{(m+2)}(x) + (x+1)u^{(m+1)}(x) + (m+1)u^{(m)}(x)$

$= 0$ …………………………………………… 1

(iii) $y'(x) = e^x[u^{(m+1)}(x) + u^{(m)}(x)]$)

$y''(x) = e^x[u^{(m+2)}(x) + 2u^{(m+1)}(x) + u^{(m)}(x)]$) ……… 1 for either

$\therefore$ $xy''(x) + (1-x)y'(x) + my(x)$

$= xe^x[u^{(m+2)}(x) + 2u^{(m+1)}(x) + u^{(m)}(x)] +$

$(1-x)e^x[u^{(m+1)}(x) + u^{(m)}(x)] + me^x u^{(m)}(x)$ ……… 1

$= e^x[xu^{(m+2)}(x) + (x+1)u^{(m+1)}(x) + (m+1)u^{(m)}(x)]$

$= 0$ by (ii). 1

$\overline{10}$