Marking Scheme

Module 2 (Algebra and Calculus)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.

Solution	Marks	Remarks
1. $\mathrm{f}(1+h)-\mathrm{f}(1)$		
$=\dfrac{10(1+h)}{7+3(1+h)^2}-\dfrac{10}{7+3}$		
$=\dfrac{10+10h}{10+6h+3h^2}-1$		
$=\dfrac{10+10h-10-6h-3h^2}{10+6h+3h^2}$		
$=\dfrac{4h-3h^2}{10+6h+3h^2}$	1	
$\mathrm{f}'(1)$		
$=\lim\limits_{h\to 0}\dfrac{\mathrm{f}(1+h)-\mathrm{f}(1)}{h}$	1M	
$=\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{4h-3h^2}{10+6h+3h^2}\right)$		
$=\lim\limits_{h\to 0}\dfrac{4-3h}{10+6h+3h^2}$	1M	withhold 1M if the step is skipped
$=\dfrac{4-3(0)}{10+6(0)+3(0)^2}$		
$=\dfrac{2}{5}$	1A	0.4
	----------(4)	
2. (a) $\mathrm{P}(x)$		
$=(x+\lambda)(x+\lambda)^2(x+\lambda)^3-(3)(5)(x+\lambda)+(1)(3)(4)-(2)(4)(x+\lambda)^2$	1M	
$=(x+\lambda)^6-8(x+\lambda)^2-15(x+\lambda)+12$		
Note that the coefficient of x^3 in the expansion of $\mathrm{P}(x)$ is 160 .		
So, we have $C_3^6\lambda^3=160$.	1M	for $C_3^6\lambda^3$
Thus, we have $\lambda=2$.	1A	
(b) Note that $\mathrm{P}'(0)$ is the coefficient of x in the expansion of $\mathrm{P}(x)$.	1M	can be absorbed
Also note that the coefficient of x is $6\lambda^5-16\lambda-15$.		
By (a), we have $\mathrm{P}'(0)=145$.	1A	
	----------(5)	

Solution	Marks	Remarks
3. (a) V		
$=\int -2t\,dt$	1M	
$=-t^2+C$, where C is a constant		
Since $V=580$ when $t=0$, we have $C=580$.	1M	
So, we have $V=580-t^2$.		
When $t=24$, we have $V=4>0$.		
Thus, the claim is correct.	1A	f.t.
(b) Let p cm be the depth of liquid X in the vessel when $t=18$.		
Since $V\big\|_{t=18}=580-18^2=256$, we have $p^2+24p=256$.	1M	
So, we have $p^2+24p-256=0$.		
Solving, we have $p=8$ or $p=-32$ (rejected) .		
Note that $\frac{dV}{dt}=(2h+24)\frac{dh}{dt}$.	1M	
Since $\left.\frac{dV}{dt}\right\|_{t=18}=-36$, we have $-36=(2(8)+24)\left.\frac{dh}{dt}\right\|_{t=18}$.		
Thus, we have $\left.\frac{dh}{dt}\right\|_{t=18}=\frac{-9}{10}$.	1A	-0.9
	----------(6)	
4. (a) $g'(x)$		
$=\dfrac{\sqrt{x}\left(\frac{1}{x}\right)-(\ln x)\left(\frac{1}{2\sqrt{x}}\right)}{x}$	1M	for quotient rule
$=\dfrac{2-\ln x}{2\sqrt{x^3}}$		
So, we have $g'(x)=0 \Leftrightarrow x=e^2$.		
<table><tr><td>x</td><td>$(0, e^2)$</td><td>e^2</td><td>$(e^2, 99)$</td></tr><tr><td>$g'(x)$</td><td>$+$</td><td>0</td><td>$-$</td></tr></table>	1M	
Therefore, G attains its maximum value only when $x=e^2$. Thus, G has only one maximum point.	1	
(b) Note that $g(x)<0$ for all $x\in(0,1)$ and $g(x)>0$ for all $x\in(1,99)$.		
So, we have $g(x)=0 \Leftrightarrow x=1$.		
The required volume		
$=\int_1^{e^2}\pi\left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$	1M	
$=\pi\int_0^2 u^2\,du$ (by letting $u=\ln x$)	1M	
$=\pi\left[\frac{u^3}{3}\right]_0^2$		
$=\frac{8\pi}{3}$	1A	
	----------(6)	

Solution	Marks	Remarks
5. (a) Note that $\frac{1}{(1)(2)}+\frac{1}{(2)(3)}=\frac{2}{3}=\frac{1+1}{(1)(2+1)}$.		
Therefore, the statement is true for $n=1$.	1	
Assume that $\sum_{k=m}^{2m}\frac{1}{k(k+1)}=\frac{m+1}{m(2m+1)}$, where m is a positive integer.	1M	
$\sum_{k=m+1}^{2m+2}\frac{1}{k(k+1)}$		
$=\sum_{k=m}^{2m}\frac{1}{k(k+1)}-\frac{1}{m(m+1)}+\frac{1}{(2m+1)(2m+2)}+\frac{1}{(2m+2)(2m+3)}$	1M	can be absorbed
$=\frac{m+1}{m(2m+1)}-\frac{1}{m(m+1)}+\frac{1}{(2m+1)(2m+2)}+\frac{1}{(2m+2)(2m+3)}$	1M	for using induction assumption
$=\frac{(m+1)^2-(2m+1)}{m(m+1)(2m+1)}+\frac{(2m+3)+(2m+1)}{(2m+1)(2m+2)(2m+3)}$		
$=\frac{m}{(m+1)(2m+1)}+\frac{2}{(2m+1)(2m+3)}$		
$=\frac{(2m+1)(m+2)}{(m+1)(2m+1)(2m+3)}$		
$=\frac{m+2}{(m+1)(2m+3)}$		
So, the statement is true for $n=m+1$ if it is true for $n=m$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(b) Putting $n=50$ in (a), we have $\sum_{k=50}^{100}\frac{1}{k(k+1)}=\frac{51}{(50)(101)}=\frac{51}{5\,050}$.	1M	either one
Putting $n=100$ in (a), we have $\sum_{k=100}^{200}\frac{1}{k(k+1)}=\frac{101}{(100)(201)}=\frac{101}{20\,100}$.		
So, we have $\sum_{k=50}^{200}\frac{1}{k(k+1)}=\frac{51}{5\,050}+\frac{101}{20\,100}-\frac{1}{(100)(101)}$.		
Thus, we have $\sum_{k=50}^{200}\frac{1}{k(k+1)}=\frac{151}{10\,050}$.	1A	
	----------(7)	

<table>
<tr><th>Solution</th><th>Marks</th><th>Remarks</th></tr>
<tr><td>6. (a) (i) Note that

$$\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix}$$

$= (\alpha)(2\alpha+1)+(-2)(\alpha)(7)+(-2)(5)(\alpha-3)-(\alpha)(\alpha-3)-(-2)(5)(2\alpha+1)-(-2)(\alpha)(7)$

$=(\alpha+4)(\alpha+10)$</td><td>1A</td><td></td></tr>
<tr><td>Since (E) has a unique solution, we have $\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix} \neq 0$.</td><td>1M</td><td></td></tr>
<tr><td>Hence, we have $(\alpha+4)(\alpha+10)\neq 0$.
So, we have $\alpha\neq -4$ and $\alpha \neq -10$.
Thus, we have $\alpha < -10$, $-10<\alpha<-4$ or $\alpha>-4$.</td><td>1A</td><td></td></tr>
<tr><td>The augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & \alpha & \alpha & 5\beta \\ 7 & \alpha-3 & 2\alpha+1 & 8\beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & \alpha+10 & \alpha+10 & 0 \\ 0 & \alpha+11 & 2\alpha+15 & \beta \end{array}\right)$$</td><td>1M</td><td></td></tr>
<tr><td>$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 1 & \alpha+5 & \beta \\ 0 & 0 & (\alpha+4)(\alpha+10) & (\alpha+10)\beta \end{array}\right)$$</td><td>1A</td><td></td></tr>
<tr><td>Since (E) has a unique solution, we have $(\alpha+4)(\alpha+10)\neq 0$.
So, we have $\alpha \neq -4$ and $\alpha\neq -10$.
Thus, we have $\alpha<-10$, $-10<\alpha<-4$ or $\alpha>-4$.</td><td>1A</td><td></td></tr>
<tr><td>(ii) Since (E) has a unique solution, we have

$$y = \frac{\begin{vmatrix} 1 & \beta & -2 \\ 5 & 5\beta & \alpha \\ 7 & 8\beta & 2\alpha+1 \end{vmatrix}}{(\alpha+4)(\alpha+10)}$$</td><td>1M</td><td>for Cramer's Rule</td></tr>
<tr><td>$$= \frac{-\beta}{\alpha+4}$$</td><td>1A</td><td></td></tr>
<tr><td>(b) When $\alpha=-4$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & -4 & -4 & 5\beta \\ 7 & -7 & -7 & 8\beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{array}\right)$$</td><td>1M</td><td></td></tr>
<tr><td>Since (E) is inconsistent, we have $\beta \neq 0$.
Thus, we have $\beta<0$ or $\beta>0$.</td><td>1A</td><td></td></tr>
<tr><td></td><td>----------(7)</td><td></td></tr>
</table>

Solution	Marks	Remarks
7. (a) $\int e^x \sin \pi x \, dx$		
$= \int \sin \pi x \, de^x$		
$= e^x \sin \pi x - \pi \int e^x \cos \pi x \, dx$	1M	
$= e^x \sin \pi x - \pi \int \cos \pi x \, de^x$		
$= e^x \sin \pi x - \pi \left(e^x \cos \pi x - \pi \int -e^x \sin \pi x \, dx \right)$	1M	
$\int e^x \sin \pi x \, dx = e^x \sin \pi x - \pi e^x \cos \pi x - \pi^2 \int e^x \sin \pi x \, dx$		
$\pi^2 \int e^x \sin \pi x \, dx + \int e^x \sin \pi x \, dx = e^x \sin \pi x - \pi e^x \cos \pi x + \text{constant}$	1M	
$(\pi^2+1) \int e^x \sin \pi x \, dx = e^x(\sin \pi x - \pi \cos \pi x) + \text{constant}$		
$\int e^x \sin \pi x \, dx = \frac{1}{\pi^2+1}(e^x(\sin \pi x - \pi \cos \pi x)) + \text{constant}$	1A	
(b) $\int_0^3 e^{3-x} \sin \pi x \, dx$		
$= \int_3^0 -e^u \sin \pi(3-u) \, du$ (by letting $u = 3 - x$)	1M	
$= \int_0^3 e^u \sin \pi u \, du$		
$= \int_0^3 e^x \sin \pi x \, dx$		
$= \frac{1}{\pi^2+1}\left[e^x(\sin \pi x - \pi \cos \pi x)\right]_0^3$ (by (a))	1M	for using the result of (a)
$= \frac{\pi(e^3+1)}{\pi^2+1}$	1A	
	----------(7)	

Solution	Marks	Remarks
8. (a) For all $x > 0$,		
$h'(x)$		
$=\frac{2}{x}\left(\left(x^2-\frac{7}{2}x+\left(\frac{7}{4}\right)^2-\left(\frac{7}{4}\right)^2\right)+4\right)$	1M	
$=\frac{2}{x}\left(\left(x-\frac{7}{4}\right)^2+\frac{15}{16}\right)$		
>0		
Thus, $h(x)$ is an increasing function.	1A	f.t.
(b) (i) y		
$=\int\left(2x-7+\frac{8}{x}\right)dx$	1M	
$=x^2-7x+8\ln x+C$, where C is a constant		
Note that $y=3$ when $x=1$.		
So, we have $1^2-7(1)+8\ln 1+C=3$.	1M	
Therefore, we have $C=9$.		
Thus, the equation of H is $y=x^2-7x+8\ln x+9$.	1A	
(ii) $h''(x)$		
$=\frac{x(4x-7)-(2x^2-7x+8)}{x^2}$	1M	
$=\frac{2x^2-8}{x^2}$		
$=\frac{2(x-2)(x+2)}{x^2}$		
Since $x>0$, we have $h''(x)=0 \Leftrightarrow x=2$.		
(see table below)	1M	
Thus, the point of inflexion of H is $(2, 8\ln 2-1)$.	1A	
	----------(8)	

x	$(0,2)$	2	$(2,\infty)$
$h''(x)$	$-$	0	$+$

Solution	Marks	Remarks
9. (a) $\frac{dy}{dx}$		
$=\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(12-x^2)^{\frac{-1}{2}}(-2x)$	1M	for chain rule
$=\frac{-x}{3\sqrt{12-x^2}}$		
So, we have $\left.\frac{dy}{dx}\right\|_{x=3}=\frac{-1}{\sqrt{3}}$.		
The equation of L is		
$y-\frac{\sqrt{3}}{3}=\frac{-1}{\sqrt{3}}(x-3)$	1M	
$x+\sqrt{3}y-4=0$	1A	
	----------(3)	
(b) (i) Putting $y=\frac{1}{\sqrt{3}}(4-x)$ in $y=\sqrt{4-x^2}$, we have	1M	
$\frac{1}{\sqrt{3}}(4-x)=\sqrt{4-x^2}$		
$x^2-2x+1=0$		
So, we have $x=1$ and $y=\sqrt{3}$.		
Thus, the point of contact of L and C is $(1,\sqrt{3})$.	1A	
(ii) When $\sqrt{4-x^2}=\frac{1}{3}\sqrt{12-x^2}$, we have $36-9x^2=12-x^2$.	1M	
So, we have $8x^2-24=0$.		
Since $0<x<2$, we have $x=\sqrt{3}$.		
Thus, the point of intersection of C and Γ is $(\sqrt{3},1)$.	1A	
(iii) The required area		
$=\int_1^{\sqrt{3}}\left(\frac{1}{\sqrt{3}}(4-x)-\sqrt{4-x^2}\right)dx+\int_{\sqrt{3}}^3\left(\frac{1}{\sqrt{3}}(4-x)-\frac{1}{3}\sqrt{12-x^2}\right)dx$	1M+1A	1M for either integral
$=\int_1^3\frac{1}{\sqrt{3}}(4-x)\,dx-\int_1^{\sqrt{3}}\sqrt{4-u^2}\,du-\int_{\sqrt{3}}^3\frac{1}{3}\sqrt{12-v^2}\,dv$		
$=\frac{4\sqrt{3}}{3}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}4\cos^2\alpha\,d\alpha-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{12\cos^2\beta}{3}d\beta$ (by letting $u=2\sin\alpha$ and $v=\sqrt{12}\sin\beta$)	1M	for either substitution
$=\frac{4\sqrt{3}}{3}-8\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos 2\theta+1}{2}d\theta$	1M	
$=\frac{4\sqrt{3}}{3}-8\left[\frac{\sin 2\theta}{4}+\frac{\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$		
$=\frac{4\sqrt{3}}{3}-\frac{2\pi}{3}$	1A	
	----------(9)	

Solution	Marks	Remarks
10. (a) $\dfrac{1}{2+\cos 2x}$		
$=\dfrac{1}{2+2\cos^2 x-1}$		
$=\dfrac{1}{2\cos^2 x+1}$		
$=\dfrac{\sec^2 x}{2+\sec^2 x}$	1	
	----------(1)	
(b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x}\,dx$		
$\displaystyle=\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{2+\sec^2 x}\,dx$ (by (a))	1M	for using (a)
$\displaystyle=\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{3+\tan^2 x}\,dx$		
$\displaystyle=\int_0^{1} \frac{1}{3+t^2}\,dt$ (by letting $t=\tan x$)	1M	
$\displaystyle=\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^1$		
$=\dfrac{\sqrt{3}\pi}{18}$	1A	
	----------(3)	
(c) By letting $u=-x$, we have $\displaystyle\int_{-a}^{0} f(x)\ln(1+e^x)\,dx=\int_a^0 -f(-u)\ln(1+e^{-u})\,du$.	1M	
So, we have $\displaystyle\int_{-a}^{0} f(x)\ln(1+e^x)\,dx=\int_0^a f(-x)\ln(1+e^{-x})\,dx$.		
$\displaystyle\int_{-a}^{a} f(x)\ln(1+e^x)\,dx$		
$\displaystyle=\int_{-a}^{0} f(x)\ln(1+e^x)\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$	1M	
$\displaystyle=\int_0^a f(-x)\ln(1+e^{-x})\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$		
$\displaystyle=-\int_0^a f(x)\ln(1+e^{-x})\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$	1M	
$\displaystyle=-\int_0^a f(x)\ln\left(\frac{e^x+1}{e^x}\right)dx+\int_0^a f(x)\ln(1+e^x)\,dx$		
$\displaystyle=-\int_0^a f(x)\ln(e^x+1)\,dx+\int_0^a x f(x)\,dx+\int_0^a f(x)\ln(1+e^x)\,dx$		
$\displaystyle=\int_0^a x f(x)\,dx$	1	
	----------(4)	

Solution	Marks	Remarks
(d) Note that $\frac{\sin 2(-x)}{(2+\cos 2(-x))^2}=\frac{-\sin 2x}{(2+\cos 2x)^2}$ for all $x \in \mathbf{R}$.	1M	withhold 1M if checking is omitted
By (c), we have $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,dx=\int_0^{\frac{\pi}{4}} \frac{x\sin 2x}{(2+\cos 2x)^2}\,dx$.	1M	for using (c)
Also note that $\frac{d}{dx}\left(\frac{1}{2(2+\cos 2x)}\right)=\frac{\sin 2x}{(2+\cos 2x)^2}$.		
$\int_0^{\frac{\pi}{4}} \frac{x\sin 2x}{(2+\cos 2x)^2}\,dx$		
$=\left[\frac{x}{2(2+\cos 2x)}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1}{2(2+\cos 2x)}\,dx$	1M	
$=\frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{1}{2+0}\right)-\frac{1}{2}\left(\frac{\sqrt{3}\pi}{18}\right)$ (by (b))	1M	for using the result of (b)
$=\frac{(9-4\sqrt{3})\pi}{144}$		
Thus, we have $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,dx=\frac{(9-4\sqrt{3})\pi}{144}$.	1A	
	----------(5)	

Solution	Marks	Remarks
11. (a) M^2		
$=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$		
$=\begin{pmatrix}-3 & -28\\ 4 & 29\end{pmatrix}$	1M	
$M^2=aM+bI$		
$\begin{pmatrix}-3 & -28\\ 4 & 29\end{pmatrix}=\begin{pmatrix}2a & 7a\\ -a & -6a\end{pmatrix}+\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}$		
$\begin{cases}-3=2a+b\\ -28=7a\\ 4=-a\\ 29=-6a+b\end{cases}$	1M	
Thus, we have $a=-4$ and $b=5$.	1A	for both correct
	----------(3)	
(b) Note that $(1-(-5))M+(5+(-5))I=6M$.	1M	
Therefore, the statement is true for $n=1$.		
Assume that $6M^k=(1-(-5)^k)M+(5+(-5)^k)I$, where k is a positive integer.		
$6M^{k+1}$		
$=M(6M^k)$		
$=M((1-(-5)^k)M+(5+(-5)^k)I)$	1M	
$=(1-(-5)^k)M^2+(5+(-5)^k)M$		
$=(1-(-5)^k)((1+(-5))M+5I)+(5+(-5)^k)M$	1M	for using the result of (a)
$=(1+(-5)-(-5)^k-(-5)^{k+1})M+(5+(-5)^{k+1})I+(5+(-5)^k)M$		
$=(1-(-5)^{k+1})M+(5+(-5)^{k+1})I$		
So, the statement is true for $n=k+1$ if it is true for $n=k$.		
By mathematical induction, we have $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$.	1	
$6M$		
$=(1-(-5))M+(5+(-5))I$	1M	
$6M^2$		
$=(1-(-5))M^2$		
$=(1-(-5))((1+(-5))M+5I)$	1M	for using the result of (a)
$=(1-(-5)^2)M+(5+(-5)^2)I$		
$6M^3$		
$=M(6M^2)$		
$=M((1-(-5)^2)M+(5+(-5)^2)I)$	1M	
$=(1-(-5)^2)M^2+(5+(-5)^2)M$		
$=(1-(-5)^2)((1+(-5))M+5I)+(5+(-5)^2)M$		
$=(1+(-5)-(-5)^2-(-5)^3)M+(5+(-5)^3)I+(5+(-5)^2)M$		
$=(1-(-5)^3)M+(5+(-5)^3)I$		
Thus, we have $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$.	1	
	----------(4)	

Solution	Marks	Remarks
(c) By (b), we have $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$ and $6M^{n+1}=(1-(-5)^{n+1})M+(5+(-5)^{n+1})I$.	1M	
$6(1-(-5)^n)M^{n+1}-6(1-(-5)^{n+1})M^n=((1-(-5)^n)(5+(-5)^{n+1})-(1-(-5)^{n+1})(5+(-5)^n))I$	1M	
$M^n\left(\frac{(1-(-5)^n)M-(1-(-5)^{n+1})I}{-6(-5)^n}\right)=\left(\frac{(1-(-5)^n)M-(1-(-5)^{n+1})I}{-6(-5)^n}\right)M^n=I$		
So, we have $(M^n)^{-1}=\frac{(1-(-5)^n)M-(1-(-5)^{n+1})I}{-6(-5)^n}$.	1M	
$(M^n)^{-1}$ $=\frac{(-5)^n-1}{6(-5)^n}M+\frac{-(-5)^{n+1}+1}{6(-5)^n}I$ $=\left(\frac{1}{6}M+\frac{5}{6}I\right)+\frac{1}{(-5)^n}\left(\frac{-1}{6}M+\frac{1}{6}I\right)$	1M	
Letting $A=\frac{1}{6}M+\frac{5}{6}I=\frac{1}{6}\begin{pmatrix}7&7\\-1&-1\end{pmatrix}$ and $B=\frac{-1}{6}M+\frac{1}{6}I=\frac{1}{6}\begin{pmatrix}-1&-7\\1&7\end{pmatrix}$, we have $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$. Thus, there exists a pair of 2×2 real matrices A and B such that $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$ for all positive integers n .	1A	f.t.
$6M^n=(1-(-5)^n)M+(5+(-5)^n)I$ (by (b)) $M^n=\frac{1}{6}(1-(-5)^n)\begin{pmatrix}2&7\\-1&-6\end{pmatrix}+\frac{1}{6}(5+(-5)^n)\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $=\begin{pmatrix}\frac{-(-5)^n+7}{6}&\frac{-7(-5)^n+7}{6}\\\frac{(-5)^n-1}{6}&\frac{7(-5)^n-1}{6}\end{pmatrix}$	1M	
$\det(M^n)=\left(\frac{-(-5)^n+7}{6}\right)\left(\frac{7(-5)^n-1}{6}\right)-\left(\frac{-7(-5)^n+7}{6}\right)\left(\frac{(-5)^n-1}{6}\right)$ $=(-5)^n$	1M	
$(M^n)^{-1}=\frac{1}{\det(M^n)}\begin{pmatrix}\frac{7(-5)^n-1}{6}&\frac{7(-5)^n-7}{6}\\\frac{-(-5)^n+1}{6}&\frac{-(-5)^n+7}{6}\end{pmatrix}$	1M	
$=\begin{pmatrix}\frac{7}{6}&\frac{7}{6}\\\frac{-1}{6}&\frac{-1}{6}\end{pmatrix}+\frac{1}{(-5)^n}\begin{pmatrix}\frac{-1}{6}&\frac{-7}{6}\\\frac{1}{6}&\frac{7}{6}\end{pmatrix}$	1M	
Letting $A=\begin{pmatrix}\frac{7}{6}&\frac{7}{6}\\\frac{-1}{6}&\frac{-1}{6}\end{pmatrix}$ and $B=\begin{pmatrix}\frac{-1}{6}&\frac{-7}{6}\\\frac{1}{6}&\frac{7}{6}\end{pmatrix}$, we have $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$. Thus, there exists a pair of 2×2 real matrices A and B such that $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$ for all positive integers n .	1A	f.t.
	-----------(5)	

Solution	Marks	Remarks
12. (a) $\left\lvert\overrightarrow{AC}\right\rvert=\left\lvert\overrightarrow{BC}\right\rvert$		
$\left\lvert\overrightarrow{OC}-\overrightarrow{OA}\right\rvert=\left\lvert\overrightarrow{OC}-\overrightarrow{OB}\right\rvert$	1M	
$\lvert -6\mathbf{i}-8\mathbf{j}+(t-2)\mathbf{k}\rvert=\lvert -8\mathbf{j}+(t-8)\mathbf{k}\rvert$		
$\sqrt{(-6)^2+(-8)^2+(t-2)^2}=\sqrt{(-8)^2+(t-8)^2}$	1M	
$t^2-4t+104=t^2-16t+128$		
$12t=24$		
$t=2$	1A	
	----------(3)	
(b) $\overrightarrow{AB}\times\overrightarrow{AC}$		
$=(-6\mathbf{i}+6\mathbf{k})\times(-6\mathbf{i}-8\mathbf{j})$		
$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -6 & 0 & 6\\ -6 & -8 & 0\end{vmatrix}$		
$=((0)(0)-(6)(-8))\mathbf{i}+((6)(-6)-(6)(0))\mathbf{j}+((-6)(-8)-(0)(-6))\mathbf{k}$	1M	
$=48\mathbf{i}-36\mathbf{j}+48\mathbf{k}$	1A	
	----------(2)	
(c) The volume of the pyramid $OABC$		
$=\frac{1}{6}\left\lvert\overrightarrow{OA}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\right\rvert$	1M	
$=\frac{1}{6}\lvert(\mathbf{i}-4\mathbf{j}+2\mathbf{k})\cdot(48\mathbf{i}-36\mathbf{j}+48\mathbf{k})\rvert$		
$=\frac{1}{6}\lvert(1)(48)+(-4)(-36)+(2)(48)\rvert$		
$=48$	1A	
The volume of the pyramid $OABC$		
$=\frac{1}{6}\left\lvert\overrightarrow{OA}\cdot(\overrightarrow{OB}\times\overrightarrow{OC})\right\rvert$	1M	
$=\frac{1}{6}\left\lvert\begin{vmatrix}1 & -4 & 2\\ -5 & -4 & 8\\ -5 & -12 & 2\end{vmatrix}\right\rvert$		
$=\frac{1}{6}\lvert(1)(-4)(2)+(-4)(8)(-5)+(2)(-5)(-12)-(1)(8)(-12)-(-4)(-5)(2)-(2)(-4)(-5)\rvert$		
$=48$	1A	
	----------(2)	
(d) (i) By (c), the volume of the pyramid $OABC$ is not equal to 0.		
So, O does not lie on Π.		
Therefore, $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ are non-zero vectors.		
Hence, we have $p\neq 0$, $q\neq 0$ and $r\neq 0$.		
Thus, we have $pqr\neq 0$.	1	

Solution	Marks	Remarks
(ii) $\overrightarrow{OD}$		
$= \overrightarrow{OA} \cdot \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{\lvert \overrightarrow{AB} \times \overrightarrow{AC} \rvert^2} (\overrightarrow{AB} \times \overrightarrow{AC})$	1M	
$= \dfrac{(6)(48)}{48^2 + (-36)^2 + 48^2} (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k})$		
$= \dfrac{288}{5\,904} (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k})$		
$= \dfrac{96}{41}\mathbf{i} - \dfrac{72}{41}\mathbf{j} + \dfrac{96}{41}\mathbf{k}$	1A	
(iii) $\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$	1M	any one
$((p-1)\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}) = 0$		
$48p - 48 - 144 - 96 = 0$		
$p = 6$		
$\overrightarrow{AQ} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$		
$(-\mathbf{i} + (q+4)\mathbf{j} - 2\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}) = 0$		
$-48 - 36q - 144 - 96 = 0$		
$q = -8$		
$\overrightarrow{AR} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$		
$(-\mathbf{i} + 4\mathbf{j} + (r-2)\mathbf{k}) \cdot (48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}) = 0$		
$-48 - 144 + 48r - 96 = 0$		
$r = 6$	1A	for all
So, we have $\overrightarrow{OE} = \dfrac{1}{6}\mathbf{i} - \dfrac{1}{8}\mathbf{j} + \dfrac{1}{6}\mathbf{k}$.		
By (b)(ii), we have $\overrightarrow{OE} = \dfrac{41}{576}\overrightarrow{OD}$.		
Thus, D , E and O are collinear.	1A	f.t.
	-----------(6)	