OUTLINES OF SOLUTIONS

The following are outlines of solutions extracted from the annual reports of past Hong Kong Advanced Level examinations. Readers should note that they are not meant to be model answers.

Outline of Solutions

1980

Paper I

Q.1 (a) (i) Using properties of matrices, we have $A\underline{0} = \underline{0}$, and for any $\underline{x}, \underline{y} \in E$, and any $\alpha, \beta \in \mathbf{R}$,

$$A(\alpha\underline{x} + \beta\underline{y}) = \alpha(A\underline{x}) + \beta(A\underline{y}) = \underline{0} .$$

Hence $\underline{0} \in E$, and for any $\underline{x}, \underline{y} \in E$, and any $\alpha, \beta \in \mathbf{R}$, $\alpha\underline{x} + \beta\underline{y} \in E$, showing that E is a subspace of V.

(ii) If $\underline{y} = \underline{p} + \underline{x}$ with $\underline{x} \in E$, then $A\underline{y} = A(\underline{p} + \underline{x}) = A\underline{p} + A\underline{x} = \underline{b}$. On the other hand, if $A\underline{y} = \underline{b}$, then putting $\underline{x} = \underline{y} - \underline{p}$, we have $A\underline{x} = A\underline{y} - A\underline{p} = \underline{0}$, showing $\underline{x} \in E$ and $\underline{y} = \underline{p} + \underline{x}$.

(b) (i) By routine method,

solution set $= \{(3t, -2t, 5t) : t \in \mathbf{R}\}$.

(ii) Applying (a)(ii) and (b)(i),

solution set $= \{(\frac{1}{2} + 3t, \frac{4}{3} - 2t, \sqrt{2} + 5t) : t \in \mathbf{R}\}$.

Q.2 (a) Reflexive: Let $I(x) \equiv 1$.
For any $f \in F$, $I * f = f * I$ and I is a polynomial, therefore $f \sim f$.

Symmetric: If $f \sim g$, then there exist polynomials $p, q \in F$ such that

$$p * f = q * g \text{ or } q * g = p * f .$$

$\therefore g \sim f$.

Transitive: If $f \sim g$ and $g \sim h$, then there exist polynomials $p, q, r, s \in F$ such that

$$p * f = q * g, \quad r * g = s * h .$$

Noting $*$ is commutative and associative, we get $(r * p) * f = (q * s) * h$ where $r * p$ and $q * s$ are both polynomials in F. Thus $f \sim h$.

Q.2 (b) (i) If $f/\sim \; = \; f_1/\sim$ and $g/\sim \; = \; g_1/\sim$, then

$$p * f = q * f_1 \quad \text{and} \quad r * g = s * g_1$$

for some polynomials p, q, r, $s \in F$. Then

$$(p * r) * (f * g) = (q * s) * (f_1 * g_1)$$

where $p * r$ and $q * s$ are polynomials in F. Thus $f * g \sim f_1 * g_1$, i.e., $(f * g)/\sim \; = (f_1 * g_1)/\sim$ or $f/\sim \circledast g/\sim \; = f_1/\sim \circledast g_1/\sim$.

(ii) Closure: follows from (b)(i).

Associativity: follows from the definition of $\circledast$ and the associativity of $*$.

Identity: $I/\sim$ is the identity since for any $f/\sim \; \in F/\sim$,
$f/\sim \circledast I/\sim \; = (f * I)/\sim \; = f/\sim$, and
$I/\sim \circledast f/\sim \; = (I * f)/\sim \; = f/\sim$.

Inverse: For any $f/\sim \; \in F/\sim$, let f^{-1} be the inverse of f in F. Then

$$f/\sim \circledast f^{-1}/\sim \; = I/\sim,$$
$$f^{-1}/\sim \circledast f/\sim \; = I/\sim.$$

Q.3 (a) Differentiating $f(x) = \dfrac{x^{p+1} - 1}{p+1} - \dfrac{x^p - 1}{p}$, we find that the absolute minimum of f on $(0, \infty)$ is 0 at $x = 1$. Hence for all $x > 0$,

$$\frac{x^{p+1} - 1}{p+1} - \frac{x^p - 1}{p} \geqslant 0$$

and the equality holds only if $x = 1$.

(b) (i) By induction on m and using (a).

(ii) If $x_{i_0} \neq 1$ for some i_0, then by (a), we have

$$\frac{x_{i_0}^{\,m} - 1}{m} > \frac{x_{i_0}^{\,m-1} - 1}{m-1}$$

and

$$\frac{x_i^{\,m} - 1}{m} \geqslant \frac{x_i^{\,m-1} - 1}{m-1} \quad \text{for } i \neq i_0$$

implying

$$\frac{\sum_{i=1}^{n} x_i^{\,m} - n}{m} > \frac{\sum_{i=1}^{n} x_i^{\,m-1} - n}{m-1} \geqslant 0$$

or $\sum_{i=1}^{n} x_i^{\,m} > n$, a contradiction.

Q.3 (c) Let $x_i = \dfrac{n y_i}{\sum_{i=1}^{n} y_i}$. Then $x_i > 0$ and $\sum_{i=1}^{n} x_i = n$.

Using (b)(i), we get

$$\frac{y_1^{\,m} + \ldots + y_n^{\,m}}{n} \geqslant \left(\frac{y_1 + \ldots + y_n}{n}\right)^m$$

By (b)(ii), if the equality holds and $m \neq 1$, then $x_1 = \ldots = x_n = 1$, giving $y_1 = \ldots = y_n$.

Q.4 (a) $y_k = \dfrac{A^{k-1} y_1 + B(1 - A^{k-1})}{(1 - A)}$.

This formula can be established by mathematical induction.

(b) (i) $x_k - a x_{k-1} = b^{k-1}(x_1 - a x_0)$.

(ii) $\dfrac{x_k}{b_k} = \dfrac{a}{b} \cdot \dfrac{x_{k-1}}{b^{k-1}} + \dfrac{x_1 - a x_0}{b}$, where $\dfrac{a}{b}$, $\dfrac{x_1 - a x_0}{b}$ are constants independent of k and $\dfrac{a}{b} \neq 1$. From (a),

$$\frac{x_k}{b_k} = \left(\frac{a}{b}\right)^{k-1} \frac{x_1}{b} + \frac{x_1 - a x_0}{b} \cdot \frac{1 - \left(\frac{a}{b}\right)^{k-1}}{1 - \frac{a}{b}},$$

and so

$$x_k = \frac{b^k - a^k}{b - a} x_1 - \frac{ab(b^{k-1} - a^{k-1})}{b - a} x_0.$$

(c) Take $a = -\dfrac{2}{3}$, $b = 1$, $\lim_{k \to \infty} x_k = \dfrac{3}{5} x_1 + \dfrac{2}{5} x_0$.

Q.5 (a) (i) By substitution, we find that $u + v$, $\omega u + \omega^2 v$, $\omega^2 u + \omega v$ are roots of the equation

$$x^3 - 3uvx - (u^3 + v^3) = 0.$$

(ii) Consider

$$\begin{cases} u^3 + v^3 = 6 \\ uv = 2 \end{cases} \quad \text{or} \quad \begin{cases} u^3 + v^3 = 6 \\ u^3 v^3 = 8 \end{cases}$$

We find that $u^3 = 2$ and $v^3 = 4$. A solution is $u = 2^{\frac{1}{3}}$, $v = 2^{\frac{2}{3}}$. By (a)(i), roots of the equation

$$x^3 - 6x - 6 = 0$$

are

$$2^{\frac{1}{3}} + 2^{\frac{2}{3}}, \quad 2^{\frac{1}{3}}\omega + 2^{\frac{2}{3}}\omega^2, \quad 2^{\frac{2}{3}}\omega + 2^{\frac{1}{3}}\omega^2.$$

Q.5 (b) (i) Let α, α, β be the roots of $(*)$. By the relations between roots and coefficients, we get

$$2\alpha + \beta = 0, \qquad \alpha^2 + 2\alpha\beta = p, \qquad -\alpha^2\beta = q .$$

Eliminating α and β, we get $27q^2 + 4p^3 = 0$.

(ii) The roots of $(*)$ are $u + v$, $\omega u + \omega^2 v$, $\omega^2 u + \omega v$, where

$$\begin{cases} u^3 + v^3 = -q , \\ uv = -\dfrac{1}{3}p \end{cases}$$

which has $u = v = -\sqrt[3]{\dfrac{q}{2}}$ as a solution if $27q^2 + 4p^3 = 0$.

Thus $\omega u + \omega^2 v = \omega^2 u + \omega v$ is a multiple root of $(*)$.

Q.6 (a) Note that $[(1+u)x - (au+b)]^{m+n} = [(x-a)u + (x-b)]^{m+n}$.
Expand the right-hand side and compare the coefficients of u^k.

(b)
$$\int_a^b (x-a)^m (x-b)^n \, dx = (-1)^{m+1} \frac{m!\,n!\,(a-b)^{m+n+1}}{(m+n+1)!} .$$

(c)
$$\frac{d^r}{dx^r}\left\{ (x-a)^m (x-b)^n \right\}\Bigg|_{x=a} = \begin{cases} \dfrac{n!\,r!}{(m+n-r)!\,(r-m)!}(a-b)^{m+n-r} & \text{if } m \le r \le m+n \\ 0 \text{ if } r > m+n \text{ or } m > r . \end{cases}$$

Q.7 (a) This can be easily verified.

(b) (i) From the properties of g we get

$$|g(z)| = |z| , \quad |g(z) - 1| = |z - 1| .$$

Using properties of complex numbers, we have

$$|g(z) - 1|^2 = |g(z)|^2 - (g(z) + \overline{g(z)}) + 1$$
$$|z - 1|^2 = |z|^2 - (z + \bar{z}) + 1 .$$

Hence

$$g(z) + \overline{g(z)} = z + \bar{z} \quad \text{or} \quad \operatorname{Re} g(z) = \operatorname{Re} z .$$

(ii) By (i), $g(i) = bi$, where b is real. Furthermore,

$$|b|^2 = |g(i)|^2 = |g(i) - g(0)|^2 = |i|^2 = 1 .$$

Therefore $b = \pm 1$.

Q.7 (c) If $g(i) = i$, then

$$|g(z) - i|^2 = |z - i|^2 .$$

Thus we get $g(z) - \overline{g(z)} = z - \bar{z}$. By (b)(i), $g(z) = z$.
If $g(i) = -i$, then

$$|g(z) + i|^2 = |z - i|^2 .$$

We get $\overline{g(z)} - g(z) = z - \bar{z}$. By (b)(i), $g(z) = \bar{z}$.

(d) g, given by $g(z) = \dfrac{f(z) - f(0)}{f(1) - f(0)}$, is an isometry satisfying the conditions in (b). Then $f(z) = ag(z) + b$ with

$$a = f(1) - f(0) \quad \text{and} \quad b = f(0) .$$

Thus $|a| = 1$. By (b) and (c), we have

$$f(z) = az + b \quad \text{or} \quad f(z) = a\bar{z} + b$$

Q.8 (a) (i)
$$P_k = \frac{1}{n^N} C_k^N (n-1)^{N-k} .$$

(ii) Since k_0 is most probable, we have

$$P_{k_0} \ge P_{k_0 - 1} \quad \text{and} \quad P_{k_0} \ge P_{k_0+1} .$$

By computation we get

$$\frac{N - n + 1}{n} \le k_0 \le \frac{N+1}{n} .$$

(iii) Mean number $= \displaystyle\sum_{k=1}^{N} k\, C_k^N \left(1 - \frac{1}{n}\right)^{N-k} \left(\frac{1}{n}\right)^k$.

Differentiating both sides of

$$(p + qx)^N = \sum_{k=0}^{N} C_k^N p^{N-k} (qx)^k$$

with respect to x and then putting $x = 1$, $p = 1 - \dfrac{1}{n}$, $q = \dfrac{1}{n}$, we get

$$\text{mean number} = \frac{N}{n} .$$

Since $\dfrac{N}{n} - \left(1 - \dfrac{1}{n}\right) \le k_0 \le \dfrac{N}{n} + \dfrac{1}{n}$, therefore $\left| k_0 - \dfrac{N}{n} \right| \le 1$.

(b) Take out one cell and consider this to be occupied by exactly k balls $(1 \le k \le N)$. There are C_k^N ways of doing so. Then the number of ways such that the remaining cells being occupied by $N - k$ balls is $A(N - k, n)$. Thus

$$A(N, n+1) = \sum_{k=1}^{N} C_k^N A(N-k, n) .$$

The second part is proved by induction on n. For $n = 1$, it can be easily verified. Assume it is true for n, i.e.

$$A(N, n) = \sum_{j=0}^{n} (-1)^j C_j^n (n - j)^N \quad \text{for any } N .$$

Q.8 (b) Now $A(N, n+1) = \sum_{k=1}^{N} C_k^N A(N-k, n)$

$$= \sum_{k=0}^{N} C_k^N \sum_{j=0}^{n} (-1)^j C_j^n (n-j)^{N-k} - A(N, n)$$

$$= \sum_{j=0}^{n} (-1)^j C_j^n (n-j)^N \left\{ \sum_{k=0}^{N} C_k^N (n-j)^{-k} \right\} - A(N, n)$$

$$= \sum_{j=0}^{n} (-1)^j C_j^n (n+1-j)^N - \sum_{j=0}^{n} (-1)^j C_j^n (n-j)^N$$

$$= (n+1)^N + \sum_{j=1}^{n+1} (-1)^j C_j^n (n+1-j)^N + \sum_{j=1}^{n+1} (-1)^j C_{j-1}^n (n+1-j)^N$$

$$= (n+1)^N + \sum_{j=1}^{n+1} (-1)^j \left\{ C_j^n + C_{j-1}^n \right\} (n+1-j)^N$$

$$= (n+1)^N + \sum_{j=1}^{n+1} (-1)^j C_j^{n+1} (n+1-j)^N$$

$$= \sum_{j=0}^{n+1} (-1)^j C_j^{n+1} (n+1-j)^N$$

implying it is also true for $n+1$.

Paper II

Q.1 (a) Since $R = P(t_1)$, $S = P(t_2)$ are the points of intersection of the line $y = m(x-a)$ with the circle, t_1 and t_2 satisfy the equation

$$\frac{2t}{1+t^2} = m\left(\frac{1-t^2}{1+t^2} - a\right)$$

so t_1 and t_2 are roots of the equation

$$m(a+1)\, t^2 + 2t + m(a-1) = 0 .$$

The result follows from the relations between roots and coefficients.

(b) RO has equation $y = \dfrac{2t_1}{1-t_1^2}\, x$.

QT has equation $y = \dfrac{2t_2}{1-t_2^2}(x-a)$.

Solving these equations, we get the coordinates of T :

$$x = \frac{at_2(1-t_1^2)}{(t_1t_2+1)(t_2-t_1)}, \quad y = \frac{2at_1t_2}{(t_1t_2+1)(t_2-t_1)} .$$

(c) By (b) and (a), the coordinates of T satisfy

$$4\left(x-\frac{a}{2}\right)^2 = \frac{1}{1-m^2a^2+m^2}, \quad 4y^2 = \frac{(1-a^2)^2m^2}{1-m^2a^2+m^2} .$$

Eliminating m, we get

$$(1-a^2)\left(x-\frac{a}{2}\right)^2 + y^2 = C, \text{ where } C = \frac{1-a^2}{4} .$$

Q.2 (a) $P_\alpha \parallel Q_\beta \iff$ there exists $m \neq 0$ such that

$$\alpha + 1 = m(\beta+2)$$
$$2\alpha + 2 = m(\beta+3)$$
$$3\alpha + 2 = m(\beta+5)$$

i.e. $$\begin{bmatrix} 1 & -m & 1-2m \\ 2 & -m & 2-3m \\ 3 & -m & 2-5m \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) (*) is solvable for α and β

$$\Rightarrow \begin{vmatrix} 1 & -m & 1-2m \\ 2 & -m & 2-3m \\ 3 & -m & 2-5m \end{vmatrix} = 0$$

$$\Rightarrow \quad m = -1 \text{ since } m \neq 0 .$$

For $m = -1$, we get $\alpha = -2$, $\beta = -1$. Hence

M_1 : $x + 2y + 4z - 2 = 0$,

M_2 : $x + 2y + 4z - 1 = 0$.

Q.2 (c) $P \perp M_2 \iff$ their normals are perpendicular

$$\iff (\alpha + 1)(1) + (2\alpha + 2)(2) + (3\alpha + 2)(4) = 0$$

$$\iff \alpha = -\frac{13}{17}.$$

$$\therefore \quad N : 4x + 8y - 5z - 29 = 0.$$

(d) $$\ell_1 \cap \ell_2 = N \cap \ell_2 = \left\{ (x, y, z) \in \mathbf{R}^3 : \begin{bmatrix} 4 & 8 & -5 \\ 1 & 1 & 1 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 29 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$\begin{vmatrix} 4 & 8 & -5 \\ 1 & 1 & 1 \\ 2 & 3 & 5 \end{vmatrix} = -21 \neq 0 \quad \therefore N \cap \ell_2 \text{ is a singleton.}$$

Q.3 (a) (i) $$|f(z)|^2 = f(z)\,\overline{f(z)}$$

$$= \left\{ \sum_{k=0}^{n} a_k r^k (\cos k\theta + i \sin k\theta) \right\} \left\{ \sum_{j=0}^{n} a_j r^j (\cos j\theta - i \sin j\theta) \right\}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j r^{k+j} [(\cos k\theta \cos j\theta + \sin k\theta \sin j\theta) + i(\sin k\theta \cos j\theta - \cos k\theta \sin j\theta)]$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j r^{k+j} \cos(k - j)\theta, \text{ since the L.H.S. is real.}$$

(ii) Putting $r = 1$, we have

$$\int_0^{2\pi} |f(\cos\theta + i \sin\theta)|^2 \, d\theta = \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j \int_0^{2\pi} \cos(k - j)\theta \, d\theta$$

$$= 2\pi \sum_{k=0}^{n} a_k^2$$

since $$\int_0^{2\pi} \cos(k - j)\theta \, d\theta = \begin{cases} 0 & \text{if } k \neq j \\ 2\pi & \text{if } k = j \end{cases}.$$

(b) Applying (a)(ii) to $f(z) = (1 + z)^n$ and noticing $|(1 + \cos\theta + i \sin\theta)|^2 = 2(1 + \cos\theta)$, we have

$$\sum_{k=0}^{n} (C_k^n)^2 = \frac{1}{2\pi} \int_0^{2\pi} 2^n (1 + \cos\theta)^n \, d\theta$$

$$= \frac{2^n}{\pi} \int_0^{\pi} (1 + \cos\theta)^n d\theta \qquad \left[\because \int_\pi^{2\pi} (1 + \cos\theta)^n d\theta = \int_0^{\pi} (1 + \cos\theta)^n d\theta \right]$$

$$= \frac{2^{2n}}{\pi} \int_0^{\pi} \left(\cos\frac{\theta}{2}\right)^{2n} d\theta$$

$$= \frac{2^{2n+1}}{\pi} \int_0^{\frac{\pi}{2}} (\cos t)^{2n} \, dt.$$

Q.4 (a) Let x denote the distance from the centre of the sphere to the base of the cone. The volume V of the cone is given by $V = \frac{\pi}{3}(a^2 - x^2)(a + x)$. By differentiation we find the maximum of V is attained at $x = \frac{a}{3}$. Hence the required height of the cone is $\frac{4a}{3}$.

(b) (i) $$r(x, \theta) = (a^2 + x^2 + 2ax\cos\theta)^{\frac{1}{2}},$$

$$g(x) = \left. \frac{-(a^2 + x^2 + 2ax\cos\theta)^{\frac{1}{2}}}{x} \right|_0^{\frac{2\pi}{3}}$$

$$= \frac{3a}{(x + a) + (a^2 + x^2 - ax)^{\frac{1}{2}}}.$$

(ii) Obviously $g(x) > 0$ on $(0, \infty)$. For $a, x > 0$,

$$(a^2 + x^2 - ax)^{\frac{1}{2}} = [(a - x)^2 + ax]^{\frac{1}{2}} > a - x.$$

$$\frac{3}{2} - g(x) = \frac{3[(a^2 + x^2 - ax)^{\frac{1}{2}} - (a - x)]}{2[(x + a) + (a^2 + x^2 - ax)]^{\frac{1}{2}}} > 0.$$

Hence $0 < g(x) < \frac{3}{2}$ on $(0, \infty)$.

Q.5 (a) $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist and are respectively $f(x_0)$, $g(x_0)$.
Since $f(x) = g(x) \; \forall x \in \mathbf{R} \setminus \{x_0\}$, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$.
Thus $f(x_0) = g(x_0)$.

(b) $(x - x_0)^m q(x) = (x - x_0)^k h(x)$ and $q(x_0) \neq 0$, $h(x_0) \neq 0$.
If $m \neq k$, then without loss of generality, let $m > k$. Dividing throughout by $(x - x_0)^k$, we have $\forall x \neq x_0$

$$(x - x_0)^{m-k} q(x) = h(x) \cdots\cdots (*)$$

Both sides are polynomials, hence continuous. From (a), $(*)$ holds for $x = x_0$. Since $m > k$, this gives $h(x_0) = 0$, a contradiction.

Thus $\quad p(x) = (x - x_0)^m q(x) = (x - x_0)^m h(x)$.

By similar argument:

$$q(x) = h(x) \quad \forall x \neq x_0.$$

By (a) again, $q(x) = h(x) \quad \forall x \in \mathbf{R}$.

Q.5 (c) 'Only if' part: $p(x) = (x - x_0)^{k+1} q(x)$ where $q(x_0) \neq 0$.

Then $p(x_0) = 0$. It can be easily verified that x_0 is a root of multiplicity k of $p'(x) = 0$.

'If' part: $p(x_0) = 0 \Rightarrow p(x) = (x - x_0)^m q(x)$ where $m \geqslant 1$ and $q(x_0) \neq 0$.

$p'(x) = (x - x_0)^{m-1} t(x)$ where

$t(x) = m q(x) + (x - x_0) q'(x)$. Hence $t(x_0) \neq 0$.

On the other hand,

$$p'(x) = (x - x_0)^k h(x) \quad \text{where } h(x_0) \neq 0.$$

By (b), $k = m - 1$. Hence $m = k + 1$.

Q.6 (a) $f(x+h) - f(x) = [g(h) - g(0)] f(x) + [f(h) - f(0)] g(x)$ since $g(0) = 1$, $f(0) = 0$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} f(x) + \lim_{h \to 0} \frac{f(h) - f(0)}{h} g(x)$$

$$= g'(0) f(x) + f'(0) g(x)$$

$$= g(x).$$

Hence $f'(x) = g(x)$.

(b) (i) Integrating by parts, we get

$$\int_0^1 f(x) f''(x)\,dx = -\int_0^1 f'(x)^2\,dx \leqslant 0.$$

Equality holds $\Rightarrow f'(x) = 0 \;\; \forall x \in [0, 1]$

$\Rightarrow f(x) =$ constant on $[0, 1]$ which must be 0 since $f(0) = f(1) = 0$.

(ii) Integrating by parts, we get

$$\int_0^1 x f(x) f'(x)\,dx = x f(x)^2 \Big|_0^1 - \int_0^1 [f(x)^2 + x f(x) f'(x)]\,dx.$$

Hence the result.

Q.7 (a) By expressing the integrands in terms of $\sin(m+n)x$, $\sin(m-n)x$, $\cos(m+n)x$, $\cos(m-n)x$ and using the given results.

(b) (i) By expanding

$$\left\{ f(x) - \sum_{i=0}^{N} \alpha_i \varphi_i(x) \right\}^2$$

and noticing that

$$\int_{-\pi}^{\pi} \varphi_i(x) \varphi_j(x)\,dx = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

we get

$$\int_{-\pi}^{\pi} \left[f(x) - \sum_{i=0}^{N} \alpha_i \varphi_i(x) \right]^2 dx$$

$$= \int_{-\pi}^{\pi} f(x)^2\,dx + \sum_{i=0}^{N} \alpha_i^2 - 2 \sum_{i=0}^{N} \alpha_i p_i$$

$$= \int_{-\pi}^{\pi} f(x)^2\,dx + \sum_{i=0}^{N} (\alpha_i - p_i)^2 - \sum_{i=0}^{N} p_i^2.$$

Hence the least value is attained when $\alpha_i = p_i$ for all i.

(ii) When $\alpha_i = p_i$, then for all integer $M \geqslant 1$,

$$0 \leqslant \int_{-\pi}^{\pi} \left[f(x) - \sum_{i=0}^{2M} \alpha_i \varphi_i(x) \right]^2 dx$$

$$= \int_{-\pi}^{\pi} f(x)^2\,dx - \sum_{i=0}^{2M} p_i^2$$

which implies the required inequality.

Q.8 (a) (i) By substituting $\begin{cases} x = kx' - hy' \\ y = hx' + ky' \end{cases}$ into the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

(ii) Use the results in (i).

(iii) $B' = 0 \iff \dfrac{2kh}{k^2 - h^2} = \dfrac{B}{A - C}$, i.e. $\tan 2\theta = \dfrac{B}{A - C}$

Q.8 (b) Here $A = 1$, $B = -2$, $C = 1$. Let $\theta = \frac{\pi}{4}$. The transformation

$$\begin{cases} x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ y' = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{cases}$$

brings the given equation into the form

$$2y'^2 + \frac{3}{\sqrt{2}}x' - \frac{4}{\sqrt{2}}y' + \frac{11}{2} = 0 .$$

Followed by the translation

$$\begin{cases} x'' = x' + \frac{3}{\sqrt{2}} \\ y'' = y' - \frac{1}{\sqrt{2}} \end{cases}$$

the original equation becomes $y''^2 + \frac{3\sqrt{2}}{4}x'' = 0$, which is that of a parabola whose line of symmetry is

$$y'' = 0 \text{ or } y' = \frac{1}{\sqrt{2}} \text{ or } x - y + 1 = 0 .$$

1981

Paper I

Q.1 (a) Solution set of (I) is

$$\{(3t + 2, 2t - 1, t) : t \in \mathbf{R}\} .$$

(b) (i) For $p \neq -5$ and $q \in \mathbf{R}$ (II) is solvable.

(ii) For $p = -5$ and $q = 1$ (II) is solvable.

(c) (i) If $p \neq -5$ the only solution for the first three equations of (III) is $(2, -1, 0)$ which does not satisfy the 4th equation. So (III) has no solution for $p \neq -5$.

(ii) If $p = -5$, the solution set for the first three equations of (III) is $\{(3t + 2, 2t - 1, t) : t \in \mathbf{R}\}$. Substituting it into the 4th equation we get

$$7t^2 + 4t - 3 = 0$$

and hence

$$t = -1 \text{ or } \frac{3}{7} .$$

Thus $(-1, -3, -1)$ and $(\frac{23}{7}, -\frac{1}{7}, \frac{3}{7})$ are the solutions of (III).

Q.2 (a) Consider the sequence $\{-b_n\}$.

Since $-b_n \leqslant -b_{n+1} \quad \forall n$ and

$-b_n \leqslant -M \quad \forall n$,

$\{-b_n\}$ converges.

$\lim_{n\to\infty} b_n = -\lim_{n\to\infty}(-b_n)$, since the last limit exists.

$\therefore \{b_n\}$ converges.

(b) Since G.M. $\leqslant$ A.M. and $x_1 < y_1$,

$x_n \leqslant y_n$ for $n = 1, 2, \ldots$.

It is obvious that $x_n, y_n \geqslant 0 \quad \forall n$.

Further, for $n \geqslant 1$

$$x_{n+1} = \sqrt{x_n y_n} \geqslant \sqrt{x_n x_n} = x_n ,$$

$$y_{n+1} = \frac{x_n + y_n}{2} \leqslant \frac{y_n + y_n}{2} = y_n .$$

Thus $\{x_n\}$ is increasing and bounded above by b, and $\{y_n\}$ is decreasing and bounded below by a. Therefore both $\{x_n\}$, $\{y_n\}$ are convergent.

Let $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$.

Since $y_{n+1} = \frac{x_n + y_n}{2}$

$$\therefore \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \frac{x_n + y_n}{2}$$

$$= \frac{1}{2}\left(\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n\right)$$

$\therefore x = y$.