CONTENT

1. FOUNDATION KNOWLEDGE AREA

1. Mathematical Induction

(1981-HL-GEN MATHS #02) (6 marks)

2. (a) Prove, by mathematical induction, that for any positive integer n ,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

(1988-HL-GEN MATHS #07) (8 marks) (Modified)

7. (b) Let $A_n = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1}n^2$

and $B_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

where n is a positive integer.

Show, by mathematical induction, that $A_n = (-1)^{n-1}B_n$ for all positive integers n .

Hence, or otherwise, find $\sum_{n=1}^{2m} A_n$ and $\sum_{n=1}^{2m+1} A_n$.

(1990-HL-GEN MATHS #05) (8 marks)

5. (a) (i) Prove by mathematical induction that for any positive integer n ,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

(ii) Find $1^3 - 2^3 + 3^3 - 4^3 + \ldots + (-1)^{r+1}r^3 + \ldots - (2n)^3$.

(1991-CE-A MATH 2 #07) (8 marks)

7. (a) Prove, by mathematical induction, that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all positive integers n .

(b) Using the formula in (a), find the sum of

$$1 \times 2 + 2 \times 3 + \ldots + n(n+1).$$

(1992-CE-A MATH 2 #01) (5 marks)

1. Prove, by mathematical induction, that

$$1 \times 2 + 2 \times 5 + 3 \times 8 + \ldots + n(3n-1) = n^2(n+1)$$

for all positive integers n .

(1993-CE-A MATH 2 #01) (5 marks)

1. Prove that

$$1^2 \times 2 + 2^2 \times 3 + \ldots + n^2(n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

for any positive integer n .

(1994-CE-A MATH 2 #05) (5 marks)

5. Prove that

$$\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}=3-\frac{2n+3}{2^n}$$

for any positive integer n .

(1997-CE-A MATH 2 #07) (6 marks)

7. Let $T_n=(n^2+1)(n!)$ for any positive integer n . Prove, by mathematical induction, that

$$T_1+T_2+\ldots+T_n=n\left[(n+1)!\right]$$

for any positive integer n .

(Note: $n!=n(n-1)(n-2)\ldots3\times2\times1$)

(1998-CE-A MATH 2 #03) (5 marks)

3. Prove, by mathematical induction, that

$$1\times2+2\times3+2^2\times4+\ldots+2^{n-1}(n+1)=2^n(n)$$

for all positive integers n .

(2000-CE-A MATH 2 #04) (6 marks)

4. Prove, by mathematical induction, that

$$1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$$

for all positive integers n .

(2001-CE-A MATH #12) (8 marks)

12. Prove, by mathematical induction, that

$$1\times2+2\times3+3\times4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$$

for all positive integers n .

Hence evaluate $1\times3+2\times4+3\times5+\ldots+50\times52$.

(2002-CE-A MATH #12) (8 marks)

12. (a) Prove, by mathematical induction, that

$$2(2)+3(2^2)+4(2^3)+\ldots+(n+1)(2^n)=n(2^{n+1})$$

for all positive integers n .

(b) Show that

$$1(2)+2(2^2)+3(2^3)+\ldots+98(2^{98})=97(2^{99})+2\ .$$

(2003-CE-A MATH #07) (5 marks)

7. Prove, by mathematical induction, that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$$

for all positive integers n .

(2005-CE-A MATH #08) (5 marks)

8. Prove, by mathematical induction, that

$$\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+\frac{n\times 2^n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1$$

for all positive integers n .

(2007-CE-A MATH #05) (5 marks)

5. Let $a\neq 0$ and $a\neq 1$. Prove by mathematical induction that

$$\frac{1}{a-1}-\frac{1}{a}-\frac{1}{a^2}-\ldots-\frac{1}{a^n}=\frac{1}{a^n(a-1)}$$

for all positive integers n .

(2008-CE-A MATH #05) (5 marks)

5. Prove, by mathematical induction, that

$$1^3+2^3+3^3+\ldots+n^3=\frac{1}{4}n^2(n+1)^2$$

for all positive integers n .

(2009-CE-A MATH #05) (5 marks)

5. Prove, by mathematical induction, that

$$1\times 4+2\times 5+3\times 6+\ldots+n(n+3)=\frac{1}{3}n(n+1)(n+5)$$

for all positive integers n .

(2012-DSE-MATH-EP(M2) #03) (5 marks)

3. Prove, by mathematical induction, that for all positive integer n ,

$$1\times 2+2\times 5+3\times 8+\ldots+n(3n-1)=n^2(n+1).$$

(2013-DSE-MATH-EP(M2) #03) (5 marks)

3. Prove, by mathematical induction, that for all positive integers n ,

$$1+\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\ldots+\frac{1}{(3n-2)\times(3n+1)}=\frac{4n+1}{3n+1}.$$

(2016-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}(-1)^k k^2=\frac{(-1)^n n(n+1)}{2}$ for all positive integers n .

(b) Using (a), evaluate $\sum_{k=3}^{333}(-1)^{k+1}k^2$.

(2018-DSE-MATH-EP(M2) #06) (7 marks)

6. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} k(k+4) = \frac{n(n+1)(2n+13)}{6}$ for all positive integers n .

(b) Using (a), evaluate $\sum_{k=333}^{555} \left(\frac{k}{112}\right)\left(\frac{k+4}{223}\right)$.

(2019-DSE-MATH-EP(M2) #05) (7 marks)

5. (a) Using mathematical induction, prove that $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$ for all positive integers n .

(b) Using (a), evaluate $\sum_{k=50}^{200} \frac{1}{k(k+1)}$.

(2020-DSE-MATH-EP(M2) #05) (7 marks)

5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all positive integers n .

(b) Using (a), evaluate $\sum_{k=4}^{123} \frac{50}{k(k+1)(k+2)}$.

(2021-DSE-MATH-EP(M2) #02) (5 marks)

2. Using mathematical induction, prove that $\sum_{k=1}^{n} \left(3k^5 + k^3\right) = \frac{n^3(n+1)^3}{2}$ for all positive integers n .

ANSWERS

(1991-CE-A MATH 2 #07)

7. (b) $\frac{1}{3}n(n+1)(n+2)$

(2001-CE-A MATH #12)

12. 45 475

(2016-DSE-MATH-EP(M2) #05)

5. (b) 55 614

(2018-DSE-MATH-EP(M2) #06)

6. (b) 1 813

(2019-DSE-MATH-EP(M2) #05)

5. (b) $\frac{151}{10050}$

(2020-DSE-MATH-EP(M2) #05)

5. (b) $\frac{387}{310}$

2. The Binomial Theorem

(1992-CE-A MATH 2 #02) (5 marks)

2. In the expansion of $(1+3x)^2(1+x)^n$, where n is a positive integer, the coefficient of x is 10 .

   (a) Find the value of n .

   (b) Find the coefficient of x^2 .

(1994-CE-A MATH 2 #03) (5 marks)

3. (a) Expand $(1-2x)^3$ and $\left(1+\frac{1}{x}\right)^5$.

   (b) Find, in the expansion of $(1-2x)^3\left(1+\frac{1}{x}\right)^5$,

   (1) the constant term, and
   (2) the coefficient of x .

(1995-CE-A MATH 2 #04) (6 marks)

4. Given $\left(x^2+\frac{1}{x}\right)^5-\left(x^2-\frac{1}{x}\right)^5=ax^7+bx+\frac{c}{x^5}$, find the values of a , b and c .

   Hence evaluate $\left(2+\frac{1}{\sqrt{2}}\right)^5-\left(2-\frac{1}{\sqrt{2}}\right)^5$.

(1997-CE-A MATH 2 #08) (7 marks)

8. Expand $(1+x)^n(1-2x)^4$ is ascending powers of x up to the term x^2 , where n is a positive integer. If the coefficient of x^2 is 54 , find the coefficient of x .

(1998-CE-A MATH 2 #01) (4 marks)

1. Find the coefficient of x^2 in the expansion of $\left(x-\frac{2}{x}\right)^6$.

(1999-CE-A MATH 2 #07) (6 marks)

7. (a) Expand $(1+2x)^n$ in ascending powers of x up to the term x^3 , where n is a positive integer.

   (b) In the expansion of $\left(x-\frac{3}{x}\right)^2(1+2x)^n$, the constant term is 210 . Find the value of n .

(2000-CE-A MATH 2 #02) (5 marks)

2. Expand $(1+2x)^7(2-x)^2$ in ascending powers of x up to the term x^2 .

(2001-CE-A MATH #04) (4 marks)

4. Find the constant term in the expansion of $\left(2x^3+\frac{1}{x}\right)^8$.

(2002-CE-A MATH #01) (4 marks)

1. If n is a positive integer and the coefficient of x^2 in the expansion of $(1+x)^n+(1+2x)^n$ is 75 , find the value(s) of n .

(2003-CE-A MATH #12) (6 marks)

12. Determine whether the expansion of $\left(2x^2+\frac{1}{x}\right)^9$ consists of

(a) a constant term,

(b) an x^2 term.

In each part, find the term if it exists.

(2004-CE-A MATH #02) (4 marks)

2. (a) Expand $(1+2x)^6$ in ascending powers of x up to the term x^3 .

(b) Find the constant term in the expansion of $\left(1-\frac{1}{x}+\frac{1}{x^2}\right)(1+2x)^6$.

(2005-CE-A MATH #02) (4 marks)

2. (a) Expand $(1+y)^5$.

(b) Using (a), or otherwise, expand $(1+x+2x^2)^5$ in ascending powers of x up to the term x^2 .

(2008-CE-A MATH #02) (4 marks)

2. (a) Expand $\left(2x+\frac{1}{x}\right)^3$.

(b) Find the coefficient of x in the expansion of $\left(3x^2-x-5\right)\left(2x+\frac{1}{x}\right)^3$.

(2009-CE-A MATH #11) (6 marks)

11. In the expansion of the binomial $\left(x^2+\frac{1}{x}\right)^{20}$, find

(a) the coefficient of x^{16} ,

(b) the constant term.

(2010-CE-A MATH #05) (5 marks)

5. The sum of the coefficients of x and x^2 in the expansion of $(1+4x)^n$ is 180, where n is a positive integer. Find the value of n and the coefficient of x^3.

(PP-DSE-MATH-EP(M2) #01) (4 marks)

1. Find the coefficient of x^5 in the expansion of $(2-x)^9$.

(2012-DSE-MATH-EP(M2) #02) (5 marks)

2. It is given that

$$(1+ax)^n = 1+6x+16x^2 + \text{terms involving higher powers of } x\ ,$$

where n is a positive integer and a is a constant. Find the values of a and n.

(2013-DSE-MATH-EP(M2) #02) (4 marks)

2. Suppose the coefficients of x and x^2 in the expansion of $(1+ax)^n$ are -20 and 180 respectively. Find the values of a and n.

(2014-DSE-MATH-EP(M2) #01) (4 marks)

1. In the expansion of $(1-4x)^2(1+x)^n$, the coefficient of x is 1.

   (a) Find the value of n.

   (b) Find the coefficient of x^2.

(2016-DSE-MATH-EP(M2) #01) (5 marks)

1. Expand $(5+x)^4$. Hence, find the constant term in the expansion of $(5+x)^4\left(1-\dfrac{2}{x}\right)^3$.

(2017-DSE-MATH-EP(M2) #02) (5 marks)

2. Let $(1+ax)^8 = \sum_{k=0}^{8} \lambda_k x^k$ and $(b+x)^9 = \sum_{k=0}^{9} \mu_k x^k$, where a and b are constants. It is given that $\lambda_2 : \mu_7 = 7:4$ and $\lambda_1 + \mu_8 + 6 = 0$. Find a.

(2018-DSE-MATH-EP(M2) #02) (5 marks)

2. Expand $(x+3)^5$. Hence, find the coefficient of x^3 in the expansion of $(x+3)^5\left(x-\dfrac{4}{x}\right)^2$.

(2020-DSE-MATH-EP(M2) #01) (4 marks)

1. (a) Expand $(1-x)^4$.

   (b) Find the constant k such that the coefficient of x^2 in the expansion of $(1+kx)^9(1-x)^4$ is -3.

(2021-DSE-MATH-EP(M2) #03) (6 marks)

1. The coefficient of x^2 in the expansion of $(1-4x)^n$ is 240, where n is a positive integer. Find

   (a) n ,

   (b) the coefficient of x^4 in the expansion of $(1-4x)^n\left(1+\dfrac{2}{x}\right)^5$.

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(1992-CE-A MATH 2 #02)

2. (a) $n = 4$

(b) 39

(1994-CE-A MATH 2 #03)

3. (a) $(1-2x)^3 = 1 - 6x + 12x^2 - 8x^3$

$$\left(1+\frac{1}{x}\right)^5 = 1 + \frac{5}{x} + \frac{10}{x^2} + \frac{10}{x^3} + \frac{5}{x^4} + \frac{1}{x^5}$$

(b) (1) 11

(2) -26

(1995-CE-A MATH 2 #04)

4. $a = 10$, $b = 20$, $c = 2$

$$\left(2+\frac{1}{\sqrt{2}}\right)^5 - \left(2-\frac{1}{\sqrt{2}}\right)^5 = \frac{401\sqrt{2}}{4}$$

(1997-CE-A MATH 2 #08)

8. $(1+x)^n(1-2x)^4$

$$= 1 + (n-8)x + \left[\frac{n(n-1)}{2} - 8n + 24\right]x^2 + \ldots$$

Coefficient of $x = 12$

(1998-CE-A MATH 2 #01)

1. 60

(1999-CE-A MATH 2 #07)

7. (a) $(1+2x)^n$

$$= 1 + 2nx + 2n(n-1)x^2 + \frac{4}{3}n(n-1)(n-2)x^3 + \ldots$$

(b) $n = 4$

(2000-CE-A MATH 2 #02)

2. $4 + 52x + 281x^2 + \ldots$

(2001-CE-A MATH #04)

4. 112

(2002-CE-A MATH #01)

1. $n = 6$

(2003-CE-A MATH #12)

12. (a) 672

(b) There is no x^2 term

(2004-CE-A MATH #02)

2. (a) $(1+2x)^6$

$= 1 + 12x + 60x^2 + 160x^3 + \ldots$

(b) 49

(2005-CE-A MATH #02)

2. (a) $(1+y)^5$

$= 1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5$

(b) $(1 + x + 2x^2)^5$

$= 1 + 5x + 20x^2 + \ldots$

(2008-CE-A MATH #02)

2. (a) $\left(2x+\frac{1}{x}\right)^3 = 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$

(b) -42

(2009-CE-A MATH #11)

11. (a) 125 970

(b) There is no constant term

(2010-CE-A MATH #05)

5. $n = 5$, the coefficient of $x^3 = 640$

(PP-DSE-MATH-EP(M2) #01)

1. -2016

(2012-DSE-MATH-EP(M2) #02)

2. $n = 9$, $a = \frac{2}{3}$

(2013-DSE-MATH-EP(M2) #02)

2. $n = 10$, $a = -2$

(2014-DSE-MATH-EP(M2) #01)

1. (a) $n = 9$

(b) -20

(2016-DSE-MATH-EP(M2) #01)

1. $(5+x)^4 = 625 + 500x + 150x^2 + 20x^3 + x^4$

   Constant term $= -735$

(2017-DSE-MATH-EP(M2) #02)

2. $a = -3$ or $\frac{-3}{7}$

(2018-DSE-MATH-EP(M2) #02)

2. $(x+3)^5$

   $= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$

   Coefficient of $x^3 = -299$

(2020-DSE-MATH-EP(M2) #01)

1. (a) $1 - 4x + 6x^2 - 4x^3 + x^4$

   (b) $\frac{1}{2}$

(2021-DSE-MATH-EP(M2) #03)

3. (a) 6

   (b) 106 240

OUT-OF-SYLLABUS

(1991-CE-A MATH 2 #01) (5 marks)

1. Given that $(1+x+ax^2)^8 = 1+8x+k_1x^2+k_2x^3+$ terms involving higher powers of x .

   (a) Express k_1 and k_2 in terms of a .

   (b) If $k_1 = 4$, find the value of a .
   Hence find the value of k_2 .

(1993-CE-A MATH 2 #03) (6 marks)

3. Given $(1+4x+x^2)^n = 1+ax+bx^2+$ other terms involving higher powers of x , where n is a positive integer.

   (a) Express a and b in terms of n .

   (b) If $a = 20$, find n and b .

(1996-CE-A MATH 2 #02) (6 marks)

2. It is given that $(1+x+ax^2)^6 = 1+6x+k_1x^2+k_2x^3+$ terms involving higher powers of x .

   (a) Express k_1 and k_2 in terms of a .

   (b) If 6 , k_1 and k_2 form an arithmetic sequence, find the value of a .

(2006-CE-A MATH #03) (5 marks)

3. It is given that

   $$(1-2x+3x^2)^n = 1-10x+kx^2+ \text{ terms involving higher powers of } x ,$$

   where n is a positive integer and k is a constant. Find the values of n and k .

(2007-CE-A MATH #12) (6 marks)

12. If the coefficient of x^2 in the expansion of $\left(1-2x+x^2\right)^n$ is 66 , find the value of n and the coefficient of x^3 .

(2011-CE-A MATH #01) (5 marks)

1. It is given that $(1+x+kx^2)^3 = 1+ax+bx^2+$ terms involving higher powers of x .

   (a) Express b in terms of k .

   (b) If 1 , a , b form a geometric sequence, find the value of k .

(1991-CE-A MATH 2 #01) (5 marks)

1. (a) $k_1 = 8a+28$, $k_2 = 56a+56$

   (b) $a = -3$, $k_2 = -112$

(1993-CE-A MATH 2 #03) (6 marks)

3. (a) $a = 4n$, $b = 8n^2 - 7n$

(b) $n = 5$, $b = 165$

(1996-CE-A MATH 2 #02) (6 marks)

2. (a) $k_1 = 6a + 15$, $k_2 = 30a + 20$

(b) $a = \frac{2}{9}$

(2006-CE-A MATH #03) (5 marks)

3. $n = 5$, $k = 55$

(2007-CE-A MATH #12) (6 marks)

12. $n = 6$, The coefficient of $x^3 = -220$

(2011-CE-A MATH #01) (5 marks)

1. (a) $b = 3(k + 1)$

(b) $k = 2$

3. (a) Trigonometric Functions (Part 1)

(1980-CE-A MATH 1 #05) (6 marks)

5. Given that $\dfrac{\sin^2 A}{1+2\cos^2 A} = \dfrac{3}{19}$, where $\dfrac{\pi}{2} < A < \pi$, find the value of $\dfrac{\sin A}{1+2\cos A}$.

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(1980-CE-A MATH 1 #05) (6 marks)

5. $\dfrac{\sin A}{1+2\cos A}=-1$

3. (b) Trigonometric Functions (Part 2)

(1979-CE-A MATH 1 #05) (6 marks)

5. The quadratic equation in x

$$(2\sin A)x^2 - 2x - \cos A = 0$$

has two equal roots. Find A , where $0° \le A < 360°$.

(1980-CE-A MATH 1 #09) (Modified) (20 marks)

9. (a) Show that $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$.

(b) Let $f(x) = 3x^3 + mx^2 - 9x + n$, where m and n are integers. When $f(x)$ is divided by $x-1$, the remainder is -8 . When $f(x)$ is divided by $x-2$, the remainder is -5 .

(i) Show that $m = -3$ and $n = 1$.

(ii) By putting $x = \tan\theta$ and using the result in (a), or otherwise, solve the equation

$$f(x) = 0 \ .$$

(Correct your answer to 2 decimal places.)

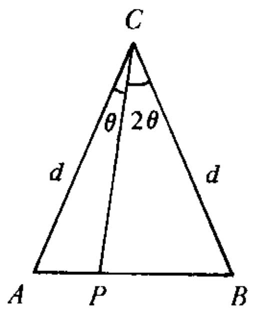
(1980-HL-GEN MATHS #03) (16 marks)

3. (a) Prove, by mathematical induction, or otherwise, that for any positive integer n ,

$$\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha = \frac{\sin\left(\frac{n+1}{2}\alpha\right)\sin\frac{n}{2}\alpha}{\sin\frac{\alpha}{2}},$$

where $\alpha \ne 2m\pi$ for any integer m .

(b) (Requires knowledge of Sine Law in the Compulsory Part)



In Figure 1, ΔABC is an isosceles triangle with $AC = BC = d$ and $AB = 1$. P is a point on AB such that $\angle ACP = \theta$ and $\angle BCP = 2\theta$. Using the sine law, show that

$$AP = \frac{1}{1 + 2\cos\theta} \ .$$

Hence, or otherwise, deduce that $\dfrac{1}{3} < AP < \dfrac{1}{2}$.

(1982-HL-GEN MATHS #04) (Modified) (16 marks)

4. (a) In ΔABC , $\angle BAC = 90°$, $BC = \ell$ and $\angle ACB = \theta$. D is a point on BC such that $AD \perp BC$ (see Figure 1).

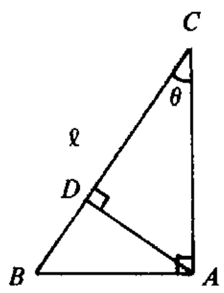


Figure 1

(i) Find the area of ΔABC in terms of ℓ and θ .

(ii) Find $\dfrac{\text{Area of } \Delta ACD}{\text{Area of } \Delta ABC}$ in terms of $\cos\theta$.

(iii) When $\dfrac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ACD} = \dfrac{1}{3}$, find the value of θ .

(b) Let $f(\theta) = 8\sin^4\dfrac{\theta}{2} + \cos 2\theta + 8\cos\theta$.

(i) Show that $f(\theta) = 4\left(\cos\theta + \dfrac{1}{2}\right)^2$.

(ii) For $0 \le \theta \le 2\pi$,

(1) find the maximum value of $f(\theta)$ and the value(s) of θ when $f(\theta)$ attains its maximum value,

(2) find the minimum value of $f(\theta)$ and the value(s) of θ when $f(\theta)$ attains its minimum value.

(1983-HL-GEN MATHS #04) (Modified) (16 marks)

4. (a) Let $f(x) = \dfrac{2(\sin^4 x - \cos^4 x - 2)}{4\cos^2 x + 5}$.

(i) Show that $f(x) = \dfrac{3}{4\cos^2 x + 5} - 1$.

(ii) Find the maximum and minimum values of $f(x)$.

(b) Solve

$$\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 0$$

for $0 \le \theta \le 2\pi$.

(1983-CE-A MATH 2 #07) (7 marks)

7. Show that $\sin^2 n\theta - \sin^2 m\theta = \sin(n+m)\theta \sin(n-m)\theta$.

Hence, or otherwise, solve the equation $\sin^2 3\theta - \sin^2 2\theta - \sin\theta = 0$ for $0 \le \theta \le \pi$.

(1984-HL GEN MATHS #05) (Modified) (16 marks)

5. (a) Express $\cot 4\theta$ in terms of $\cot\theta$.

Hence solve the equation $x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$.

(Give your answers in terms of π.)

(b) (i) If $\cos\theta - \cos\phi = a$ and $\sin\theta - \sin\phi = b$ ($b \neq 0$),

show that

$$\frac{1}{2}(2 - a^2 - b^2) = \cos(\theta - \phi) \text{ and } \frac{-a}{b} = \tan\frac{\theta + \phi}{2} .$$

(ii) Solve the system of equations

$$\begin{cases} \cos\theta - \cos\phi = 1 \\ \sin\theta - \sin\phi = \sqrt{3} \end{cases}$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$.

(1985-HL GEN MATHS #05) (8 marks)

5. (b) (i) Show that

$$(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2 = 1 - (\sin 2B + \sin 2C - \sin 2A).$$

(ii) If $A + B + C = 2\pi$, deduce, from (b) (i), that

$$(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2 = 1 - 4\sin A\cos B\cos C.$$

Furthermore, if $A = \frac{\pi}{2}$, find the greatest value of $\cos B\cos C$.

(1986-CE-A MATH 2 #05) (Modified) (16 marks) (Requires knowledge of Sine Law in the Compulsory Part)

5. In ΔABC , the lengths of the sides a , b and c form an arithmetic sequence, i.e.

$b - a = c - b$ where a is the length of the shortest side.

The difference between the greatest angle A and the smallest angle C is 90° .

(a) (i) Using the sine law, or otherwise, show that

$$\sin B = \frac{1}{2}(\sin A + \sin C).$$

(ii) Using the relation $\angle A - \angle C = 90^\circ$, show that

$$\sin A + \sin C = \sqrt{2}\cos\frac{B}{2} .$$

(iii) Hence deduce that $\sin\frac{B}{2} = \frac{\sqrt{2}}{4}$ and $\sin B = \frac{\sqrt{7}}{4}$.

(b) Show that $\angle B = 90^\circ - 2\angle C$.

Hence deduce that $\sin C = \frac{\sqrt{7} - 1}{4}$.

(c) Show that the lengths of the sides of ΔABC are in the ratios

$\sqrt{7} + 1 : \sqrt{7} : \sqrt{7} - 1$.

(1987-HL-GEN MATHS #05) (Modified) (16 marks)

5. (a) Let ΔABC be an acute-angled triangle.

(i) Show that $\cos^2 A + \cos^2 B = \frac{1}{2}(\cos 2A + \cos 2B) + 1$.

(ii) Show that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.

(b) (i) Prove, by mathematical induction, that for any positive integers n ,

$$\cos\phi + \cos 3\phi + \cos 5\phi + \ldots + \cos(2n-1)\phi = \frac{\sin 2n\phi}{2\sin\phi} ,$$

where ϕ is not a multiple of π .

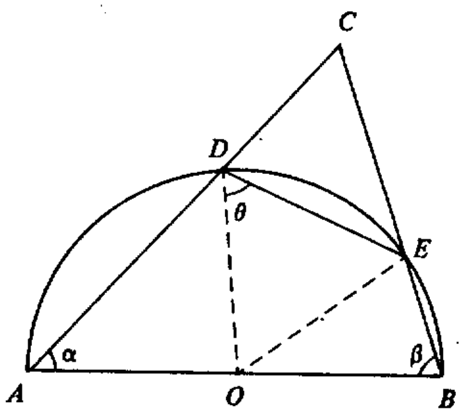
(1987-CE-A MATH 2 #06) (6 marks)

6. Express $\sin 3\theta$ in terms of $\sin\theta$. Hence find the three roots of the equation $8x^3 - 6x + 1 = 0$ to 2 significant figures.

(1988-HL-GEN MATHS #05) (16 marks)

5.

Figure 1



In Figure 1, $ADEB$ is a semi-circle with diameter AB and centre O . BE and AD are produced to meet at C . $AB = c$, $AC = b$, $BC = a$, $\angle A = \alpha$, $\angle B = \beta$ and $\angle ODE = \theta$.

(a) (i) By considering ΔDOE , find DE in terms of c and θ .

(ii) Show that $\angle C = \theta$ and ΔEDC is similar to ΔABC .

Hence express CD and CE in terms of a , b and θ .

(b) (i) Show that the area of $\Delta CED = \frac{1}{2}ab\cos^2\theta\sin\theta$

and hence the area of the quadrilateral $ADEB = \frac{1}{2}ab\sin^3\theta$.

(ii) If the area of ΔCED : the area of quadrilateral $ADEB$ = 1 : 3, find θ .

Suppose further that $ab = c^2$, show that ΔABC is equilateral.

(1988-CE-A MATH 2 #07) (7 marks)

7. (a) Without using calculators, show that $\frac{\pi}{10}$ is a root of $\cos 3\theta = \sin 2\theta$.

(b) Given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, find the value of $\sin\frac{\pi}{10}$, expressing the answer in surd form.

(1989-HL-GEN MATHS #05) (Modified) (16 marks)

5. (a) Find the solution of $\sin x - \sin 2x + \sin 3x = 0$ for $0 < \theta < 2\pi$.

(b) Let $f(\theta) = \sin 2\theta + \sin\theta + \cos\theta$.

(i) Express $f(\theta)$ in terms of p , where $p = \sin\theta + \cos\theta$.

(ii) Using (i) and the method of completing the square, find the smallest value of $f(\theta)$. For $0 < \theta < \pi$, find also the value of θ such that $f(\theta)$ attains its smallest value.

(1989-CE-A MATH 2 #05) (5 marks)

5. Let $y = 5\sin\theta - 12\cos\theta + 7$.

(a) Express y in the form $r\sin(\theta - \alpha) + p$, where r , α and p are constants and $0^\circ \le \alpha \le 90^\circ$.

(b) Using the result in (a), find the least value of y .

(1990-CE-A MATH 2 #06) (5 marks)

6. (a) If $\cos\theta + \sqrt{3}\sin\theta = r\cos(\theta - \alpha)$, where $r > 0$ and $0^\circ \le \alpha \le 90^\circ$, find r and α .

(b) Let $x = \dfrac{1}{\cos\theta + \sqrt{3}\sin\theta + 5}$, find the range of values of x .

(1992-CE-A MATH 2 #12) (16 marks) (Requires knowledge of Cosine Law in the Compulsory Part)

12. (a) Using the identity $2\cos x\sin y = \sin(x+y) - \sin(x-y)$, show that

$$2\left[\cos\theta + \cos(\theta+2\alpha) + \cos(\theta+4\alpha) + \cos(\theta+6\alpha) + \cos(\theta+8\alpha)\right]\sin\alpha = \sin(\theta+9\alpha) - \sin(\theta-\alpha).$$

Hence show that

$$\cos\theta + \cos\left(\theta+\frac{2\pi}{5}\right) + \cos\left(\theta+\frac{4\pi}{5}\right) + \cos\left(\theta+\frac{6\pi}{5}\right) + \cos\left(\theta+\frac{8\pi}{5}\right) = 0.$$

(b)

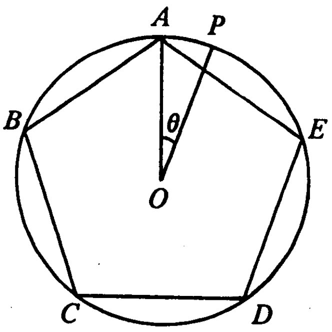


Figure 4

A , B , C , D and E are the vertices of a regular pentagon inscribed in a circle of radius r and centred at O . P is a point on the circumference of the circle such that $\angle POA = \theta$, as shown in Figure 4.

(i) By considering ΔOPD , show that

$$PD^2 = 2r^2 - 2r^2\cos\left(\theta+\frac{6\pi}{5}\right).$$

(ii) Show that $PA^2 + PB^2 + PC^2 + PD^2 + PE^2 = 10r^2$.

(iii) QP is a line perpendicular to the plane of the circle such that $QP = 2r$.
Find $QA^2 + QB^2 + QC^2 + QD^2 + QE^2$.

(1997-CE-A MATH 2 #01) (4 marks)

1. Show that $\dfrac{\sin 3\theta}{\sin\theta} + \dfrac{\cos 3\theta}{\cos\theta} = 4\cos 2\theta$.

(2002-CE-A MATH #08) (5 marks)

8. Given $0 < x < \dfrac{\pi}{2}$. Show that $\dfrac{\tan x - \sin^2 x}{\tan x + \sin^2 x} = \dfrac{4}{2+\sin 2x} - 1$.

Hence, or otherwise, find the least value of $\dfrac{\tan x - \sin^2 x}{\tan x + \sin^2 x}$.

(2003-CE-A MATH #10) (5 marks)

10. Given two acute angles α and β . Show that $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\left(\dfrac{\alpha+\beta}{2}\right)$.

If $3\sin\alpha - 4\cos\alpha = 4\cos\beta - 3\sin\beta$, find the value of $\tan(\alpha+\beta)$.

(2006-CE-A MATH #02) (3 marks)

2. Prove the identity $\cos^2 x - \cos^2 y = -\sin(x+y)\sin(x-y)$.

(2008-CE-A MATH #03) (4 marks)

3. Find the value of $\tan 22.5°$ in surd form.

(2008-CE-A MATH #09) (5 marks)

9. (a) Express $\sin x + \sqrt{3}\cos x$ in the form $r\sin(x + \alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(b) Using (a), find the least and the greatest values of $\sin x + \sqrt{3}\cos x$ for $0° \leq x \leq 90°$.

(2011-CE-A MATH #07) (6 marks)

7. Solve $\sin 5x + \sin x = \cos 2x$ for $0° \leq x \leq 90°$.

(SP-DSE-MATH-EP(M2) #05) (4 marks)

5. By considering $\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$, find the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$.

(PP-DSE-MATH-EP(M2) #04) (5 marks)

4. (a) Let $x = \tan\theta$, show that $\frac{2x}{1+x^2} = \sin 2\theta$.

(b) Using (a), find the greatest value of $\frac{(1+x)^2}{1+x^2}$, where x is real.

(2012-DSE-MATH-EP(M2) #10) (6 marks) (Requires knowledge of Sine Law)

10.

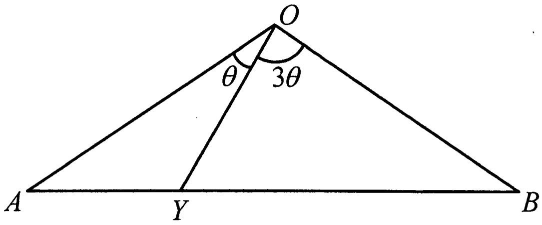


Figure 5

In Figure 5, OAB is an isosceles triangle with $OA = OB$, $AB = 1$, $AY = y$, $\angle AOY = \theta$ and $\angle BOY = 3\theta$.

(a) Show that $y = \frac{1}{4}\sec^2\theta$.

(b) Find the range of values of y .

(Hint: you may use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)

(2013-DSE-MATH-EP(M2) #07) (5 marks)

7. (a) Prove the identity $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$.

(b) Using (a), prove the identity $\tan y = \dfrac{\sin 8y \cos 4y \cos 2y}{(1+\cos 8y)(1+\cos 4y)(1+\cos 2y)}$.

(2015-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) Prove that $\sin^2 x \cos^2 x = \dfrac{1-\cos 4x}{8}$.

(b) Let $f(x) = \sin^4 x + \cos^4 x$.

(i) Express $f(x)$ in the form $A\cos Bx + C$, where A , B and C are constants.

(ii) Solve the equation $8f(x) = 7$, where $0 \le x \le \dfrac{\pi}{2}$.

(2015-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) Using mathematical induction, prove that $\sin\dfrac{x}{2}\sum_{k=1}^{n}\cos kx = \sin\dfrac{nx}{2}\cos\dfrac{(n+1)x}{2}$ for all positive integers n .

(b) Using (a), evaluate $\sum_{k=1}^{567}\cos\dfrac{k\pi}{7}$.

(2016-DSE-MATH-EP(M2) #06) (6 marks)

6. (a) Prove that $x+1$ is a factor of $4x^3+2x^2-3x-1$.

(b) Express $\cos 3\theta$ in terms of $\cos\theta$.

(c) Using the results of (a) and (b), prove that $\cos\dfrac{3\pi}{5} = \dfrac{1-\sqrt{5}}{4}$.

(2017-DSE-MATH-EP(M2) #07) (8 marks)

7. (a) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

(b) Let $\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$.

(i) Prove that $\dfrac{\sin 3\left(x-\dfrac{\pi}{4}\right)}{\sin\left(x-\dfrac{\pi}{4}\right)} = \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$.

(ii) Solve the equation $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$.

(2018-DSE-MATH-EP(M2) #03) (5 marks)

3. (a) If $\cot A = 3\cot B$, prove that $\sin(A+B) = 2\sin(B-A)$.

   (b) Using (a), solve the equation $\cot\left(x+\frac{4\pi}{9}\right) = 3\cot\left(x+\frac{5\pi}{18}\right)$, where $0 \le x \le \frac{\pi}{2}$.

(2020-DSE-MATH-EP(M2) #03) (6 marks)

3. (a) Let x be an angle which is not a multiple of $30°$. Prove that

   (i) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$,

   (ii) $\tan x\ \tan(60° - x)\ \tan(60° + x) = \tan 3x$.

   (b) Using (a) (ii), prove that $\tan 55°\ \tan 65°\ \tan 75° = \tan 85°$.

(2021-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) Prove that $\cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x - 1$.

   (b) Solve the equation $\cos 4\theta + \cos 8\theta + \cos 12\theta = -1$, where $0 \le \theta \le \frac{\pi}{2}$.

ANSWERS

(1979-CE-A MATH 1 #05)

5. $A = 135°$ or $315°$

(1980-CE-A MATH 1 #09)

9. (b) (ii) $x = 0.11, 2.26, -1.37$

(1982-HL-GEN MATHS #04) (Modified)

4. (a) (i) $\frac{1}{2}\ell^2 \sin\theta \cos\theta$

(ii) $\cos^2\theta$

(iii) $\frac{\pi}{6}$

(b) (ii) (1) Maximum value = 9

$\theta = 0, 2\pi$

(1) Minimum value = 0

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

(1983-HL-GEN MATHS #04) (Modified)

4. (a) (ii) Maximum value $= \frac{-2}{5}$

Minimum value $= \frac{-2}{3}$

(b) $x = \frac{\pi}{6}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{6\pi}{5}, \frac{3\pi}{2}, \frac{8\pi}{5}, \frac{11\pi}{6}$

(1983-CE-A MATH 2 #07)

7. $\theta = 0, \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}$ or π

(1984-HL GEN MATHS #05)

5. (a) $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$

$x = \cot\frac{\pi}{16}, \cot\frac{5\pi}{16}, \cot\frac{9\pi}{16}, \cot\frac{13\pi}{16}$

(b) (ii) $\theta = \frac{\pi}{3}, \phi = \frac{4\pi}{3}$

(1985-HL GEN MATHS #05)

5. (b) (ii) $\frac{1}{2}$

(1987-CE-A MATH 2 #06)

6. $x = 0.17\ ,\ 0.77\ ,\ -0.94$

(1988-HL-GEN MATHS #05)

5. (a) (i)

(ii)

(b) (ii)

(1988-CE-A MATH 2 #07)

7. (b) $\sin\frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}$

(1989-HL-GEN MATHS #05) (Modified)

5. (a)

(b) (i)

(ii)

(1989-CE-A MATH 2 #05)

5. (a) $y = 13\sin(\theta - 67.4°) + 7$

(b) The least value of $y = -6$

(1990-CE-A MATH 2 #06)

6. (a) $r = 2, \alpha = 60°$

(b) $\frac{1}{7} \le x \le \frac{1}{3}$

(1992-CE-A MATH 2 #12)

12. (b)

(iii) $QA^2 + QB^2 + QC^2 + QD^2 + QE^2 = 30r^2$

(2002-CE-A MATH #08)

8. The least value $= \frac{1}{3}$

(2003-CE-A MATH #10)

10. $\tan(\alpha + \beta) = \frac{-24}{7}$

(2008-CE-A MATH #03)

3. $\tan 22.5° = \sqrt{2} - 1$

(2008-CE-A MATH #09)

9. (a) $\sin x + \sqrt{3}\cos x = 2\sin(x + 60°)$

(b) Least value = 1, Greatest value = 2

(2011-CE-A MATH #07)

7. $x = 10°$ or $45°$ or $50°$

(SP-DSE-MATH-EP(M2) #05)

5. $\frac{1}{8}$

(PP-DSE-MATH-EP(M2) #04)

4. (b) The greatest value = 2

(2012-DSE-MATH-EP(M2) #10)

10. (b) $\frac{1}{4} < y < \frac{1}{2}$

(2015-DSE-MATH-EP(M2) #07)

7. (b) (i) $f(x) = \frac{1}{4}\cos 4x + \frac{3}{4}$

(ii) $x = \frac{\pi}{12}$ or $\frac{5\pi}{12}$

(2015-DSE-MATH-EP(M2) #08)

8. (b) -1

(2016-DSE-MATH-EP(M2) #06)

6. (b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(2017-DSE-MATH-EP(M2) #07)

7. (b) (ii) $x = \frac{5\pi}{12}$

(2018-DSE-MATH-EP(M2) #03)

3. (b) $\frac{7\pi}{18}$

(2021-DSE-MATH-EP(M2) #04)

4. (b) $\theta = \frac{\pi}{12}$, $\theta = \frac{\pi}{8}$, $\theta = \frac{\pi}{4}$,

$\theta = \frac{3\pi}{8}$ or $\theta = \frac{5\pi}{12}$

2. CALCULUS AREA

1. Limits and Derivatives

(1991-CE-A MATH 1 #02) (5 marks)

2. Let $f(x)=\dfrac{1}{1+x}$. Find $f'(x)$ from the first principles.

(1993-CE-A MATH 1 #01) (5 marks)

1. (a) Simplify $\left(\sqrt{2(x+\Delta x)}-\sqrt{2x}\right)\left(\sqrt{2(x+\Delta x)}+\sqrt{2x}\right)$.

   (b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{2x}\right)$ from the first principles.

(1996-CE-A MATH 1 #02) (4 marks)

2. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2)$ from first principles.

(1998-CE-A MATH 1 #01) (4 marks)

1. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right)$ from first principles.

(2000-CE-A MATH 1 #03) (5 marks)

3. (a) Show that $\dfrac{1}{\sqrt{x+\Delta x}}-\dfrac{1}{\sqrt{x}}=\dfrac{-\Delta x}{\sqrt{x}\left(\sqrt{x+\Delta x}\right)\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}$.

   (b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\sqrt{x}}\right)$ from first principles.

(2003-CE-A MATH #02) (4 marks)

2. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)$ from first principles.

(2005-CE-A MATH #03) (4 marks)

3. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x}\right)$ from first principles.

(2007-CE-A MATH #04) (4 marks)

4. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)$ from first principles.

(2009-CE-A MATH #08) (4 marks)

8. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x+1}\right)$ from first principles.

(SP-DSE-MATH-EP(M2) #01) (4 marks)

1. Find $\frac{d}{dx}\left(\sqrt{2x}\right)$ from first principles.

(PP-DSE-MATH-EP(M2) #06) (4 marks)

6. Find $\frac{d}{dx}\left(\frac{1}{x}\right)$ from first principles.

(2012-DSE-MATH-EP(M2) #01) (3 marks)

1. Let $f(x) = e^{2x}$. Find $f'(0)$ from first principles.

(2013-DSE-MATH-EP(M2) #01) (4 marks)

1. Find $\frac{d}{dx}(\sin 2x)$ from first principles.

(2015-DSE-MATH-EP(M2) #01) (4 marks)

1. Find $\frac{d}{dx}\left(x^5 + 4\right)$ from first principles.

(2016-DSE-MATH-EP(M2) #02) (5 marks)

2. Prove that $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+h}} = \frac{h}{(x+h)\sqrt{x} + x\sqrt{x+h}}$. Hence, find $\frac{d}{dx}\sqrt{\frac{3}{x}}$ from first principles.

(2017-DSE-MATH-EP(M2) #01) (5 marks)

1. Find $\frac{d}{d\theta}\sec 6\theta$ from first principles.

(2018-DSE-MATH-EP(M2) #01) (4 marks)

1. Let $f(x) = (x^2 - 1)e^x$. Express $f(1+h)$ in terms of h . Hence, find $f'(1)$ from first principles.

(2019-DSE-MATH-EP(M2) #01) (4 marks)

1. Let $f(x) = \frac{10x}{7+3x^2}$. Prove that $f(1+h) - f(1) = \frac{4h - 3h^2}{10 + 6h + 3h^2}$. Hence, find $f'(1)$ from first principles.

(2020-DSE-MATH-EP(M2) #02) (4 marks)

2. Define $f(x) = \frac{x}{\sqrt{2+x}}$ for all $x > -2$. Find $f'(2)$ from first principles.

(2021-DSE-MATH-EP(M2) #01) (4 marks)

1. Let $f(x) = \frac{1}{3x^2 + 4}$. Find $f'(x)$ from first principles.

ANSWERS

(1991-CE-A MATH 1 #02)

2. $f'(x) = \dfrac{-1}{(1+x)^2}$

(1993-CE-A MATH 1 #01)

1. (a) $2\Delta x$

   (b) $\dfrac{1}{\sqrt{2x}}$

(1996-CE-A MATH 1 #02)

2. $2x$

(1998-CE-A MATH 1 #01)

1. $\dfrac{1}{2\sqrt{x}}$

(2000-CE-A MATH 1 #03)

3. (b) $-\dfrac{1}{2}x^{-\frac{3}{2}}$

(2003-CE-A MATH #02)

2. $3x^2$

(2005-CE-A MATH #03)

3. $\dfrac{-1}{x^2}$

(2007-CE-A MATH #04)

4. $2x$

(2009-CE-A MATH #08)

8. $\dfrac{1}{2\sqrt{x+1}}$

(SP-DSE-MATH-EP(M2) #01)

1. $\dfrac{1}{\sqrt{2x}}$

(PP-DSE-MATH-EP(M2) #06)

6. $\dfrac{-1}{x^2}$

(2012-DSE-MATH-EP(M2) #01)

1. 2

(2013-DSE-MATH-EP(M2) #01)

1. $2\cos 2x$

(2015-DSE-MATH-EP(M2) #01)

1. $5x^4$

(2016-DSE-MATH-EP(M2) #02)

2. $\dfrac{-\sqrt{3}}{2}x^{-\frac{3}{2}}$

(2017-DSE-MATH-EP(M2) #01)

1. $6\sec 6\theta\ \tan 6\theta$

(2018-DSE-MATH-EP(M2) #01)

1. $f(1+h) = h(h+2)e^{1+h}$

   $f'(1) = 2e$

(2019-DSE-MATH-EP(M2) #01)

1. $\dfrac{2}{5}$

(2020-DSE-MATH-EP(M2) #02)

2. $\dfrac{3}{8}$

(2020-DSE-MATH-EP(M2) #02)

1. $\dfrac{-6x}{\left(3x^2+4\right)^2}$

2. Differentiation

(1994-CE-A MATH 1 #04) (6 marks)

4. Let $y = \tan\left(\frac{1}{x}\right)$.

   Show that $x^2\frac{\mathrm{d}y}{\mathrm{d}x} + (y^2 + 1) = 0$.

   Hence show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{2(x + y)}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

(1996-CE-A MATH 1 #01) (3 marks)

1. Let $\mathrm{f}(x) = \sin^3 x$.

   Find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(1997-CE-A MATH 1 #01) (3 marks)

1. Let $\mathrm{f}(x) = \sqrt{3 + x^2}$. Find $\mathrm{f}'(-1)$.

(1997-CE-A MATH 1 #02) (3 marks)

2. $P(8,1)$ is a point on the curve $y^2 + \sqrt[3]{x}\,y - 3 = 0$. Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at P .

(1999-CE-A MATH 1 #01) (4 marks)

1. Find

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x^2 + 1)$,

   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin(x^2 + 1)}{x}\right]$.

(2000-CE-A MATH 1 #02) (4 marks)

2. Find

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x$,

   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin^2(3x + 1)$.

(2001-CE-A MATH #01) (3 marks)

1. Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2x + 1}\right)$.

(2002-CE-A MATH #03) (4 marks)

3. Let $x \sin y = 2002$.

   Find $\frac{dy}{dx}$.

(2003-CE-A MATH #04) (4 marks)

4. Given that $3x^2 + 3y^2 - 2xy = 12$, find $\frac{dy}{dx}$ when $x = 2$, $y = 0$.

(2005-CE-A MATH #09) (6 marks)

9. (a) Find $\frac{d}{dx}\sin^3(x^2 + 1)$.

   (b) Let $xy + y^2 = 2005$. Find $\frac{dy}{dx}$.

(2006-CE-A MATH #01) (3 marks)

1. Find $\frac{d}{dx}\left[\frac{\sin(2x + 1)}{x}\right]$.

(2010-CE-A MATH #01) (4 marks)

1. Find

   (a) $\frac{d}{dx}\cos(x^3 + 1)$,

   (b) $\frac{d}{dx}\left[x\cos(x^3 + 1)\right]$.

(PP-DSE-MATH-EP(M2) #07) (5 marks)

7. Let $f(x) = e^x(\sin x + \cos x)$.

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Find the value of x such that $f''(x) - f'(x) + f(x) = 0$ for $0 \le x \le \pi$.

(2014-DSE-MATH-EP(M2) #04) (3 marks)

4. Let $x = 2y + \sin y$. Find $\frac{d^2y}{dx^2}$ in terms of y .

(2015-DSE-MATH-EP(M2) #02) (5 marks)

2. Let $y = x \sin x + \cos x$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(b) Let k be a constant such that $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + k\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0$ for all real values of x . Find the value of k .

ANSWERS

(1994-CE-A MATH 1 #04) (6 marks)

(1996-CE-A MATH 1 #01) (3 marks)

1. $f'(x) = 3\sin^2 x \cos x$

   $f''(x) = 6\sin x\cos^2 x - 3\sin^3 x$

(1997-CE-A MATH 1 #01) (3 marks)

1. $-\dfrac{1}{2}$

(1997-CE-A MATH 1 #02) (3 marks)

2. $\left.\dfrac{dy}{dx}\right|_{(8,1)} = \dfrac{-1}{48}$

(1999-CE-A MATH 1 #01) (4 marks)

1. (a) $2x\cos(x^2+1)$

   (b) $\dfrac{2x^2\cos(x^2+1) - \sin(x^2+1)}{x^2}$

(2000-CE-A MATH 1 #02) (4 marks)

2. (a) $2\sin x\cos x$

   (b) $6\sin(3x+1)\cos(3x+1)$

(2001-CE-A MATH #01) (3 marks)

1. $\dfrac{2x(x+1)}{(2x+1)^2}$

(2002-CE-A MATH #03) (4 marks)

3. $\dfrac{dy}{dx} = \dfrac{-\tan y}{x}$

(2003-CE-A MATH #04) (4 marks)

4. 3

(2005-CE-A MATH #09) (6 marks)

9. (a) $6x\sin^2(x^2+1)\cos(x^2+1)$

   (b) $\dfrac{dy}{dx} = \dfrac{-y}{x+2y}$

(2006-CE-A MATH #01) (3 marks)

1. $\dfrac{2x\cos(2x+1) - \sin(2x+1)}{x^2}$

(2010-CE-A MATH #01) (4 marks)

1. (a) $-3x^2\sin(x^3+1)$

   (b) $-3x^3\sin(x^3+1) + \cos(x^3+1)$

(PP-DSE-MATH-EP(M2) #07) (5 marks)

7. (a) $f'(x) = 2e^x\cos x$

   $f''(x) = 2e^x(\cos x - \sin x)$

   (b) $x = \dfrac{\pi}{4}$

(2014-DSE-MATH-EP(M2) #04) (3 marks)

4. $\dfrac{d^2y}{dx^2} = \dfrac{\sin y}{(2+\cos y)^3}$

(2015-DSE-MATH-EP(M2) #02) (5 marks)

2. (a) $\dfrac{dy}{dx} = x\cos x$

   $\dfrac{d^2y}{dx^2} = -\sin x + \cos x$

   (b) $k = -2$

3. Applications of Differentiation

3. (a) Tangents ~~and Normals~~ to curves

(1991-CE-A MATH 1 #06) (7 marks) (Modified)

6. Let C be the curve $y = \frac{1}{x} + x$, where $x \neq 0$. $P(1,2)$ and $Q\left(\frac{1}{2}, \frac{5}{2}\right)$ are two points on C .

   (a) Find equations of the tangent ~~and normal~~ to C at P .

   (b) Show that the tangent to C at Q passes through the point $A(0,4)$.

(1992-CE-A MATH 1 #05) (6 marks)

5. The curve $(x-2)(y^2+3) = -8$ cuts the y-axis at two points. Find

   (a) the coordinates of the two points;

   (b) the slope of the tangent to the curve at each of the two points.

(1993-CE-A MATH 1 #07) (7 marks)

7. Given the curve $C : x^2 - 2xy^2 + y^3 + 1 = 0$.

   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (b) Find the equation of the tangent to C at the point $(2, -1)$.

(1994-CE-A MATH 1 #06) (7 marks)

6. Given the curve $C : x^2 + y\cos x - y^2 = 0$.

   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (b) $P\left(\frac{\pi}{2}, \frac{-\pi}{2}\right)$ is a point on the curve C . Find the equation of the tangent to the curve at P .

(1995-CE-A MATH 1 #06) (7 marks)

6. $P(4,1)$ is a point on the curve $y^2 + y\sqrt{x} = 3$, where $x > 0$.

   (a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at P .

   ~~(b) Find the equation of the normal to the curve at P .~~

(1996-CE-A MATH 1 #06) (7 marks)

6. Find the equations of the two tangents to the curve $C : y = \dfrac{6}{x+1}$ which are parallel to the line $x + 6y + 10 = 0$.

(1997-CE-A MATH 1 #02) (3 marks)

2. $P(8,1)$ is a point on the curve $y^2 + \sqrt[3]{x}\,y - 3 = 0$. Find the value of $\dfrac{dy}{dx}$ at P .

(1998-CE-A MATH 1 #08) (7 marks)

8. $P(0,2)$ is a point on the curve $x^2 - xy + 3y^2 = 12$.

(a) Find the value of $\dfrac{dy}{dx}$ at P .

~~(b) Find the equation of the normal to the curve at P .~~

(1999-CE-A MATH 1 #06) (6 marks)

6. The point $P(a, a)$ is on the curve $3x^2 - xy - y^2 - a^2 = 0$, where a is a non-zero constant.

(a) Find the value of $\dfrac{dy}{dx}$ at P .

(b) Find the equation of the tangent to the curve at P .

(2000-CE-A MATH 1 #04) (5 marks)

4. $P(-1,2)$ is a point on the curve $(x+2)(y+3) = 5$. Find

(a) the value of $\dfrac{dy}{dx}$ at P .

(b) the equation of the tangent to the curve at P .

(2001-CE-A MATH #07) (5 marks)

7. $P(2,0)$ is a point on the curve $x - (1 + \sin y)^5 = 1$. Find the equation of the tangent to the curve at P .

(2002-CE-A MATH #02) (4 marks)

2. Find the equation of the tangent to the curve $C : y = (x-1)^4 + 4$ which is parallel to the line $y = 4x + 8$.

(2004-CE-A MATH #09) (6 marks)

9.

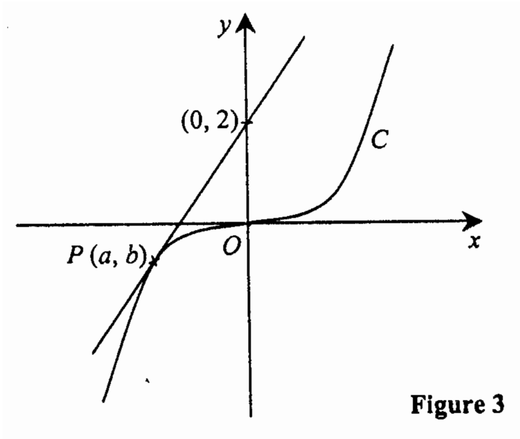


Figure 3

In Figure 3, $P(a, b)$ is a point on the curve $C : y = x^3$. The tangent to C at P passes through the point $(0,2)$.

(a) Show that $b = 3a^3 + 2$.

(b) Find the value of a and b .

(2006-CE-A MATH #12) (5 marks)

12. (a) Let $x^2 - xy + y^2 = 7$. Find $\frac{dy}{dx}$.

~~(b) Find the equation of the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(1,3)$.~~

(2008-CE-A MATH #06) (5 marks)

6. Find the equation of the tangent to the curve $y = \frac{3x}{x^2 + 2}$ at the point $(2,1)$.

(2009-CE-A MATH #06) (5 marks)

6. Let C be the curve $y^3 + x^3 y = 10$.

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent to C at the point $(1,2)$.

(2010-CE-A MATH #10) (6 marks)

10. It is given that P is a point on the curve $C : y = x^3$. If the y-intercept of the tangent L to C at P is -16 , find the equation of L .

(2011-CE-A MATH #06) (5 marks)

~~6. Find the equation of the normal to the curve $y = \frac{x^2 + 1}{x + 1}$ at $x = 1$.~~

(SP-DSE-MATH-EP(M2) #06) (5 marks)

6. Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$.

   Find the equation of the tangent to C at the point $(1,1)$.

(PP-DSE-MATH-EP(M2) #09) (6 marks)

9. Find the equation of the two tangents to the curve $x^2 - xy - 2y^2 - 1 = 0$ which are parallel to the straight line $y = 2x + 1$.

(2014-DSE-MATH-EP(M2) #03) (5 marks)

3. Find the equation of tangent to the curve $x \ln y + y = 2$ at the point where the curve cuts the y-axis.

ANSWERS

(1991-CE-A MATH 1 #06) (7 marks)

6. (a) Tangent is $y = 2$
Normal is $x = 1$

(1992-CE-A MATH 1 #05) (6 marks)

5. (a) (0,1) and (0, − 1)
(b) $\left.\frac{dy}{dx}\right|_{(0,1)} = 1$, $\left.\frac{dy}{dx}\right|_{(0,-1)} = -1$

(1993-CE-A MATH 1 #07) (7 marks)

7. (a) $\frac{dy}{dx} = \frac{2x - 2y^2}{4xy - 3y^2}$
(b) $y = -\frac{2}{11}x - \frac{7}{11}$

(1994-CE-A MATH 1 #06) (7 marks)

6. (a) $\frac{dy}{dx} = \frac{y \sin x - 2x}{\cos x - 2y}$
(b) $y = -\frac{3}{2}x + \frac{\pi}{4}$

(1995-CE-A MATH 1 #06) (7 marks)

6. (a) $\left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-1}{16}$
(b) $y = 16x - 63$

(1996-CE-A MATH 1 #06) (7 marks)

6. $x + 6y - 11 = 0$ or $x + 6y + 13 = 0$

(1998-CE-A MATH 1 #08) (7 marks)

8. (a) $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{1}{6}$
(b) $6x + y - 2 = 0$

(1999-CE-A MATH 1 #06) (6 marks)

6. (a) $\left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5}{3}$
(b) $5x - 3y - 2a = 0$

(2000-CE-A MATH 1 #04) (5 marks)

4. (a) $\left.\frac{dy}{dx}\right|_{(-1,2)} = -5$
(b) $5x + y + 3 = 0$

(2001-CE-A MATH #07) (5 marks)

7. $x - 5y - 2 = 0$

(2002-CE-A MATH #02) (4 marks)

2. $y = 4x - 3$

(2004-CE-A MATH #09) (6 marks)

9. (b) $a = b = -1$

(2006-CE-A MATH #12) (5 marks)

12. (a) $\frac{dy}{dx} = \frac{y - 2x}{2y - x}$
(b) $y = -5x + 8$

(2008-CE-A MATH #06) (5 marks)

6. $x + 6y - 8 = 0$

(2009-CE-A MATH #06) (5 marks)

6. (a) $\frac{dy}{dx} = \frac{-3x^2y}{x^3 + 3y^2}$
(b) $6x + 13y - 32 = 0$

(2010-CE-A MATH #10) (6 marks)

10. $y = 12x - 16$

(2011-CE-A MATH #06) (5 marks)

6. $2x + y - 3 = 0$

(SP-DSE-MATH-EP(M2) #06) (5 marks)

6. $x - 5y + 4 = 0$

(PP-DSE-MATH-EP(M2) #09) (6 marks)

9. $y = 2x + 2$ or $y = 2x - 2$

(2014-DSE-MATH-EP(M2) #03) (5 marks)

3. $y = -x \ln 2 + 2$

3. (b) Curve Sketching

(1991-CE-A MATH 1 #04) (7 marks)

4. Let $y = x + \sin 2x$, where $0 \le x \le \pi$.

Find

(a) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

(b) the maximum and minimum values of y .

(1994-CE-A MATH 1 #09) (Modified) (16 marks)

9. Given the curve $C : y = \frac{x^2}{1+x} - \frac{4}{3}$, where $x \ne -1$.

(a) Find the x- and y-intercepts of the curve C .

(b) Find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = \frac{2}{(1+x)^3}$.

(c) Find the turning point(s) of the curve C .

For each turning point, test whether it is a maximum or a minimum point.

(d) Sketch the curve C for

(i) $-5 \le x < -1$;

(ii) $-1 < x \le 3$.

(1995-CE-A MATH 1 #03) (5 marks)

3. Using the information in the following table, sketch the graph of $y = f(x)$, where $f(x)$ is a polynomial.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$f(x)$		1		2		1	
$f'(x)$	< 0	0	> 0	0	< 0	0	> 0

(1996-CE-A MATH 1 #09) (Modified) (16 marks)

9. C_1 is the curve $y = \dfrac{4x-3}{x^2+1}$.

(a) Find

(i) the x- and y-intercepts of the curve C_1 ;

(ii) the range of values of x for which $\dfrac{4x-3}{x^2+1}$ is decreasing;

(iii) the turning point(s) of C_1 , stating whether each point is a maximum or a minimum point. (Testing for maximum/minimum is not required.)

(b) Sketch the curve C_1 for $-10 \le x \le 10$.

(c) C_2 is the curve $y = \dfrac{|4x-3|}{x^2+1}$.

Using the result of (b), sketch the curve C_2 for $-10 \le x \le 10$ on the same graph.

Hence write down the greatest and least values of $\dfrac{|4x-3|}{x^2+1}$ for $-10 \le x \le 10$.

(1997-CE-A MATH 1 #10) (Modified) (16 marks)

10. The function $f(x) = \dfrac{x^2+kx+9}{x^2+1}$, where k is a constant, attains a stationary value at $x = 3$.

(a) Find $f'(x)$ in terms of k and x .

Hence show that $k = -6$.

(b) (i) Find the x- and y-intercepts of the curve $y = f(x)$.

(ii) Find the maximum and minimum points of the curve $y = f(x)$.

(c) Sketch the graph of $y = f(x)$ for $-6 \le x \le 6$.

Hence sketch the graph of $y = -f(x) - 1$ for $-6 \le x \le 6$ on the same graph.

(2000-AL-P MATH 2 #09) (Modified) (12 marks)

9. Let $f(x) = \dfrac{x}{(1+x^2)^2}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f''(x) > 0$.

(c) Find all relative extreme points, points of inflexion and asymptotes of $y = f(x)$.

(d) Sketch the graph of $f(x)$.

(2000-CE-A MATH 1 #10) (Modified) (16 marks)

10. Let $f(x) = \dfrac{7-4x}{x^2+2}$.

(a) (i) Find the x- and y-intercepts of the curve $y = f(x)$.

(ii) Find the range of values of x for which $f(x)$ is decreasing.

(iii) Show that the maximum and minimum values of $f(x)$ are 4 and $\dfrac{-1}{2}$ respectively.

(b) Sketch the curve $y = f(x)$ for $-2 \leq x \leq 5$.

(c) Let $p = \dfrac{7-4\sin\theta}{\sin^2\theta + 2}$, where θ is real.

From the graph in (b), a student concludes that the greatest and least values of p are 4 and $\dfrac{-1}{2}$ respectively.

Explain whether the student is correct. If not, what should be the greatest and least values of p ?

(2001-CE-A MATH #18) (12 marks)

18. Let $f(x)$ be a polynomial, where $-2 \le x \le 10$. Figure 5 (a) shows a sketch of the curve $y = f'(x)$, where $f'(x)$ denotes the first derivative of $f(x)$.

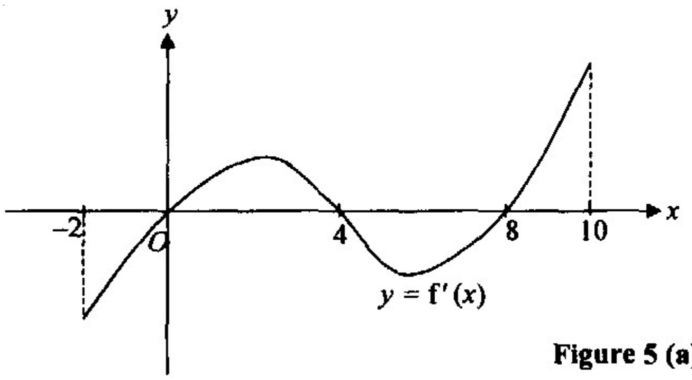


Figure 5 (a)

(a) (i) Write down the range of values of x for which $f(x)$ is increasing.

(ii) Find the x-coordinates of the maximum and minimum points of the curve $y = f(x)$.

(iii) In Figure 5 (b), draw a possible sketch of the curve $y = f(x)$.

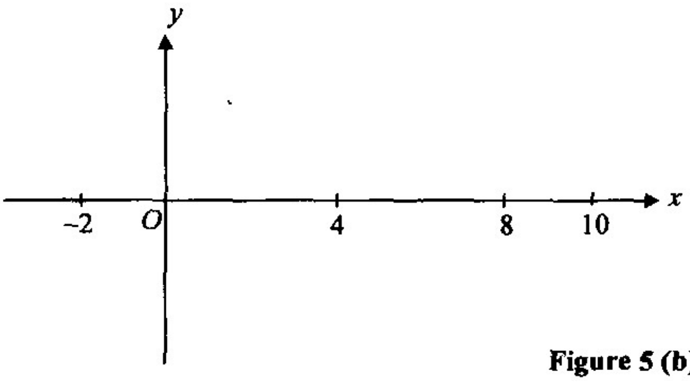


Figure 5 (b)

(b) In Figure 5 (c), sketch the curve $y = f''(x)$.

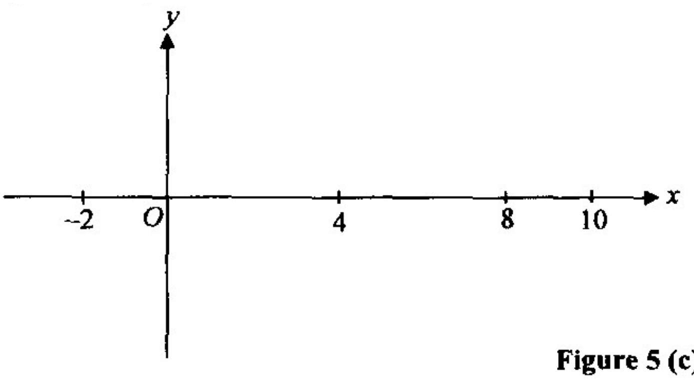


Figure 5 (c)

(c) Let $g(x) = f(x) + x$, where $-2 \le x \le 10$.

(i) In Figure 5 (a), sketch the curve $y = g'(x)$.

(ii) A student makes the following note:

Since the functions $f(x)$ and $g(x)$ are different, the graphs of $y = f''(x)$ and $y = g''(x)$ should be different.

Explain whether the student is correct or not.

(2002-AL-P MATH 2 #08) (Modified) (10 marks)

8. Let $f(x) = x^2 - \dfrac{8}{x-1}$. ($x \neq 1$)

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine the range of values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f'(x) < 0$,

(iii) $f''(x) > 0$,

(iv) $f''(x) < 0$.

(c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

(d) Find the asymptote(s) of the graph of $f(x)$.

(e) Sketch the graph of $f(x)$.

(2003-CE-A MATH #13) (7 marks)

13. Let $f(x) = 2\sin x - x$ for $0 \leq x \leq \pi$. Find the greatest and least values of $f(x)$.

(2007-AL-P MATH 2 #07)

7. Let $f(x) = \dfrac{(x+15)(x+1)^2}{(x-6)^2}$. ($x \neq 6$)

(a) Find $f'(x)$ and $f''(x)$.

(b) Solve each of the following inequalities:

(i) $f'(x) > 0$,

(ii) $f''(x) > 0$.

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = f(x)$.

(d) Find the asymptote(s) of the graph of $y = f(x)$.

(e) Sketch the graph of $y = f(x)$.

(2007-CE-A MATH #10) (5 marks)

10.

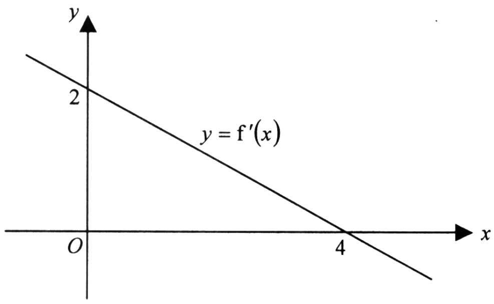


Figure 4

Let $f(x)$ be a function of x . Figure 4 shows the graph of $y = f'(x)$ which is a straight line with x- and y-intercepts 4 and 2 respectively.

(a) Find the slope of the tangent to the curve $y = f(x)$ at $x = 1$.

(b) Find the x-coordinate(s) of all the turning point(s) of the curve $y = f(x)$. For each turning point, determine whether it is a minimum point or a maximum point.

(2008-CE-A MATH #13) (7 marks)

13. Let $f(x) = x(x-6)^2$.

(a) Find the maximum and minimum points of the graph of $y = f(x)$.

(b) Sketch the graph of $y = f(x)$.

(2010-AL-P MATH 2 #07) (15 marks)

7. Let $f : \mathbb{R}\backslash\{-3\} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x-1}{(x+3)^3}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Solve

(i) $f(x) > 0$,

(ii) $f'(x) > 0$,

(iii) $f''(x) > 0$.

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = f(x)$.

(d) Find the asymptote(s) of the graph of $y = f(x)$.

(e) Sketch the graph of $y = f(x)$.

(f) Let $n(k)$ be the number of points of intersection of the graph of $y = f(x)$ and the horizontal line $y = k$. Using the graph of $y = f(x)$, find $n(k)$ for any $k \in \mathbb{R}$.

(2010-CE-A MATH #08) (4 marks)

2. It is given that $f(x)$ is a polynomial with the following properties:

(1) $f(0) = f(2) = f(4) = 0$;

(2) For $0 \le x \le 4$, the minimum and maximum values of $f(x)$ are -2 and 2 respectively;

(3)

	$0 \le x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$3 < x \le 4$
$f'(x)$	> 0	0	< 0	0	> 0

Using the above information, sketch the graph of $y = f(x)$ for $0 \le x \le 4$.

(2011-CE-A MATH #08) (6 marks)

8. Let $f(x) = (x + 2)(x^2 + 1)$.

(a) Find the maximum and minimum points of the graph of $y = f(x)$.

(b) Sketch the graph of $y = f(x)$.

(2012-DSE-MATH-EP(M2) #05) (6 marks)

5. Find the minimum point(s) and asymptote(s) of the graph of $y = \dfrac{x^2 + x + 1}{x + 1}$.

(2013-DSE-MATH-EP(M2) #05) (6 marks)

5. Consider a continuous function $f(x) = \dfrac{3 - 3x^2}{3 + x^2}$. It is given that

x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$

('+' and '−' denote 'positive value' and 'negative value' respectively.)

(a) Find all the maximum and/or minimum point(s) and point(s) of inflexion.

(b) Find the asymptote(s) of the graph of $y = f(x)$.

(c) Sketch the graph of $y = f(x)$.

(2014-DSE-MATH-EP(M2) #02) (5 marks)

2. Consider the curve $C : y = x^3 - 3x$.

(a) Find $\dfrac{dy}{dx}$ from first principles.

(b) Find the range of x where C is decreasing.

(2016-DSE-MATH-EP(M2) #04) (7 marks)

4. Define $f(x) = \dfrac{2x^2 + x + 1}{x - 1}$ for all $x \neq 1$. Denote the graph of $y = f(x)$ by G . Find

(a) the asymptote(s) of G ,

(b) the slope of the normal to G at the point $(2, 11)$.

(2018-DSE-MATH-EP(M2) #08) (8 marks)

8. Define $f(x) = \dfrac{A}{x^2 - 4x + 7}$ for all real numbers x , where A is a constant. It is given that the extreme value of $f(x)$ is 4 .

(a) Find $f'(x)$.

(b) Someone claims that there are at least two asymptotes of the graph of $y = f(x)$. Do you agree? Explain your answer.

(c) Find the point(s) of inflexion of the graph of $y = f(x)$.

(2021-DSE-MATH-EP(M2) #05) (7 marks)

5. Define $r(x) = \dfrac{x^3 - x^2 - 2x + 3}{(x - 1)^2}$ for all real numbers $x \neq 1$.

(a) Find the asymptote(s) of the graph of $y = r(x)$.

(b) Find $\dfrac{d}{dx} r(x)$.

(c) Someone claims that there is only one point of inflexion of the graph of $y = r(x)$. Do you agree? Explain your answer.

ANSWERS

(1991-CE-A MATH 1 #04) (7 marks)

4. (a) $\dfrac{dy}{dx} = 1 + 2\cos 2x$

$\dfrac{d^2y}{dx^2} = -4\sin 2x$

(b) $y_{max} = \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$, $y_{min} = \dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}$

(1994-CE-A MATH 1 #09) (16 marks)

9. (a) x-intercept $= 2$ or $\dfrac{-2}{3}$, y-intercept $= \dfrac{-4}{3}$

(b) $\dfrac{dy}{dx} = \dfrac{2x + x^2}{(1+x)^2}$

(c) Minimum point = $\left(0, \dfrac{-4}{3}\right)$

Maximum point = $\left(-2, \dfrac{-16}{3}\right)$

(1996-CE-A MATH 1 #09) (16 marks)

9. (a) (i) x-intercept $= \dfrac{3}{4}$, y-intercept $= -3$

(ii) $x \geq 2$ or $x \leq \dfrac{-1}{2}$

(iii) Minimum point = $\left(\dfrac{-1}{2}, -4\right)$

Maximum point = (2,1)

(1997-CE-A MATH 1 #10) (16 marks)

10. (a) $f'(x) = \dfrac{-kx^2 - 16x + k}{(x^2+1)^2}$

(b) (i) x-intercept = 3, y-intercept = 9

(ii) Minimum point = (3,0)

Maximum point = $\left(\dfrac{-1}{3}, 10\right)$

(2000-AL-P MATH 2 #09) (12 marks)

9. (a) $f'(x) = \dfrac{1 - 3x^2}{(1+x^2)^3}$

$f''(x) = \dfrac{-12x(1-x^2)}{(1+x^2)^4}$

(b) (i) $-\dfrac{\sqrt{3}}{3} < x < \dfrac{\sqrt{3}}{3}$

(ii) $-1 < x < 0$ or $x > 1$

(c) Minimum point = $\left(\dfrac{-\sqrt{3}}{3}, \dfrac{-3\sqrt{3}}{16}\right)$

Maximum point = $\left(\dfrac{-1}{3}, 10\right)$

(2000-CE-A MATH 1 #10) (16 marks)

10. (a) (i) x-intercept $= \dfrac{7}{4}$, y-intercept $= \dfrac{7}{2}$

(ii) $\dfrac{-1}{2} \leq x \leq 4$

(c) Greatest value of $p = 4$

Least value of $p = 1$

(2001-CE-A MATH #18) (12 marks)

18. (a) (i) $0 \leq x \leq 4$, $x \geq 8$

(ii) Minimum point at $x = 0$ and 8

Maximum point at $x = 4$

(2002-AL-P MATH 2 #08)

8. (a) $f'(x) = 2x + \dfrac{8}{(x-1)^2}$

$f''(x) = 2 + \dfrac{16}{(1-x)^3}$

(b) (i) $x > -1$ and $x \neq 1$

(ii) $x < -1$

(iii) $x < 1$ or $x > 3$

(iv) $1 < x < 3$

(c) Minimum point = $(-1,5)$

Point of inflexion = (3,5)

(d) Vertical asymptote is $x = 1$

(2003-CE-A MATH #13) (7 marks)

13. Greatest value $= \sqrt{3} - \frac{\pi}{3}$

Least values of $= -\pi$

(2007-AL-P MATH 2 #07)

7. (a) $f'(x) = \frac{(x+1)(x+8)(x-27)}{(x-6)^3}$

$f''(x) = \frac{686(x+3)}{(x-6)^4}$

(b) (i) $x < -8, -1 < x < 6$ or $x > 27$

(ii) $-3 < x < 6$ or $x > 6$

(c) Minimum points = $(-1,0)$ and $\left(27, \frac{224}{3}\right)$

Maximum point = $\left(-8, \frac{7}{4}\right)$

Point of inflexion = $\left(-3, \frac{16}{27}\right)$

(d) Vertical asymptote is $x = 6$

Oblique asymptote is $y = x + 29$

(2007-CE-A MATH #10) (5 marks)

10. (a) Slope $= f'(1) = \frac{3}{2}$ at $x = 1$

(b) x-coordinate = 4, maximum point

(2008-CE-A MATH #13) (7 marks)

13. (a) Maximum point = (2, 32)

Minimum point = (6, 0)

(2010-AL-P MATH 2 #07) (15 marks)

7. (a) $f'(x) = \frac{-2(x-3)}{(x+3)^4}$

$f''(x) = \frac{6(x-5)}{(x+3)^5}$

(b) (i) $x < -3$ or $x > 1$

(ii) $x < -3$ or $-3 < x < 3$

(iii) $x < -3$ or $x > 5$

(c) Maximum point = $\left(3, \frac{1}{108}\right)$

Point of inflexion = $\left(5, \frac{1}{128}\right)$

(d) Vertical asymptote is $x = -3$

Horizontal asymptote os $y = 0$

(f) $n(k) = \begin{cases} 1 \text{ when } k \le 0 \text{ or } k > \frac{1}{108} \\ 2 \text{ when } k = \frac{1}{108} \\ 3 \text{ when } 0 < k < \frac{1}{108} \end{cases}$

(2011-CE-A MATH #08) (6 marks)

8. (a) Maximum point = $(-1,2)$

Minimum point = $\left(\frac{-1}{3}, \frac{50}{27}\right)$

(2012-DSE-MATH-EP(M2) #05) (6 marks)

5. Minimum point = (0,1)

Vertical asymptote is $x = -1$

Oblique asymptote $y = x$

(2013-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) Maximum point = (0,1)

Points of inflexion = (1,0) and $(-1,0)$

(b) Horizontal asymptote is $y = -3$

(2014-DSE-MATH-EP(M2) #02) (5 marks)

2. (a) $\frac{dy}{dx} = 3x^2 - 3$

(b) $-1 \le x \le 1$

(2016-DSE-MATH-EP(M2) #04) (7 marks)

4. (a) Vertical asymptote is $x = 1$

Oblique asymptote is $y = 2x + 3$

(b) Slope = $\frac{1}{2}$

(2018-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $\frac{48 - 24x}{(x^2 - 4x + 7)^2}$

(b) Horizontal asymptote is $y = 0$

The claim is disagreed.

(c) (1,3) and (3,3)

(2021-DSE-MATH-EP(M2) #05) (7 marks)

5. (a) Vertical asymptote is $x = 1$

Oblique asymptote is $y = x + 1$

(b) $1 + \frac{x - 3}{(x - 1)^3}$

(c) The claim is agreed.

3. (c) Optimization and Rates of Change problems

(1991-CE-A MATH 1 #11) (Modified) (16 marks)

11.

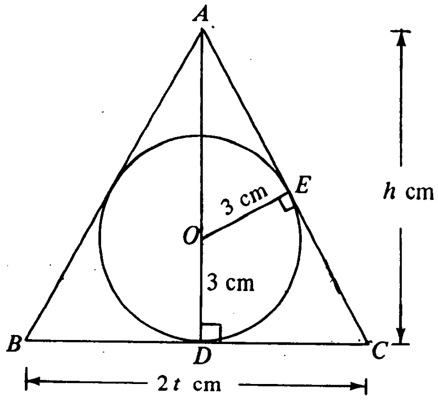


Figure 2(a)

ABC is a variable isosceles triangle with $AB = AC$ such that the radius of its inscribed circle is 3 cm. The height AD and the base BC of ΔABC are h cm and $2t$ cm respectively, where $h > 6$. (See Figure 2(a).) Let p cm be the perimeter of ΔABC.

(a) Show that $t^2 = \dfrac{9h}{h-6}$.

(b) Show that $p = \dfrac{2h^{\frac{3}{2}}}{(h-6)^{\frac{1}{2}}}$.

(c) Find

(i) the range of values of h for which $\dfrac{dp}{dh}$ is positive.

(ii) the minimum value of p.

(d) (i) Sketch the graph of p against h for $h > 6$.

(ii) Hence write down the range of values of p for which two different isosceles triangles whose inscribed circles are of radii 3 cm can have the same perimeter p cm.

(1991-CE-A MATH 1 #12) (16 marks)

12.

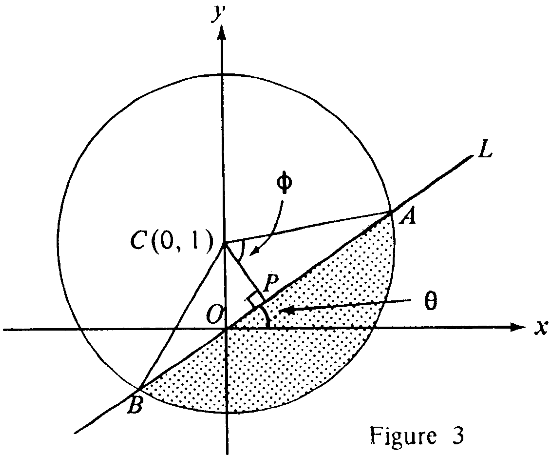


Figure 3

Figure 3 shows a circle of radius 2 centred at the point $C(0,1)$. A variable straight line L with positive slope passes through the origin O and makes an angle θ with the positive x-axis. L intersects the circle at points A and B . Let S be the area of the shaded segment. P is the point on L such that CP is perpendicular to AB . Let $\angle PCA = \varphi$.

(a) (i) Find the length of CP in terms of θ .

Hence show that $\cos\theta = 2\cos\varphi$.

(ii) Show that $S = 4\varphi - 2\sin 2\varphi$.

(b) (i) Find $\frac{\mathrm{d}\varphi}{\mathrm{d}\theta}$ in terms of θ and φ .

(ii) Hence find $\frac{\mathrm{d}S}{\mathrm{d}\theta}$ in terms of θ .

(c) L rotates about O in the clockwise direction such that θ decreases steadily at a rate of $\frac{1}{30}$ radian per second.

Find the rate of change of S with respect to time when $\theta = \frac{\pi}{3}$.

(1992-CE-A MATH 1 #07) (7 marks)

7.

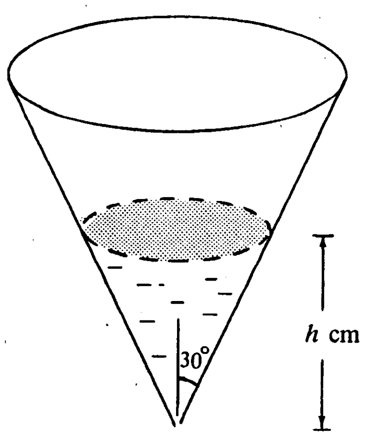


Figure 2

Figure 2 shows a vessel in the shape of a right circular cone with semi-vertical angle $30°$. Water is flowing out of the cone through its apex at a constant rate of π cm^3 s^{-1} .

(a) Let V cm^3 be the volume of water in the vessel when the depth of water is h cm . Express V in terms of h .

(b) How fast is the water level falling when the depth of water is 4 cm ?

(1992-CE-A MATH 1 #11) (16 marks)

11.

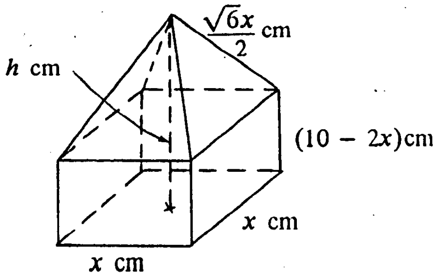


Figure 4 (a)

Figure 4 (a) shows a solid consisting of a right pyramid and a cuboid with a common face which is a square of side x cm . The slant edge of the pyramid is $\frac{\sqrt{6}x}{2}$ cm and the height of the cuboid is $(10-2x)$ cm , where $0<x<5$.

(a) Let h cm be the height of the solid. Show that $h=10-x$.

(b) Let V cm^3 be the volume of the solid.

(i) Show that $V=10x^2-\frac{5}{3}x^3$.

(ii) Find the range of values of x for which V is increasing.
Hence write down the range of values of x for which V is decreasing.

(c)

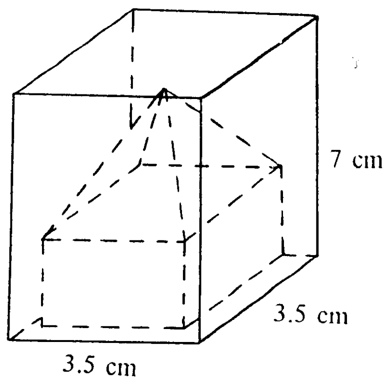


Figure 4 (b)

The solid is placed COMPLETELY inside a rectangular box as shown in Figure 4(b). The base of the box is a square of side 3.5 cm and the height of the box is 7 cm .

(i) Show that $3 \le x \le 3.5$.

(ii) Hence find, correct to one decimal place, the greatest volume of the solid.

(d) The side of the square base of the box in (c) is now changed to 4.7 cm and the height 5.5 cm . Find, correct to one decimal place, the greatest volume of the solid that can be placed COMPLETELY inside the box.

(1993-CE-A MATH 1 #09) (16 marks)

9.

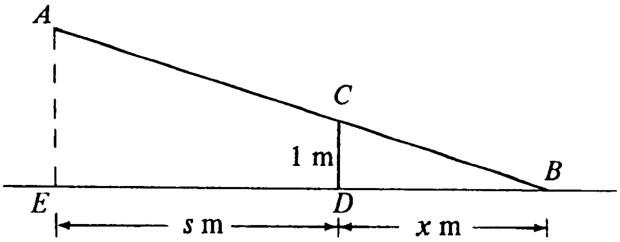


Figure 2

Figure 2 shows a straight rod AB of length 8 m resting on a vertical wall CD of height 1 m . The end B is free to slide along a horizontal rail such that AB is vertically above the rail. Let E be the projection of A on the rail, $DE = s$ m and $BD = x$ m , where $0 < x < 3\sqrt{7}$.

(a) Show that $s = \dfrac{8x}{\sqrt{1+x^2}} - x$.

(b) Find the maximum value of s .

(c) Let P m² be the area of the trapezium $CAED$.

(i) Show that $P = \dfrac{32x}{1+x^2} - \dfrac{x}{2}$.

(ii) Does P attain a maximum when s attains its maximum? Explain your answer.

(1994-AS-M & S #02) (5 marks)

2. The population size x of an endangered species of animals is modelled by the equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0,$$

where t denotes the time.

It is known that $x = 100e^{kt}$ where k is a negative constant. Determine the value of k .

(1994-CE-A MATH 1 #12) (16 marks)

12.

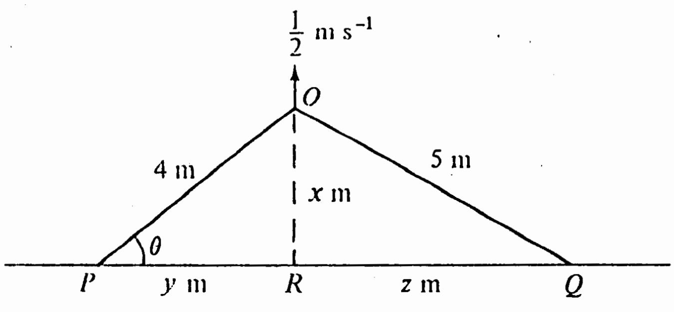


Figure 3

In Figure 3, two rods OP and OQ are hinged at O. The lengths of OP and OQ are 4 m and 5 m respectively. The end O is pushed upwards at a constant rate of $\frac{1}{2}$ ms^{-1} along a fixed vertical axis, and the ends P and Q move along a horizontal rail. R is the projection of O on the rail. At time t seconds, $OR = x$ m and $\angle OPQ = \theta$ where $0 < \theta < \frac{\pi}{2}$.

(a) Express x in terms of θ.
Hence find the rate of change of θ with respect to t in terms of θ.

(b) Let $PR = y$ m, $RQ = z$ m.
Express $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}z}{\mathrm{d}t}$ in terms of θ.
Hence find the rate of change of PQ with respect to t when $\theta = \frac{\pi}{6}$, giving your answer correct to 3 significant figures.

(c) Find the value of θ such that the area of ΔOPR is a maximum.
By considering the value of $\angle OQR$, find the value of θ such that the area of ΔORQ is a maximum, giving your answer correct to 3 significant figures.

(1995-CE-A MATH 1 #09) (16 marks)

9.

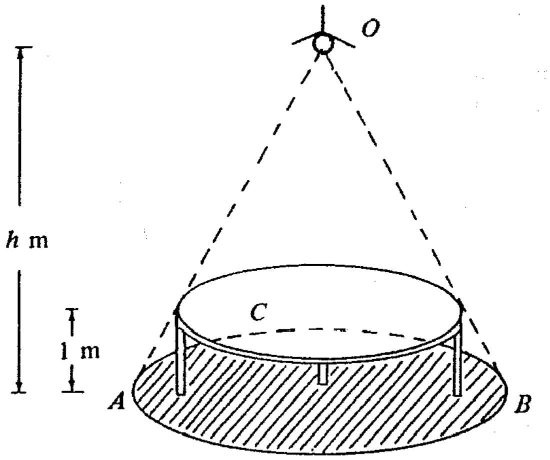


Figure 2

A small lamp O is placed h m above the ground, where $1 < h \leq 5$. Vertically below the lamp is the centre of a round table of radius 2 m and height 1 m . The lamp casts a shadow ABC of the table on the ground (see Figure 2). Let S m^2 be the area of the shadow.

(a) Show that $S = \dfrac{4\pi h^2}{(h-1)^2}$.

(b) If the lamp is lowered vertically at a constant rate of $\dfrac{1}{8}$ ms^{-1} , find the rate of change of S with respect to time when $h = 2$.

(c) Let V m^3 be the volume of the cone $OABC$.

(i) Show that $V = \dfrac{4\pi h^3}{3(h-1)^2}$.

(ii) Find the minimum value of V as h varies.
Does S attain a minimum when V attains its minimum? Explain your answer.

(1995-CE-A MATH 1 #12) (16 marks)

12.

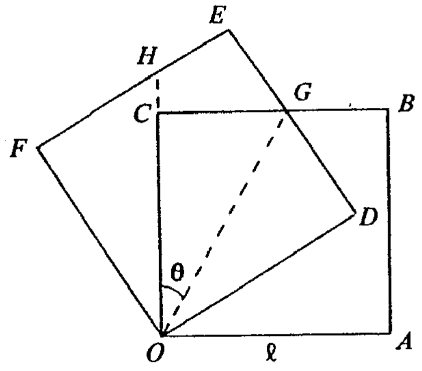


Figure 4

In Figure 4, $OABC$ is the position of a square of side ℓ . The square is rotated anticlockwise about O to a new position $ODEF$. BC cuts DE at G and OC produced cuts EF at H. Let $\angle COG = \theta$, where $\frac{\pi}{8} < \theta < \frac{\pi}{4}$.

(a) Name a triangle which is congruent to ΔOCG .

Hence show that the area of ΔOFH is $\dfrac{\ell^2}{2\tan 2\theta}$.

(b) Let S be the sum of the areas of ΔOFH and the quadrilateral $ODGC$.

(i) Show that $S = \dfrac{\ell^2}{2}\left(\dfrac{2-\cos 2\theta}{\sin 2\theta}\right)$.

(ii) Find the range of values of θ for which S is

(1) increasing,

(2) decreasing.

Hence find the minimum value of S .

(c) Find the maximum value of the area of the quadrilateral $CGEH$.

(1996-CE-A MATH 1 #11) (16 marks)

11.

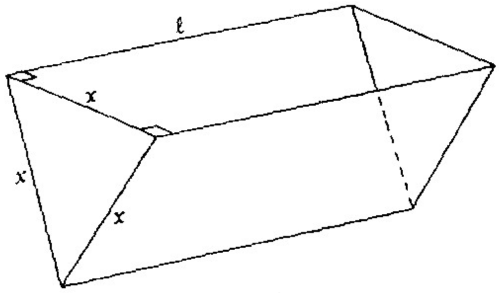


Figure 3(a)

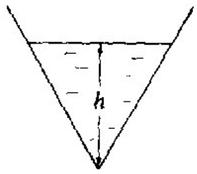


Figure 3(b)

Figure 3(a) shows a vessel with a capacity of 24 cubic units. The length of the vessel is ℓ and its vertical cross-section is an equilateral triangles of side x . The vessel is made of thin metal plates and has no lid. Let S be the total area of metal plates used to make the vessel.

(a) Show that $S = \frac{\sqrt{3}}{2}x^2 + \frac{64\sqrt{3}}{x}$.

(b) Find the values of x and ℓ such that the area of metal plates used to make the vessel is minimum.

(c) At time $t = 0$, the vessel described in part (b) is completely filled with water. Suppose the water evaporates at a rate proportional to the area of water surface at that instant such that $\frac{dV}{dt} = -\frac{1}{10}A$, where V and A are respectively the volume of water and the area of water surface at time t .

(i) Let h be the depth of water in the vessel at time t . (See Figure 3(b).) Show that $A = 4h$ and $V = 2h^2$. Hence, or otherwise, find $\frac{dh}{dt}$.

(ii) Find the time required for the water in the vessel to evaporate completely.

(1997-CE-A MATH 1 #04) (5 marks)

4.

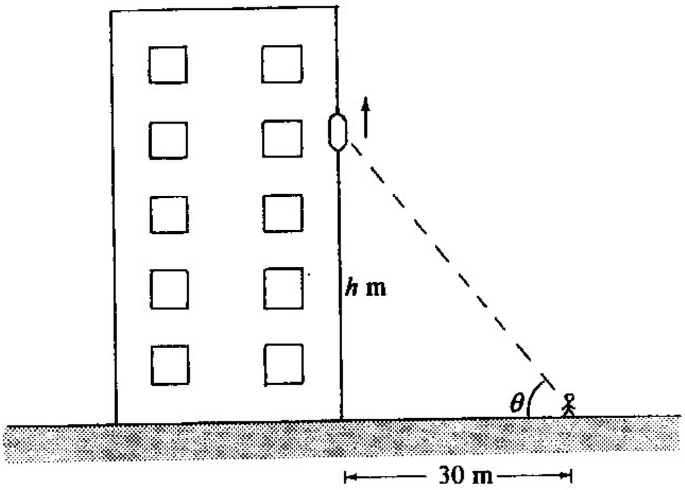


Figure 1

A man stands at a horizontal distance of 30 m from a sight-seeing elevator of a building as shown in Figure 1. The elevator is rising vertically with a uniform speed of 1.5 ms^{-1} . When the elevator is at a height h m above the ground, its angle of elevation from the man is θ . Find the rate of change of θ with respect to time when the elevator is at a height $30\sqrt{3}$ m above the ground. (Note: You may assume that the sizes of the elevator and the man are negligible.)

(1997-CE-A MATH 1 #12) (16 marks)

12.

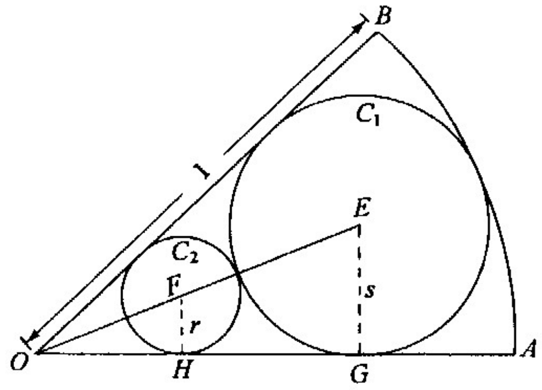


Figure 4

In Figure 4, OAB is a sector of unit radius and $\angle AOB = 2\theta$, where $0 < \theta < \dfrac{\pi}{2}$. C_1 is an inscribed circle of radius s in the sector. C_2 is another circle of radius r touching OA , OB and C_1 . Let E and F be the centres of C_1 and C_2 respectively. OA touches C_1 and C_2 at G and H respectively.

(a) Show that $s = \dfrac{\sin\theta}{1+\sin\theta}$.

Hence find $\dfrac{ds}{d\theta}$.

(b) By considering ΔOFH and ΔOEG , express r in terms of s .

Hence show that $\dfrac{dr}{d\theta} = \dfrac{\cos\theta(1-3\sin\theta)}{(1+\sin\theta)^3}$.

(c) By considering the ranges of values of θ for which r is

(i) increasing, and

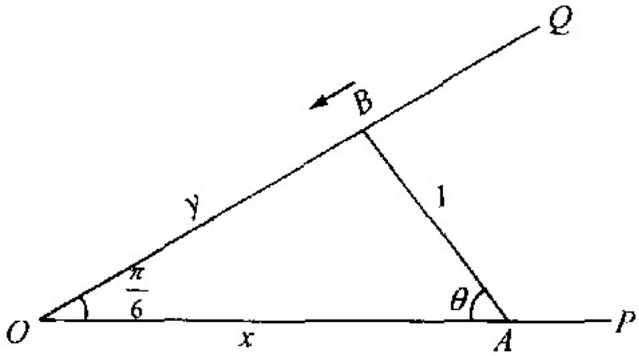
(ii) decreasing,

find the maximum area of circle C_2 . (Note : You may give your answers correct to three significant figures.)

(d) Does the area of circle C_1 attain a minimum when the area of the circle C_2 attains its maximum? Explain your answer.

(1998-CE-A MATH 1 #13) (16 marks)

13.



In Figure 5, POQ is a rail and $\angle POQ = \frac{\pi}{6}$. AB is a rod of length 1 m which is free to slide on the rail with end A on OP and end B on OQ . Initially, end A is at the point on OP such the $\angle OAB = \frac{4\pi}{9}$. End B is pushed towards O at a constant speed. After t seconds, $OA = x$ m , $OB = y$ m and $\angle OAB = \theta$, where $0 \le \theta \le \frac{4\pi}{9}$.

(a) Express x and y in terms of θ .

(b) Let S m² be the area of ΔOAB .

Show that $\frac{dS}{d\theta} = \sin\left(\frac{5\pi}{6} - 2\theta\right)$.

Hence find the value of θ such that S is a maximum.

(c) Using (a), show that $\frac{dx}{dt} = \frac{-\cos\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}\frac{dy}{dt}$.

(d) A student makes the following prediction regarding the motion of end A of the rod:

> As end B moves from its initial position to point O , end A will first move away from O and then it will change its direction and move towards O .

Is the student's prediction correct? Explain your answer.

(1999-CE-A MATH 1 #08) (7 marks)

8.

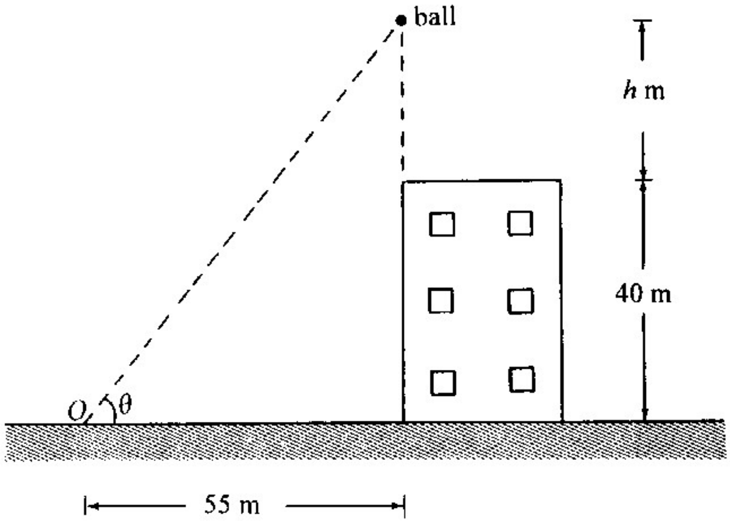


Figure 1

A ball is thrown vertically upwards from the roof of a building 40 m in height. After t seconds, the height of the ball above the roof is h m, where $h = 20t - 5t^2$. At this instant, the angle of elevation of the ball from a point O, which is at a horizontal distance of 55 m from the building, is θ. (See Figure 1.)

(a) Find

(i) $\tan\theta$ in terms of t.

(ii) the value of θ when $t = 3$.

(b) Find the rate of change of θ with respect to time when $t = 3$.

(1999-CE-A MATH 1 #12) (16 marks)

12.

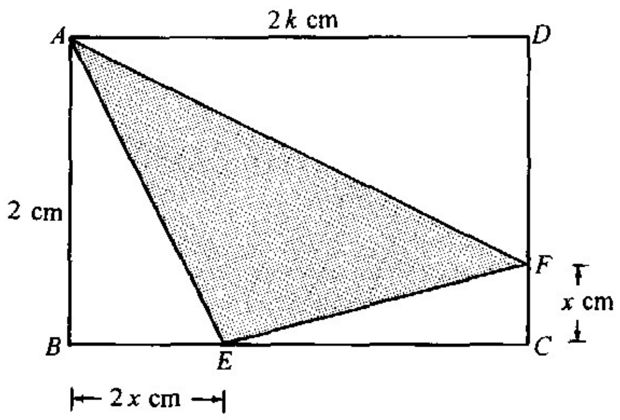


Figure 5

Figure 5 shows a rectangle $ABCD$ with $AB = 2$ cm and $AD = 2k$ cm , where k is a positive number. E and F are two variable points on the sides BC and CD respectively such that $CF = x$ cm and $BE = 2x$ cm , where x is a non-negative number. Let S cm^2 denote the area of ΔAEF .

(a) Show that $S = x^2 - 2x + 2k$.

(b) Suppose $k = \dfrac{3}{2}$.

(i) By considering that points E and F lie on the sides BC and CD respectively, show that $0 \le x \le \dfrac{3}{2}$.

(ii) Find the least value of S and the corresponding value of x .

(iii) Find the greatest value of S .

(c) Suppose $k = \dfrac{3}{8}$. A student says that S is least when $x = 1$.

(i) Explain whether the student is correct.

(ii) Find the least value of S .

(1999-CE-A MATH 1 #13) (16 marks)

13.

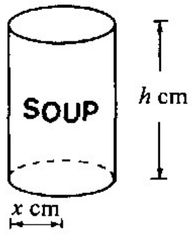


Figure 6

A food company produces cans of instant soup. Each can is in the form of a right cylinder with a base radius of x cm and a height of h cm (see Figure 6) and its capacity is V cm^3, where V is constant. The cans are made of thin metal sheets. The cost of the curved surface of the can is 1 cent per cm^2 and the cost of the plane surfaces is k cents per cm^2. Let C cents be the production cost of one can. For economic reasons, the value of C is minimized.

(a) Express h in terms of π, x and V.

Hence show that $C = \dfrac{2V}{x} + 2\pi k x^2$.

(b) If $\dfrac{dC}{dx} = 0$, express x^3 in terms of π, k and V.

Hence show that C is minimum when $\dfrac{x}{h} = \dfrac{1}{2k}$.

(c) Suppose $k = 2$ and $V = 256\pi$.

(i) Find the values of x and h.

(ii) If the value of k increases, how would the dimensions of the can be affected? Explain your answer.

(d) The company intends to produce a bigger can of capacity $2V$ cm^3, which is also in the form of a right cylinder. Suppose the costs of the curved surface and plane surfaces of the bigger can are maintained at 1 cent and k cents per cm^2 respectively. A worker suggests that the ratio of base radius to height of the bigger can should be twice that of the smaller can in order to minimize the production cost. Explain whether the worker is correct.

(2000-CE-A MATH 1 #13) (16 marks)

13.

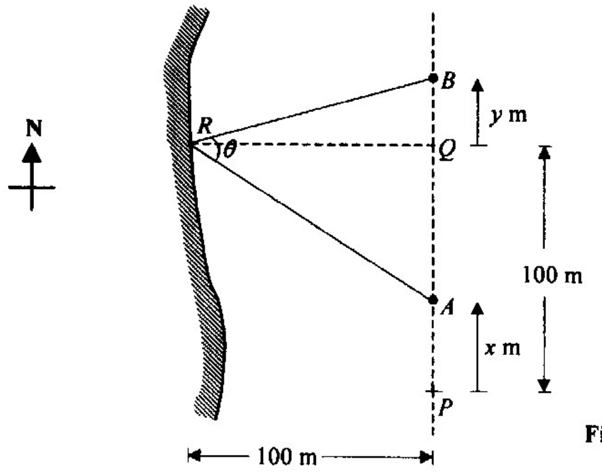


Figure 5

Two boats A and B are initially located at points P and Q in a lake respectively, where Q is at a distance 100 m due north of P . R is a point on the lakeside which is at a distance 100 m due west of Q . (See Figure 5.) Starting from time (in seconds) $t = 0$, boats A and B sail northwards. At time t , let the distances traveled by A and B be x m and y m respectively, where $0 \leq x \leq 100$. Let $\angle ARB = \theta$.

(a) Express $\tan\angle ARQ$ in terms of x .

Hence show that $\tan\theta = \dfrac{100(100 - x + y)}{10000 - 100y + xy}$.

(b) Suppose boat A sails with a constant speed of 2 ms^{-1} and B adjust its speed continuously so as to keep the value of $\angle ARB$ unchanged.

(i) Using (a), show that $y = \dfrac{100x}{200 - x}$.

(ii) Find the speed of boat B at $t = 40$.

(iii) Suppose the maximum speed of boat B is 3 ms^{-1} . Explain whether it is possible to keep the value of $\angle ARB$ unchanged before boat A reaches Q .

(2002-CE-A MATH #14) (12 marks)

14.

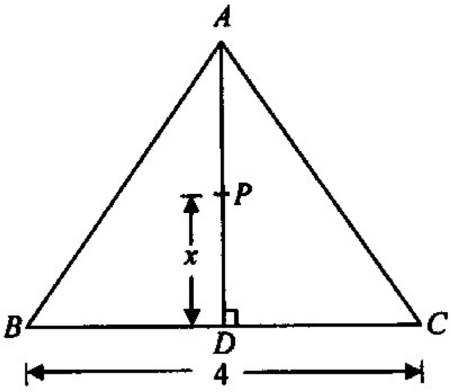


Figure 5 shows an isosceles triangle ABC with $AB = AC$ and $BC = 4$. D is the foot of perpendicular from A to BC and P is a point on AD . Let $PD = x$ and $r = PA + PB + PC$, where $0 \le x \le AD$.

(a) Suppose that $AD = 3$.

(i) Show that $\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{2x}{\sqrt{x^2+4}} - 1$.

(ii) Find the range of values of x for which

(1) r is increasing.

(2) r is decreasing.

Hence, or otherwise, find the least value of r .

(iii) Find the greatest value of r .

(b) Suppose that $AD = 1$. Find the least value of r .

(2003-AL-P MATH 2 #02)

2. (a) Let $f(x) = x^{\frac{1}{x}}$ for all $x \ge 1$. Find the greatest value of $f(x)$.

(b) Using (a) or otherwise, find a positive integer m , such that $m^{\frac{1}{m}} \ge n^{\frac{1}{n}}$ for all positive integers n .

(2004-CE-A MATH #16) (12 marks)

16.

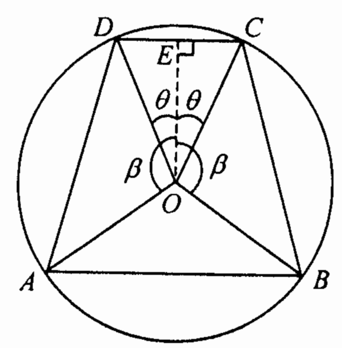


Figure 9

In Figure 9, $ABCD$ is a quadrilateral inscribed in a circle centred at O and with radius r, such that $AB // DC$ and O lies inside the quadrilateral. Let $\angle COD = 2\theta$ and reflex $\angle AOB = 2\beta$, where $0 < \theta < \frac{\pi}{2} < \beta < \pi$. Point E denotes the foot of perpendicular from O to DC. Let S be the area of $ABCD$.

(a) Show that $S = \frac{r^2}{2}\left[\sin 2\theta - \sin 2\beta + 2\sin(\beta - \theta)\right]$.

(b) Suppose β is fixed. Let S_β be the greatest value of S as θ varies.

Show that $S_\beta = 2r^2\sin^3\left(\frac{2\beta}{3}\right)$ and the corresponding value of θ is $\frac{\beta}{3}$.

(Hint: You may use the identity $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$.)

(c) A student says:

Among all possible values of β, the quadrilateral $ABCD$ becomes a square when S_β in (b) attains its greatest value.

Determine whether the student is correct or not.

(2005-CE-A MATH #18) (12 marks)

18.

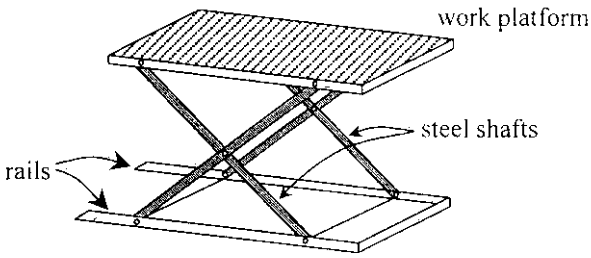


Figure 10

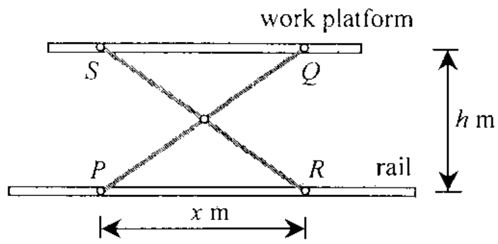


Figure 11

Figure 10 shows an elevating platform for lifting workers to work at different heights. The horizontal work platform is supported by two identical pairs of steel shafts. Figure 11 shows a cross-section of the elevating platform in a vertical plane containing one pair of shafts PQ and RS. The two shafts, each of length 4 m, are hinged at their mid-points. The ends P and R of the shafts can move along a straight horizontal rail with identical uniform speed and in opposite directions. Suppose that the elevating platform is operated under the following conditions:

(*) Initially, $PR = 3.6$ m. The work platform is the lifted upward by moving the ends P and R of the shafts towards each other such that both PR and SQ decrease at a uniform rate of $\frac{1}{2}$ ms^{-1}. Let $PR = x$ m at time t s. It is given that $0.8 \le x \le 3.6$.

In this question, numerical answers should be correct to three significant figures.

(a) Let h m be the height of the work platform above the rail at time t s.

(i) Find the range of possible values of h.

(ii) Show that $\frac{dh}{dt} = \frac{x}{2\sqrt{16 - x^2}}$.

(b) Suppose that the operation of elevating platform has to comply with the following safety regulation:

At any instant, the elevating speed of work platforms should not exceed 2 ms^{-1}.

(i) Determine whether the operation of the above elevating platform under the conditions (*) will comply with this regulation.

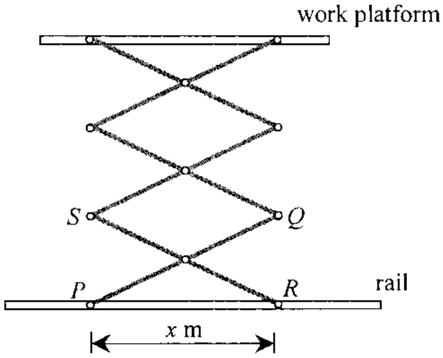


Figure 12

(ii) Figure 12 shows a vertical cross-section of a scissors-type elevating platform which can bring workers to a greater height. Two more identical pairs of shafts are installed on each side of the elevating platform as shown. Suppose that this elevating platform is operated under the same conditions (*) as described above. Do you think the operation of this elevating platform will comply with the safety regulation?

If "Yes", state your reasoning.

If "No", find the range of possible values of $\frac{dx}{dt}$ in order for the operation of this elevating platform to comply with the safety regulation.

(2006-CE-A MATH #15) (12 marks)

15.

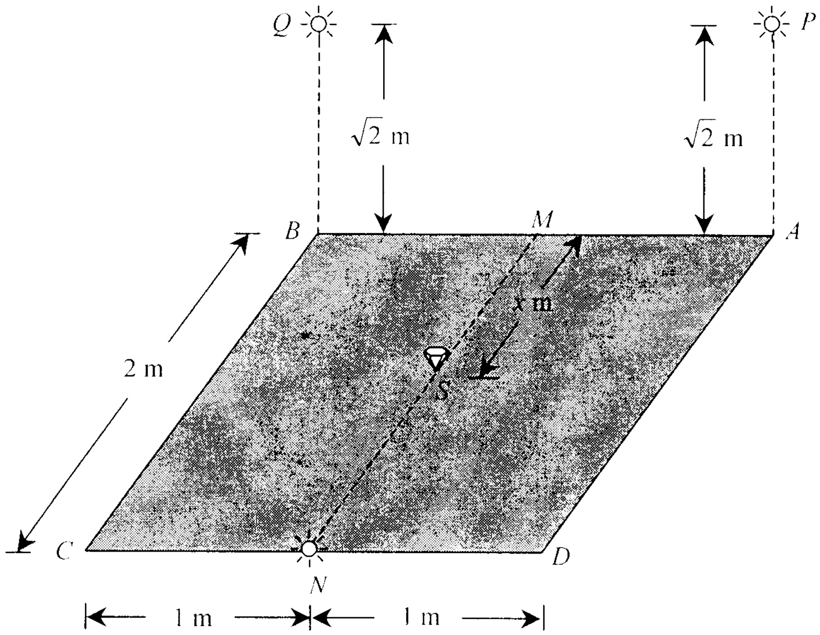


Figure 5

In Figure 5, $ABCD$ is a horizontal square board of side 2 m for displaying diamonds. Let M, N be the mid-points of BA and CD respectively. Three identical small bulbs are located at points N, P and Q respectively for illumination purpose, where P and Q are at a height $\sqrt{2}$ m vertically above A and B respectively. A diamond is placed at a point S along MN and $MS = x$ cm, where $0 \le x \le \frac{3}{2}$. Let $PS + QS + NS = \ell$ m.

(a) Express ℓ in terms of x.

Hence show that $\frac{d\ell}{dx} = \frac{2x}{\sqrt{x^2+3}} - 1$.

(b) Find the values of x at which ℓ attains

(i) the least value, and

(ii) the greatest value.

(c) Suppose that the intensity of light entry received by the diamond from each bulb varies inversely as the square of the distance of the bulb from the diamond, with k (> 0, in suitable unit) being the variation constant. Let E (in suitable unit) be the total intensity of light energy received by the diamond from the three bulbs.

(i) Express E in terms of k and x.

(ii) A student guesses that when ℓ attains its least value, E will attain its greatest value.

Explain whether the student's guess is correct or not.

(2007-CE-A MATH #09) (5 marks)

9.

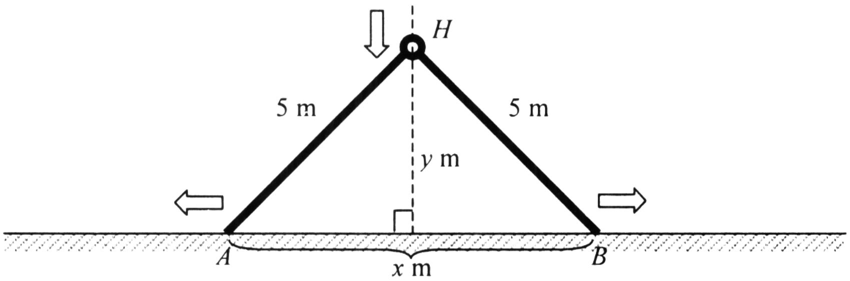


Figure 3

Two rods HA and HB, each of length 5 m, are hinged at H. The rods slide such that A, B, H are on the same vertical plane and A, B move in opposite directions on the horizontal floor, as shown in Figure 3. Let AB be x m and the distance of H from the floor be y m.

(a) Write down an equation connecting x and y.

(b) When H is 3 m from the ground, its falling speed is $2\ \text{ms}^{-1}$. Find the rate of change of the distance between A and B with respect to time at that moment.

(2007-CE-A MATH #16) (12 marks)

16.

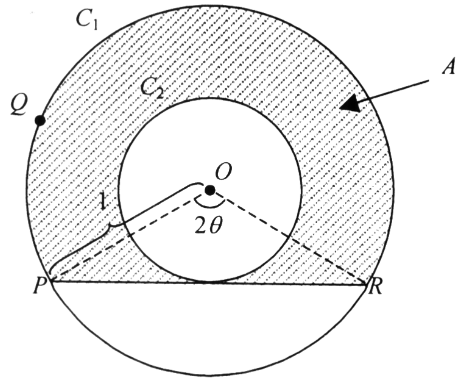


Figure 9

C_1 is a circle with centre O and radius 1. PR is a variable chord which subtends an angle 2θ at O, where $0 < \theta < \dfrac{\pi}{2}$. C_2 is a circle with centre O and touches PR. Let the area of the shaded region bounded by C_1, C_2 and PR be A (see Figure 9).

(a) Show that

(i) $A = \pi \sin^2\theta - \theta + \dfrac{1}{2}\sin 2\theta$,

(ii) $\dfrac{dA}{d\theta} = (\pi - \tan\theta)\sin 2\theta$.

(b) When A attains its greatest value, find the value of $\tan\theta$.

(c) A student guesses that when A attains its greatest value, the perimeter of the shaded region will also attain its greatest value. Explain whether the student's guess is correct or not.

(Note: the perimeter of the shaded region $= P\overset{\frown}{Q}R + PR +$circumference of C_2 .)

(2008-CE-A MATH #18) (12 marks)

18.

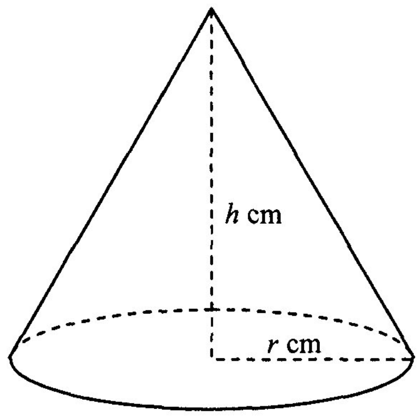


Figure 7

In a Winter Carnival, a display item is in the shape of a right circular cone. It is made of ice and a stabilizer so that the display remains in the shape of a right circular cone with the volume remaining constant. Within the duration of the Carnival, the height of the cone decreased at a constant rate of 2 cm per day. At time t days after the beginning of the Carnival, the base radius and height of the cone are r cm and h cm respectively (see Figure 7).

(a) Show that $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r}{h}$.

(b) Let S cm^2 be the curved surface area of the cone.

(i) Show that $\frac{\mathrm{d}}{\mathrm{d}t}(S^2) = \frac{2\pi^2 r^2}{h}(2r^2 - h^2)$.

(ii) At the beginning of the Carnival, the height of the cone is 1.2 times the base radius. The gatekeeper of the Carnival claims that the curved surface area of the display increases during the whole period of the Carnival. Do you agree with the gatekeeper? Explain your answer.

(2009-CE-A MATH #16) (12 marks)

16. (a) Let $f(x) = (14 - x)(x^2 + 9)$.

(i) Find the coordinates of all the maximum and minimum points of the curve $y = f(x)$.

(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 14$ in the answer book.

(Note: the suggested range of values of y is $0 \le y \le 500$.)

(b)

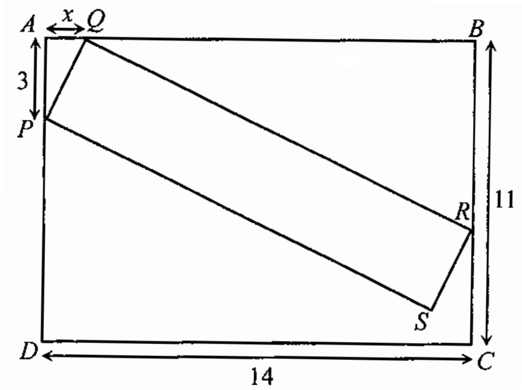


Figure 8

Figure 8 shows a rectangular cardboard $ABCD$ with $BC = 11$ and $DC = 14$. A variable rectangle $PQRS$ is cut from the cardboard according to the following rules:

(1) P is a fixed point on AD such that $AP = 3$,

(2) Q and R are points on AB and BC respectively.

Let x be the length of AQ and $g(x)$ be the area of the rectangle $PQRS$.

(i) By considering ΔAPQ and ΔBQR, express BR in terms of x.

Hence show that $g(x) = \dfrac{(14 - x)(9 + x^2)}{3}$.

(ii) By considering the fact that point S lies inside the cardboard $ABCD$, show that the range of values of x is given by

$$0 \le x \le 2 \text{ or } 12 \le x \le 14 .$$

(iii) Using (a)(ii), find the greatest value of $g(x)$ in the range shown in (b)(ii).

(2010-CE-A MATH #13) (12 marks)

13.

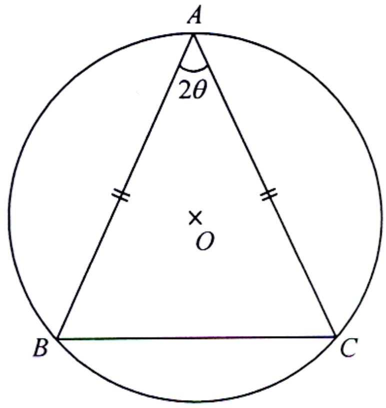


Figure 3

Figure 3 shows a circle with centre O and radius 1. A triangle ABC is inscribed in the circle with $AB = AC$. Let $\angle BAC = 2\theta$, where $0 < \theta < \frac{\pi}{4}$.

(a) Let S be the area of ΔABC .

(i) Show that $S = \frac{\sin 4\theta}{2} + \sin 2\theta$.

(ii) Find the maximum area of ΔABC .

(b)

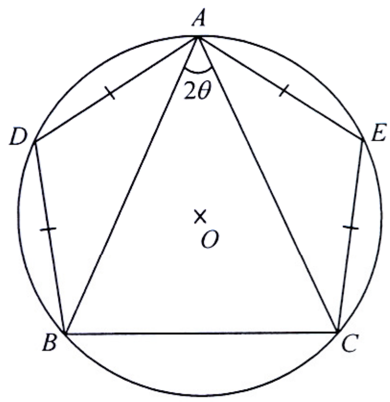


Figure 4

Two points D and E are added on the circle in Figure 3 such that $AD = BD = AE = CE$ (see Figure 4). When the area of ΔABC attains its maximum, will the area of pentagon $ADBCE$ also attain the maximum? Explain your answer.

(2011-CE-A MATH #15) (12 marks)

15. (a)

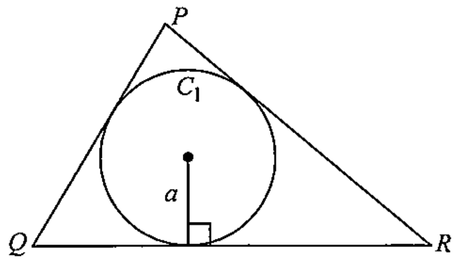


Figure 8

Figure 8 shows a triangle PQR with perimeter $2s$ and area A . A circle C_1 of radius a is inscribed in the triangle. Show that $a = \frac{A}{s}$.

(b)

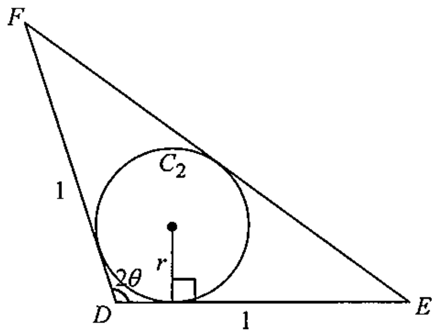


Figure 9

Figure 9 shows an isosceles triangle DEF with $DE = DF = 1$ and $\angle EDF = 2\theta$, where $0 < \theta < \frac{\pi}{2}$.

A circle C_2 of radius r is inscribed in the triangle.

(i) Using (a), show that $r = \cos\theta - \frac{\cos\theta}{1+\sin\theta}$.

(ii) Find θ , correct to 3 decimal places, which maximizes the area of C_2 .

(iii) Frankie studies the relationship between the area of C_2 and the perimeter of ΔDEF when $\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$. Frankie claims that:

"When the perimeter of ΔDEF is the least, the area of the inscribed circle is also the least."

Do you agree with Frankie? Explain your answer.

(SP-DSE-MATH-EP(M2) #02) (4 marks)

2. A snowball in a shape of sphere is melting with its volume decreasing at a constant rate of $4\text{ cm}^3\text{s}^{-1}$. When its radius is 5 cm, find the rate of change of its radius.

(2012-DSE-MATH-EP(M2) #06) (6 marks)

6.

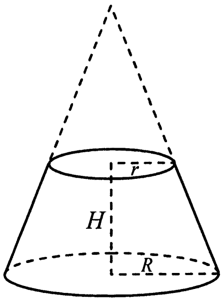


Figure 1

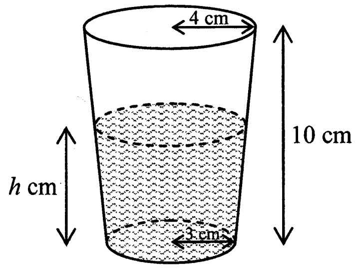


Figure 2

A frustum of height H is made by cutting off a right circular cone of base radius r from a right circular cone of base radius R (see Figure 1). It is given that the volume of the frustum is $\frac{\pi}{3}H\left(r^2 + rR + R^2\right)$. An empty glass is in the form of an inverted frustum described above with height 10 cm, the radii of the rim and the base 4 cm and 3 cm respectively. Water is being poured into the glass. Let $h\text{ cm}$ $(0 \le h \le 10)$ be the depth of the water inside the glass at time $t\text{ s}$ (see Figure 2).

(a) Show that the volume $V\text{ cm}^3$ of water inside the glass at time $t\text{ s}$ is given by

$$V = \frac{\pi}{300}(h^3 + 90h^2 + 2700h) .$$

(b) If the volume of water in the glass is increasing at the rate $7\pi\text{ cm}^3\text{s}^{-1}$, find the rate of increase of depth of water at the instant when $h = 5$.

(2013-DSE-MATH-EP(M2) #12) (13 marks)

12.

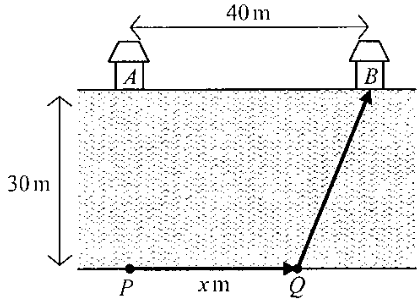


Figure 3

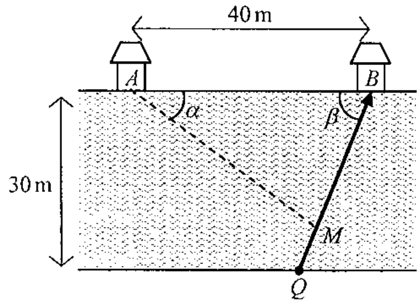


Figure 4

In Figure 3, the distance between two houses A and B lying on a straight river bank is 40 m . The width of the river is always 30 m . In the beginning , Mike stands at the starting point P in the opposite bank which is 30 m from A . Mike's wife, situated at A , is watching him running along the bank of x m at a constant speed of $7\ \text{ms}^{-1}$ to point Q then swimming at a constant speed of $1.4\ \text{ms}^{-1}$ along a straight path to teach B .

(a) Let T seconds be the time that Mike travels from P to B .

(i) Express T in terms of x .

(ii) When T is minimum, show that x satisfies the equation $2x^2 - 160x + 3125 = 0$.

Hence show that $QB = \dfrac{25\sqrt{6}}{2}$ m .

(b) In Figure 4, Mike is swimming from Q to B with QB equals to the value mentioned in (a)(ii).

Let $\angle MAB = \alpha$ and $\angle ABM = \beta$, where M is the position of Mike.

(i) By finding $\sin\beta$ and $\cos\beta$, show that $MB = \dfrac{200\tan\alpha}{\tan\alpha + 2\sqrt{6}}$.

(ii) Find the rate of change of α when $\alpha = 0.2$ radian . Correct your answer to 4 decimal places.

(2014-DSE-MATH-EP(M2) #10) (12 marks)

10.

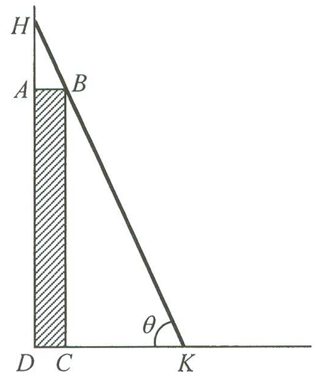


Figure 2

Thomas has a bookcase of dimensions 100 cm × 24 cm × 192 cm at the corner in his room. He want to hang a decoration on the wall above the bookcase. Therefore, he find a ladder to climb up. Initially, the ladder touches the wall, the edge of the top of the bookcase and the floor at the same time. Let rectangle $ABCD$ be the side-view of the bookcase and HK be the side-view of the ladder, so that $AB = 24$ cm and $BC = 192$ cm (see Figure 2). Let $\angle HKD = \theta$.

(a) Find the length of HK in terms of θ .

(b) Prove that the shortest length of the ladder is $120\sqrt{5}$ cm .

(c)

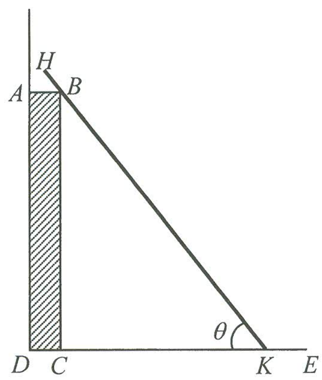


Figure 3

Suppose the length of the ladder is 270 cm . Suddenly, the ladder slides down so that the end of the ladder, K , moves towards E (see Figure 3). The ladder touches the edge of the top of the bookcase and the floor at the same time. Let x cm and y cm be the horizontal distances from H and K respectively to the wall.

(i) When $CK = 160$ cm , the rate of change of θ is -0.1 rad s^{-1}. Find the rate of change of x at this moment, correct to 4 significant figures.

(ii) Thomas claims that K is moving towards E at a speed faster than the horizontal speed H is leaving the wall. Do you agree? Explain your answer.

(2016-DSE-MATH-EP(M2) #03) (5 marks)

3. Consider the curve $C : y = 2e^x$, where $x > 0$. It is given that P is a point lying on C . The horizontal line which passes through P cuts the y-axis at the point Q . Let O be the origin. Denote the x-coordinate of P by u .

   (a) Express the area of ΔOPQ in term of u .

   (b) If P moves along C such that OQ increases at a constant rate of 6 units per second, find the rate of change of the area of ΔOPQ when $u = 4$.

(2017-DSE-MATH-EP(M2) #06) (7 marks)

6. A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.

   (a) Let A cm² be the wet curved surface area of the container and h cm be the depth of water in the container. Prove that $A = \dfrac{15}{16}\pi h^2$.

   (b) The depth of water in the container increases at a constant rate of $\dfrac{3}{\pi}$ cm/s . Find the rate of change of the wet curved surface area of the container when the volume of water in the container is 96π cm³ .

(2018-DSE-MATH-EP(M2) #09) (12 marks)

9. Consider the curve $C : y = \ln\sqrt{x}$, where $x > 1$. Let P be a moving point lying on C . The normal to C at P cuts the x-axis at the point Q while the vertical line passing through P cuts the x-axis at the point R .

   (a) Denote the x-coordinate of P by r . Prove that the x-coordinate of Q is $\dfrac{4r^2 + \ln r}{4r}$.

   (b) Find the greatest area of ΔPQR .

   (c) Let O be the origin. It is given that OP increases at a rate not exceeding $32e^2$ units per minute. Someone claims that the area of ΔPQR increases at a rate lower than 2 square units per minute when the x-coordinate of P is e . Is the claim correct? Explain your answer.

(2020-DSE-MATH-EP(M2) #06) (7 marks)

6. Consider the curve $C_1 : y = 2^{x-1}$, where $x > 0$. Denote the origin by O . Let $P(u, v)$ be a moving point on C_1 such that the area of the circle with OP as a diameter increases at a constant rate of 5π square units per second.

   (a) Define $S = u^2 + v^2$. Does S increases at a constant rate? Explain your answer.

   (b) Let C_2 be the curve $y = 2^x$, where $x > 0$. The vertical line passing through P cuts C_2 at the point Q . Find the rate of change of the area of ΔOPQ when $u = 2$.

(2021-DSE-MATH-EP(M2) #10) (13 marks)

10. Denote the graph of $y = \sqrt{x^2 + 36}$ and the graph of $y = -\sqrt{(20 - x)^2 + 16}$ by F and G respectively, where $0 < x < 20$. Let P be a moving point on F . The vertical line passing through P cuts G at the point Q . Denote the x-coordinate of P by u . It is given that the length of PQ attains its minimum value when $u = a$.

(a) Find a .

(b) The horizontal line passing through P cuts the y-axis at the point R while the horizontal line passing through Q cuts the y-axis at the point S .

(i) Someone claims that the area of the rectangle $PQSR$ attains its minimum value when $u = a$. Do you agree? Explain your answer.

(ii) The length of OP increases at a constant rate of 28 units per minute. Find the rate of change of the perimeter of the rectangle $PQSR$ when $u = a$.

ANSWERS

(1991-CE-A MATH 1 #11) (16 marks)

11. (c) (i) $h > 9$

(ii)

(d) (ii)

(1991-CE-A MATH 1 #12) (16 marks)

12. (a) (i) $CP = \cos\theta$

(b) (i) $\dfrac{d\phi}{d\theta} = \dfrac{\sin\theta}{2\sin\phi}$

(ii) $\dfrac{dS}{d\theta} = 4\sin\theta\sqrt{1-\dfrac{1}{4}\cos^2\theta}$

(c) $\dfrac{\sqrt{5}}{20}$ per second

(1992-CE-A MATH 1 #07) (7 marks)

7. (a) $V = \dfrac{\pi}{9}h^3$

(b) $\dfrac{3}{16}$ cm/s

(1992-CE-A MATH 1 #11) (16 marks)

11. (b) (ii) V is increasing when $0 < x \le 4$

V is decreasing when $4 \le x < 5$

(c) (ii) 51.0 cm^3

(d) 50.6 cm^3

(1993-CE-A MATH 1 #09) (16 marks)

9. (b) $3\sqrt{3}$

(c) (ii) No

(1994-AS-M & S #02) (5 marks)

2. $k = -1$

(1994-CE-A MATH 1 #12) (16 marks)

12. (a) $x = 4\sin\theta$

$\dfrac{d\theta}{dt} = \dfrac{1}{8\cos\theta}$

(b) $\dfrac{dy}{dt} = \dfrac{-\tan\theta}{2}$, $\dfrac{dz}{dt} = \dfrac{-2\sin\theta}{\sqrt{25-16\sin^2\theta}}$

Rate $= -0.507$ m/s

(c) Area of ΔOPR is maximum when $\theta = \dfrac{\pi}{4}$

Area of ΔORQ is maximum when $\theta = 1.08$

(1995-CE-A MATH 1 #09) (16 marks)

9. (b) $\dfrac{dS}{dt} = 2\pi$ per second

(c) (ii) Minimum value of $V = 9\pi$

S does not attain a minimum when V attains its minimum.

(1995-CE-A MATH 1 #12) (16 marks)

12. (a) $\Delta ODG \cong \Delta OCG$

(b) (ii) (1) $\dfrac{\pi}{6} < \theta < \dfrac{\pi}{4}$

(2) $\dfrac{\pi}{8} < \theta < \dfrac{\pi}{6}$

Minimum value of $S = \dfrac{\sqrt{3}}{2}\ell^2$

(c) $\left(1-\dfrac{\sqrt{3}}{2}\right)\ell^2$

(1996-CE-A MATH 1 #11) (16 marks)

11. (b) $x = 4$, $\ell = 2\sqrt{3}$

(c) (i) $\dfrac{dh}{dt} = \dfrac{-1}{10}$

(ii) $20\sqrt{3}$

(1997-CE-A MATH 1 #04) (5 marks)

4. $\dfrac{1}{80}$ per second

(1997-CE-A MATH 1 #12) (16 marks)

12. (a) $\dfrac{ds}{d\theta} = \dfrac{\cos\theta}{(1+\sin\theta)^2}$

(c) (i) $0 < \theta \le 0.340$

(ii) $0.340 \le \theta < \dfrac{\pi}{2}$

Maximum area of circle $C_2 = \dfrac{\pi}{64}$

(d) No

(1998-CE-A MATH 1 #13) (16 marks)

13. (a) $x = 2\sin\left(\dfrac{5\pi}{6} - \theta\right)$, $y = 2\sin\theta$

(b) $\theta = \dfrac{5\pi}{12}$

(d) Yes

(1999-CE-A MATH 1 #08) (7 marks)

8. (a) (i) $\tan\theta = \dfrac{4t - t^2 + 8}{11}$

(ii) $\dfrac{\pi}{4}$

(b) $\dfrac{1}{11}$ per second

(1999-CE-A MATH 1 #12) (16 marks)

12. (b) (ii) Least value of $S = 2$

Corresponding value of $x = 1$

(iii) 3

(c) (i) Incorrect

(ii) $\dfrac{9}{64}$

(1999-CE-A MATH 1 #13) (16 marks)

13. (a) $h = \dfrac{V}{\pi x^2}$

(b) $x^3 = \dfrac{V}{2\pi k}$

(c) (i) $x = 4$, $h = 16$

(ii) The base radius decreases,

the height increases.

(d) Incorrect

(2000-CE-A MATH 1 #13) (16 marks)

13. (a) $\tan\angle ARQ = \dfrac{100 - x}{100}$

(b) (ii) $\dfrac{25}{9}$ m/s

(iii) Impossible

(2002-AS-M & S #02) (5 marks)

2. (a) 173.35 m^3/h

(b) 920.49 m^3

(2002-CE-A MATH #14) (12 marks)

14. (a) (ii) (1) $\dfrac{2}{\sqrt{3}} \le x \le 3$

(2) $0 \le x \le \dfrac{2}{\sqrt{3}}$

Least value of $r = 2\sqrt{3} + 3$

(iii) $2\sqrt{13}$

(b) $2\sqrt{5}$

(2003-AL-P MATH 2 #02)

2. (a) $e^{\frac{1}{e}}$

(b) $m = 3$

(2004-CE-A MATH #16) (12 marks)

16. (c) Correct

(2005-CE-A MATH #18) (12 marks)

18. (a) (i) $1.74 \le h \le 3.92$

(b) (i) Yes

(ii) No

(2006-CE-A MATH #15) (12 marks)

15. (a) $\ell = 2\sqrt{x^2+3} + 2 - x$

(b) (i) 1

(ii) 0

(c) (i) $E = \dfrac{k}{(2-x)^2} + \dfrac{2k}{x^2+3}$

(ii) No

(2007-CE-A MATH #09) (5 marks)

9. (a) $\frac{x^2}{4}+y^2=25$

(b) 3 m/s

(2007-CE-A MATH #16) (12 marks)

16. (b) π

(c) No

(2008-CE-A MATH #18) (12 marks)

18. (b) (ii) Agreed

(2009-CE-A MATH #16) (12 marks)

16. (a) (i) Minimum point $=\left(\frac{1}{3},\frac{3362}{27}\right)$

Maximum point $=(9,450)$

(b) (i) $BR=\frac{x(14-x)}{3}$

(iii) 102

(2010-CE-A MATH #13) (12 marks)

13. (a) (ii) $\frac{3\sqrt{3}}{4}$

(b) No

(2011-CE-A MATH #15) (12 marks)

15. (b) (ii) 0.666 rad

(iii) No

(SP-DSE-MATH-EP(M2) #02) (4 marks)

2. $\frac{-1}{25\pi}$ cm/s

(2012-DSE-MATH-EP(M2) #06) (6 marks)

6. (b) $\frac{4}{7}$ cm/s

(2013-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) (i) $T=\frac{x+5\sqrt{x^2-80x+2500}}{7}$

(b) (i) $\sin\beta=\frac{2\sqrt{6}}{5}$, $\cos\beta=\frac{1}{5}$

(ii) -0.0357 rad/s

(2014-DSE-MATH-EP(M2) #10) (12 marks)

10. (a) $HK=\left(\frac{24}{\cos\theta}+\frac{192}{\sin\theta}\right)$ cm

(c) (i) 11.79 cm/s

(ii) Agreed

(2016-DSE-MATH-EP(M2) #03) (5 marks)

3. (a) ue^u

(b) 15 sq. unit per second

(2017-DSE-MATH-EP(M2) #06) (7 marks)

6. (b) 45 cm^2/s

(2018-DSE-MATH-EP(M2) #09) (12 marks)

9. (b) $\frac{1}{4e^2}$ square units

(c) $\left.\frac{dA}{dt}\right|_{r=e}\le\frac{4e}{\sqrt{4e^2+1}}<2$

(2020-DSE-MATH-EP(M2) #06) (7 marks)

6. (a) ... Yes

(b) 5 square units per second

(2021-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) $a=12$

(b) (i) The claim is disagreed.

(ii) 42 units per minutes

4. Indefinite Integration

(1979-CE-A MATH 1 #07) (5 marks) (Modified)

7. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Hence, find the indefinite integral $\int 4\sin^3\theta \, \mathrm{d}\theta$.

(1980-CE-A MATH 1 #04) (6 marks)

4. Find $\int \frac{2}{\cot\frac{x}{2}+\tan\frac{x}{2}} \, \mathrm{d}x$.

(1980-CE-A MATH 2 #02) (5 marks) (Modified)

2. Find the indefinite integral $\int (x+2)\sqrt{x-1} \, \mathrm{d}x$.

(1981-CE-A MATH 2 #01) (5 marks)

1. Find the indefinite integral $\int (1+\cos\theta)^2 \, \mathrm{d}\theta$.

(1982-CE-A MATH 2 #01) (5 marks) (Modified)

1. Find the indefinite integral $\int \frac{x}{\sqrt{x+9}} \, \mathrm{d}x$.

(1983-CE-A MATH 2 #02) (5 marks) (Modified)

2. Find the indefinite integral $\int x\sin^2(x^2) \, \mathrm{d}x$.

(1986-CE-A MATH 2 #07) (6 marks)

7. Let $y = \frac{\tan^3\theta}{3} - \tan\theta$.

   Find $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ in terms of $\tan\theta$.

   Hence, or otherwise, find $\int \tan^4\theta \, \mathrm{d}\theta$.

(1989-CE-A MATH 2 #04) (5 marks)

4. (a) Find $\int \cos^2 2x \, \mathrm{d}x$.

   (b) Using the result of (a), find $\int \sin^2 2x \, \mathrm{d}x$.

(1990-CE-A MATH 2 #03) (5 marks)

3. Using the substitution $u = \sin^2 x$, find $\displaystyle\int \frac{\sin x \cos x}{\sqrt{9\sin^2 x + 4\cos^2 x}}\, dx$.

(1990-AL-P MATH 2 #04) (Part) (3 marks)

4. (b) Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x + 2}}$.

(1992-CE-A MATH 2 #04) (6 marks)

4. The slope of the tangent to a curve C at any point (x, y) on C is $x^2 - 2$. C passes through the point $(3, 4)$.

   (a) Find an equation of C .

   (b) Find the coordinates of the point on C at which the slope of the tangent is -2 .

(1993-AL-P MATH 2 #05) (7 marks)

5. Evaluate $\displaystyle\int e^{2x}(\sin x + \cos x)^2\, dx$.

(1993-CE-A MATH 2 #06) (7 marks)

6. The slope of a curve C at any point (x, y) on C is $3x^2 - 6x - 1$. C passes through the point $(1, 0)$.

   (a) Find the equation of C .

   (b) Find the equation of the tangent to C at the point where C cuts the y-axis.

(1994-AL-P MATH 2 #02) (Modified) (6 marks)

1. (a) Evaluate $\displaystyle\int \tan^3 x\, dx$,

   (b) Let $\dfrac{x^2 - x + 2}{x(x-2)^2} \equiv \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$, where A , B , and C are constants. Find the values of A , B and C , hence find $\displaystyle\int \frac{x^2 - x + 2}{x(x-2)^2}\, dx$.

(1994-CE-A MATH 2 #01) (4 marks)

1. Find $\displaystyle\int (\sin x - \cos x)^2\, dx$.

(1994-CE-A MATH 2 #08) (7 marks)

8. The slope at any point (x, y) of a curve C is given by

$$\frac{dy}{dx} = 8 - 10x$$

and C passes through the point $A(1, 13)$.

(a) Find the equation of C .

(b) Find the equation of the normal to C at the point where C cuts the y-axis.

(1995-AL-P MATH 2 #02) (5 marks)

2. (a) Using the substitution $x = \sin^2\theta$ ($0 < \theta < \frac{\pi}{2}$) , prove that

$$\int \frac{f(x)}{\sqrt{x(1-x)}}\,dx = 2\int f(\sin^2\theta)\,d\theta$$

(b) Hence, or otherwise, evaluate $\int \frac{dx}{\sqrt{x(1-x)}}$ and $\int \sqrt{\frac{x}{1-x}}\,dx$.

(1995-CE-A MATH 2 #01) (5 marks)

1. The slope at any point (x, y) of a curve C is given by

$$\frac{dy}{dx} = 2x\sqrt{x^2+1}$$

and C cuts the y-axis at the point $(0, 1)$. Find the equation of C .

(Hint : Let $u = x^2 + 1$.)

(1996-CE-A MATH 2 #06) (6 marks)

6. The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = \tan^3 x \sec x$. If the curve passes through the origin, find its equation.

(Hint : Let $u = \sec x$.)

(1997-AL-P MATH 2 #01) (5 marks)

1. (a) Show that $\frac{d}{dx}\tan\frac{x}{2} = \frac{1}{1+\cos x}$.

(b) Using (a), or otherwise, find $\int \frac{x+\sin x}{1+\cos x}\,dx$.

(1997-CE-A MATH 2 #02) (4 marks)

2. Find $\int x\sqrt{x-1}\,dx$. (Hint : Let $u = x - 1$.)

(1997-CE-A MATH 2 #05) (5 marks)

5. The slope at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = 6x + \frac{1}{x^2} \ ,$$

where $x > 0$. If the curve cuts the x-axis at the point $(1, 0)$, find its equation.

(1998-CE-A MATH 2 #04) (5 marks)

4. The slope at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = \cos^2 x \ .$$

If the curve passes through the point $\left(\frac{\pi}{2}, \pi\right)$, find its equation.

(1999-CE-A MATH 2 #02) (4 marks)

2. Evaluate $\int x(x+2)^{99}\,dx$.

(1999-CE-A MATH 2 #06) (6 marks)

6. The slope at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = 3x^2 - 2x + k \ .$$

If the curve touches the x-axis at the point $(2, 0)$, find

(a) the value of k ,

(b) the equation of the curve.

(2000-CE-A MATH 2 #01) (4 marks)

1. Find $\int \sqrt{2x+1}\,dx$.

(2000-CE-A MATH 2 #06) (7 marks)

6.

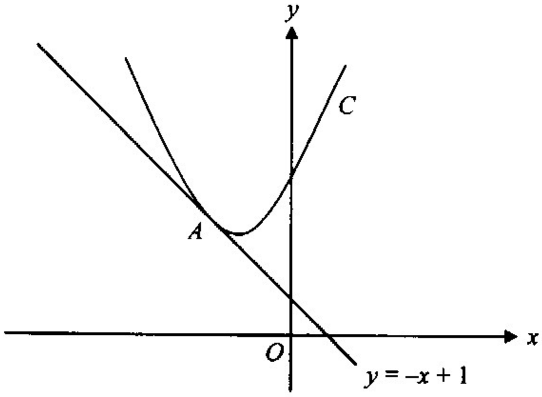


Figure 3

The slope at any point (x, y) of a curve C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 3$. The line $y = -x + 1$ is a tangent to the curve at point A . (See Figure 3.) Find

(a) the coordinates of A ,

(b) the equation of C .

(2001-AL-P MATH 2 #02) (5 marks)

2. Evaluate

(a) $\int \frac{x^3}{1+x^2}\,\mathrm{d}x$.

(b) $\int x^2 \tan^{-1} x\,\mathrm{d}x$.

(2001-AL-P MATH 2 #12) (Part / Modified) (6 marks)

12. (a) (i) Evaluate $\displaystyle\int \frac{1}{x^2-x+1}\,\mathrm{d}x$,

(ii) Let $\dfrac{x^2+1}{(x^2-x+1)(x^2+x+1)} \equiv \dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+x+1}$, where A , B , C and D are constants. Find A , B , C and D . Hence, evaluate $\displaystyle\int \frac{x^2+1}{(x^2-x+1)(x^2+x+1)}\,\mathrm{d}x$.

(2001-CE-A MATH #02) (4 marks)

2. Find $\displaystyle\int \frac{x}{\sqrt{3x^2+1}}\,\mathrm{d}x$. (Hint : Let $u = 3x^2+1$.)

(2003-CE-A MATH #01) (3 marks)

1. Find $\displaystyle\int \cos^2\theta\,\mathrm{d}\theta$.

(2004-CE-A MATH #01) (4 marks)

1. Find

(a) $\displaystyle\int \cos(3x+1)\,\mathrm{d}x$,

(b) $\displaystyle\int (2-x)^{2004}\,\mathrm{d}x$.

(2004-CE-A MATH #03) (4 marks)

3. The slope at any point (x, y) of a curve C is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2+1$. If the x-intercept of C is 1 , find the equation of C .

(2005-CE-A MATH #01) (2 marks)

1. Find $\displaystyle\int (2x-3)^7\,\mathrm{d}x$.

(2005-CE-A MATH #10) (6 marks)

10. (a) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x(x+1)^n\right] = (x+1)^{n-1}\left[(n+1)x+1\right]$, where n is a rational number.

(b) The slope at any point (x, y) of a curve C is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x+1)^{2004}(2006x+1)$. If C passes through the point $(-1,1)$, find the equation of C .

(2006-AL-P MATH 2 #04) (Part)

4. (a) Using the substitution $t = \sqrt{1+x^2}$, find $\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x$.

(2006-CE-A MATH #10) (5 marks)

10. The slope at any point (x, y) of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 + 2\cos 2x$. If the curve passes through the point $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, find its equation.

(2007-AL-P MATH 2 #04) (Part)

4. (a) Using integration by parts, find $\int e^x \sin x\,\mathrm{d}x$.

(2007-CE-A MATH #01) (3 marks)

1. Find $\int \frac{x^4+1}{x^2}\,\mathrm{d}x$.

(2008-CE-A MATH #01) (2 marks)

1. Find $\int (8x+5)^{250}\,\mathrm{d}x$.

(2009-CE-A MATH #01) (5 marks)

1. Find

(a) $\int (4x+1)^2\,\mathrm{d}x$,

(b) $\int \sin 3\theta \cos\theta\,\mathrm{d}\theta$.

(2011-CE-A MATH #05) (5 marks)

5. (a) Find $\int (2x+1)^2\,\mathrm{d}x$.

(b) The slope at any point (x, y) of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x+1)^2$. If the curve passes through the point $(-1,0)$, find its equation.

(SP-DSE-MATH-EP(M2) #03) (4 marks)

3. The slope at any point (x, y) of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\ln(x^2+1)$. It is given that the curve passes through the point $(0,1)$. Find the equation of the curve.

(SP-DSE-MATH-EP(M2) #04) (4 marks)

4. Find $\int \left(x^2 - \frac{1}{x}\right)^4 \mathrm{d}x$.

(PP-DSE-MATH-EP(M2) #08) (5 marks)

8. (a) Using integration by substitution, find $\int \frac{\mathrm{d}x}{\sqrt{4 - x^2}}$.

(b) Using integration by parts, find $\int \ln x \, \mathrm{d}x$.

(2012-DSE-MATH-EP(M2) #04) (5 marks)

4. (a) Find $\int \frac{x + 1}{x} \mathrm{d}x$.

(b) Using the substitution $u = x^2 - 1$, find $\int \frac{x^3}{x^2 - 1} \mathrm{d}x$.

(2013-DSE-MATH-EP(M2) #04) (5 marks)

4. The slope at any point (x, y) of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x - 1$. It is given that the curve passes through the point $(1, e)$.

(a) Find the equation of the curve.

(b) Find the equation of tangent to the curve at the point where the curve cuts the y-axis.

(2014-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) Find $\int \frac{\mathrm{d}x}{\sqrt{9 - x}}$, where $x < 9$.

(b) Using integration by substitution, find $\int \frac{\mathrm{d}x}{\sqrt{9 - x^2}}$, where $-3 < x < 3$.

(2015-DSE-MATH-EP(M2) #04) (7 marks)

4. (a) Using integration by parts, find $\int x^2 \ln x \, \mathrm{d}x$.

(b) At any point (x, y) on the curve Γ , the slope of the tangent to Γ is $9x^2 \ln x$. It is given that Γ passes through the point $(1, 4)$. Find the equation of Γ .

(2017-DSE-MATH-EP(M2) #08) (8 marks)

8. Let $f(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. Denote the curve $y = f(x)$ by Γ. It is given that Γ passes through the point $P(e^3, 7)$ and $f'(x) = \dfrac{1}{x}\ln x^2$ for all $x > 0$. Find

(a) the equation of the tangent to Γ at P,

(b) the equation of Γ,

(c) the point(s) of inflexion of Γ.

(2018-DSE-MATH-EP(M2) #05) (7 marks)

5. (a) Using integration by substitution, find $\displaystyle\int x^3\sqrt{1+x^2}\,\mathrm{d}x$.

(b) At any point (x, y) on the curve Γ, the slope of the tangent to Γ is $15x^3\sqrt{1+x^2}$. The y-intercept of Γ is 2. Find the equation of Γ.

(2019-DSE-MATH-EP(M2) #03) (6 marks)

3. A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580 cm³ of liquid X. The researcher finds that during the experiment, $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -2t$, where V cm³ is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

(a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

(b) It is given that $V = h^2 + 24h$, where h cm is the depth of liquid X in the vessel. Find the value of $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ when $t = 18$.

(2019-DSE-MATH-EP(M2) #08) (8 marks)

8. Let $h(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. It is given that $h'(x) = \dfrac{2x^2 - 7x + 8}{x}$ for all $x > 0$.

(a) Is $h(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y = h(x)$ by H. It is given that H passes through the point $(1,3)$. Find

(i) the equation of H,

(ii) the point(s) of inflexion of H.

(2020-DSE-MATH-EP(M2) #07) (8 marks)

7. Let $f(x)$ be a continuous function defined on $\mathbf{R}$. Denote the curve $y = f(x)$ be Γ . It is given that Γ passes through the point $(1, 2)$ and $f'(x) = -2x + 8$ for all $x \in \mathbf{R}$.

   (a) Find the equation of Γ .

   (b) Let L be a tangent to Γ such that L passes through the point $(5, 14)$ and the slope of L is negative. Denote the point of contact of Γ and L by P . Find

      (i) the coordinates of P ,

      (ii) the equation of the normal to Γ at P .

ANSWERS

(1979-CE-A MATH 1 #07) (5 marks) (Modified)

7. $-3\cos\theta+\dfrac{1}{3}\cos 3\theta$+constant

(1980-CE-A MATH 1 #04) (6 marks)

4. $-\cos x$+constant

(1980-CE-A MATH 2 #02) (5 marks) (Modified)

2. $\dfrac{2}{5}(x-1)^{\frac{5}{2}}+2(x-1)^{\frac{3}{2}}$+constant

(1981-CE-A MATH 2 #01) (5 marks)

1. $\dfrac{3}{2}\theta+2\sin\theta+\dfrac{1}{4}\sin 2\theta$+constant

(1982-CE-A MATH 2 #01) (5 marks) (Modified)

1. $\dfrac{2}{3}(x+9)^{\frac{3}{2}}-18(x+9)^{\frac{1}{2}}$+constant

(1983-CE-A MATH 2 #02) (5 marks) (Modified)

2. $\dfrac{x^2}{4}-\dfrac{1}{8}\sin 2x^2$+constant

(1986-CE-A MATH 2 #07) (6 marks)

7. $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\tan^4\theta-1$

$$\int\tan^4\theta\,\mathrm{d}\theta=\frac{\tan^3\theta}{3}-\tan\theta+\theta+\text{constant}$$

(1989-CE-A MATH 2 #04) (5 marks)

4. (a) $\dfrac{1}{2}x+\dfrac{1}{8}\sin 4x$+constant

   (b) $\dfrac{1}{2}x-\dfrac{1}{8}\sin 4x$+constant

(1990-CE-A MATH 2 #03) (5 marks)

3. $\dfrac{1}{5}\sqrt{5\sin^2 x+4}$+constant

(1990-AL-P MATH 2 #04) (Part) (3 marks)

4. (b) $\ln\left(x+2+\sqrt{x^2+4x+2}\right)$+constant

(1992-CE-A MATH 2 #04) (6 marks)

4. (a) $y=\dfrac{1}{3}x^3-2x+1$

   (b) (0,1)

(1993-AL-P MATH 2 #05) (7 marks)

5. $\dfrac{1}{2}e^{2x}+\dfrac{1}{4}e^{2x}(\sin 2x-\cos 2x)$+constant

(1993-CE-A MATH 2 #06) (7 marks)

6. (a) $y=x^3-3x^2-x+3$

   (b) $y=-x+3$

(1994-AL-P MATH 2 #02) (Modified) (6 marks)

1. (a) $\dfrac{1}{2}\tan^2 x+\ln|\cos x|$+constant

   (b) $\dfrac{1}{2}\ln|x|+\dfrac{1}{2}\ln|x-2|-\dfrac{2}{x-2}$+constant

(1994-CE-A MATH 2 #01) (4 marks)

1. $x-\sin^2 x$+constant

(1994-CE-A MATH 2 #08) (7 marks)

8. (a) $y=8x-5x^2+10$

   (b) $y=\dfrac{-1}{8}x+10$

(1995-AL-P MATH 2 #02) (5 marks)

2. (b) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x(1-x)}}=2\sin^{-1}\sqrt{x}$+constant

$$\int\sqrt{\frac{x}{1-x}}\,\mathrm{d}x=\sin^{-1}\sqrt{x}-\sqrt{x(1-x)}+\text{constant}$$

(1995-CE-A MATH 2 #01) (5 marks)

1. $y=\dfrac{2}{3}(x^2+1)^{\frac{3}{2}}+\dfrac{1}{3}$

(1996-CE-A MATH 2 #06) (6 marks)

6. $y=\dfrac{1}{3}\sec^3 x-\sec x+\dfrac{2}{3}$

(1997-AL-P MATH 2 #01) (5 marks)

1. (b) $x\tan\frac{x}{2}+\text{constant}$

(1997-CE-A MATH 2 #02) (4 marks)

2. $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+\text{constant}$

(1997-CE-A MATH 2 #05) (5 marks)

5. $y=3x^2-\frac{1}{x}-2$

(1998-CE-A MATH 2 #04) (5 marks)

4. $y=\frac{x}{2}+\frac{\sin 2x}{4}+\frac{3\pi}{4}$

(1999-CE-A MATH 2 #02) (4 marks)

2. $\frac{(x+2)^{101}}{101}-\frac{(x+2)^{100}}{50}+\text{constant}$

(1999-CE-A MATH 2 #06) (6 marks)

6. (a) $k=-8$

   (b) $y=x^3-x^2-8x+12$

(2000-CE-A MATH 2 #01) (4 marks)

1. $\frac{1}{3}(2x+1)^{\frac{3}{2}}+\text{constant}$

(2000-CE-A MATH 2 #06) (7 marks)

6. (a) $A=(-2,3)$

   (b) $y=x^2+3x+5$

(2001-AL-P MATH 2 #02) (5 marks) (Modified)

2. (a) (i) $\frac{x^2}{2}-\frac{1}{2}\ln(1+x^2)+\text{constant}$

   (ii) $\tan^{-1}x+\text{constant}$

   (b) $\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\ln(1+x^2)+\text{constant}$

(2001-AL-P MATH 2 #12) (Part / Modified) (6 marks)

12. (a) (i) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+\text{constant}$

   (ii) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\text{constant}$

(2001-CE-A MATH #02) (4 marks)

2. $\frac{1}{3}(3x^2+1)^{\frac{1}{2}}+\text{constant}$

(2003-CE-A MATH #01) (3 marks)

1. $\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+\text{constant}$

(2004-CE-A MATH #01) (4 marks)

1. (a) $\frac{1}{3}\sin(3x+1)+\text{constant}$

   (b) $\frac{-(2-x)^{2005}}{2005}+\text{constant}$

(2004-CE-A MATH #03) (4 marks)

3. $y=x^3+x-2$

(2005-CE-A MATH #01) (2 marks)

1. $\frac{1}{16}(2x-3)^8+\text{constant}$

(2005-CE-A MATH #10) (6 marks)

10. (b) $y=x(x+1)^{2005}+1$

(2006-AL-P MATH 2 #04) (Part)

4. (a) $\frac{1}{3}(1+x^2)^{\frac{3}{2}}-(1+x^2)^{\frac{1}{2}}+\text{constant}$

(2006-CE-A MATH #10) (5 marks)

10. $y=3x+\sin 2x-1$

(2007-AL-P MATH 2 #04) (Part)

4. (a) $\frac{e^x}{2}(\sin x-\cos x)+\text{constant}$

(2007-CE-A MATH #01) (3 marks)

1. $\frac{x^3}{3} - \frac{1}{x}$+constant

(2008-CE-A MATH #01) (2 marks)

1. $\frac{(8x+5)^{251}}{2008}$+constant

(2009-CE-A MATH #01) (5 marks)

1. (a) $\frac{(4x+1)^3}{12}$+constant

   (b) $\frac{-\cos 4\theta}{6} - \frac{\cos 2\theta}{4}$+constant

(2011-CE-A MATH #05) (5 marks)

5. (a) $\frac{(2x+1)^3}{6}$+constant

   (b) $y = \frac{(2x+1)^3+1}{6}$

(SP-DSE-MATH-EP(M2) #03) (4 marks)

3. $y = (x^2+1)\ln(x^2+1) - x^2 + 1$

(SP-DSE-MATH-EP(M2) #04) (4 marks)

4. $\frac{x^9}{9} - \frac{2x^6}{3} + 2x^3 - 4\ln|x| - \frac{1}{3x^3}$+constant

(PP-DSE-MATH-EP(M2) #08) (5 marks)

8. (a) $\sin^{-1}\frac{x}{2}$+constant

   (b) $x\ln x - x$+constant

(2012-DSE-MATH-EP(M2) #04) (5 marks)

4. (a) $x + \ln|x|$+constant

   (b) $\frac{1}{2}(x^2-1) + \frac{1}{2}\ln\left|x^2-1\right|$+constant

(2013-DSE-MATH-EP(M2) #04) (5 marks)

4. (a) $y = e^x - x + 1$

   (b) $y = 2$

(2014-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) $-2\sqrt{9-x}$+constant

   (b) $\sin^{-1}\frac{x}{3}$+constant

(2015-DSE-MATH-EP(M2) #04) (7 marks)

4. (a) $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3$+constant

   (b) $y = 3x^3\ln x - x^3 + 5$

(2017-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $6x - e^3y + e^3 = 0$

   (b) $y = (\ln x)^2 - 2$

   (c) $(e, -1)$

(2018-DSE-MATH-EP(M2) #05) (7 marks)

5. (a) $\frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}(1+x^2)^{\frac{3}{2}}$+constant

   (b) $y = 3(1+x^2)^{\frac{5}{2}} - 5(1+x^2)^{\frac{3}{2}} + 4$

(2019-DSE-MATH-EP(M2) #03) (6 marks)

3. (a) Correct

   (b) -0.9

(2019-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $h(x)$ is an increasing function

   (b) (i) $y = x^2 - 7x + 8\ln|x| + 9$

   (ii) $(2, 8\ln 2 - 1)$

(2020-DSE-MATH-EP(M2) #07) (8 marks)

7. (a) $y = -x^2 + 8x - 5$

   (b) (i) $(7, 2)$

   (ii) $x - 6y + 5 = 0$

5. Definite Integration

(1979-CE-A MATH 2 #07) (20 marks)

7. (a) Evaluate $\displaystyle\int_1^5 \frac{x}{\sqrt{4x+5}}\,\mathrm{d}x$.

(b) Given that $x^2 + xy + y^2 = a^2$, where $a \neq 0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and deduce that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{3x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y}{(x+2y)^2} \ .$$

Hence evaluate $(x+2y)^3 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

(1980-CE-A MATH 2 #12) (20 marks)

12. (a) Given that $\mathrm{f}(x) = \mathrm{f}(a-x)$ for all real values of x, by using the substitution $u = a - x$, show that

$$\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x = a\int_0^a \mathrm{f}(u)\,\mathrm{d}u - \int_0^a u\,\mathrm{f}(u)\,\mathrm{d}u \ .$$

Hence deduce that

$$\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x = \frac{a}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x \ .$$

(b) By using the substitution $u = x - \dfrac{\pi}{2}$, show that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^4 u}{\sin^4 u + \cos^4 u}\,\mathrm{d}u \ .$$

By using this result and

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x + \int_c^b \mathrm{f}(x)\,\mathrm{d}x \ ,$$

evaluate

$$\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x \ .$$

(c) Using (a) and (b), evaluate

$$\int_0^{\pi} \frac{x\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x \ .$$

(1981-CE-A MATH 2 #03) (6 marks) (Modified)

3. Evaluate $\displaystyle\int_0^9 \frac{x}{\sqrt{9-x}}\,\mathrm{d}x$.

(1981-CE-A MATH 2 #08) (20 marks) (Modified)

8. (a) Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x \, dx$.

(b) (i) Show that $\frac{1}{x^2+3} - \frac{1}{(x+1)^2} \equiv \frac{2(x-1)}{(x^2+3)(x+1)^2}$ for $x \neq -1$.

(ii) Using the substitution $x = \sqrt{3} \tan \theta$, show that $\int_0^3 \frac{dx}{x^2+3} = \frac{\pi\sqrt{3}}{9}$.

(iii) Using the results of (i) and (ii), evaluate $\int_0^3 \frac{2(x-1)}{(x^2+3)(x+1)^2} dx$.

(1982-CE-A MATH 2 #05) (6 marks) (Modified)

5. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos x + \sin x} dx$. Evaluate I .

(1983-CE-A MATH 2 #03) (5 marks) (Modified)

3. Evaluate $\int_0^1 x^3 \sqrt{1+3x^2} \, dx$.

(1983-CE-A MATH 2 #11) (20 marks)

11. (a) Show that $\frac{\sin 3\theta}{\sin \theta} = 2\cos 2\theta + 1$.

By putting $\theta = \frac{\pi}{4} + \phi$ in the above identity, show that

$$\frac{\cos 3\phi - \sin 3\phi}{\cos \phi + \sin \phi} = 1 - 2\sin 2\phi \ .$$

(b) Using the substitution $\phi = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos \phi + \sin \phi} d\phi = \int_{\frac{\pi}{2}}^0 \frac{\sin 3u}{\cos u + \sin u} du \ .$$

Hence, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos \phi + \sin \phi} d\phi = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos 3\phi - \sin 3\phi}{\cos \phi + \sin \phi} d\phi.$$

(c) Using the results in (a) and (b), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos \phi + \sin \phi} d\phi \ .$$

(1983-CE-A MATH 2 #12) (20 marks)

12. Let $\mathrm{f}(x)$ be a function of x and let k and s be constants.

(a) By using the substitution $y = x + ks$, show that

$$\int_0^s \mathrm{f}(x + ks)\,\mathrm{d}x = \int_{ks}^{(k+1)s} \mathrm{f}(x)\,\mathrm{d}x \ .$$

Hence show that, for any positive integer n ,

$$\int_0^s \left[\mathrm{f}(x) + \mathrm{f}(x + s) + \cdots + \mathrm{f}\left(x + (n-1)s\right)\right] \mathrm{d}x = \int_0^{ns} \mathrm{f}(x)\,\mathrm{d}x \ .$$

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{\mathrm{d}x}{\sqrt{1 - x^2}}$ by using the substitution $x = \sin\theta$.

Using the result together with (a), evaluate

$$\int_0^{\frac{1}{2n}} \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1 - \left(x + \frac{1}{2n}\right)^2}} + \frac{1}{\sqrt{1 - \left(x + \frac{2}{2n}\right)^2}} + \cdots + \frac{1}{\sqrt{1 - \left(x + \frac{n-1}{2n}\right)^2}} \right) \mathrm{d}x \ .$$

(1984-CE-A MATH 2 #05) (8 marks) (Modified)

5. By considering $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan^3\theta)$, find $\displaystyle\int \tan^2\theta \sec^2\theta \,\mathrm{d}\theta$.

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4\theta \,\mathrm{d}\theta$.

(1984-CE-A MATH 2 #07) (20 marks)

7. (a) Prove that $\dfrac{1}{x^3} + \dfrac{3}{(2-3x)^2} = \dfrac{3x^3 + 9x^2 - 12x + 4}{9x^5 - 12x^4 + 4x^3}$.

Hence find the value of $\displaystyle\int_1^2 \frac{3x^3 + 9x^2 - 12x + 4}{9x^5 - 12x^4 + 4x^3}\,\mathrm{d}x$.

(b) (i) Find $\displaystyle\int \frac{\cos\phi}{\sin^4\phi}\,\mathrm{d}\phi$.

(ii) Using the substitution $x = \tan\phi$ and the result of (i), evaluate $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{1} \frac{3\sqrt{1+x^2}}{x^4}\,\mathrm{d}x$.

(1985-CE-A MATH 2 #03) (5 marks) (Modified)

3. Evaluate $\displaystyle\int_3^4 \frac{x}{\sqrt{25-x^2}}\,dx$.

(1985-CE-A MATH 2 #08) (20 marks)

8. (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt$.

(b) By using the substitution $t = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \cos^3 t \sin^4 t \, dt = \int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt \ .$$

(c) Show that $\displaystyle\int_{-\frac{\pi}{2}}^{0} \cos^3 t \sin^4 t \, dt = \int_0^{\frac{\pi}{2}} \cos^3 t \sin^4 t \, dt$ and $\displaystyle\int_{-\frac{\pi}{2}}^{0} \sin^3 t \cos^4 t \, dt = -\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt$.

(d) Using the above results, or otherwise, evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{8} \sin^3 2t \, (\sin t + \cos t) \, dt \ .$$

(1986-CE-A MATH 2 #08) (20 marks)

8. (a) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(b) Using the result in (a), or otherwise, evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi} \cos^{2n+1} x \, dx$, where n is a positive integer,

(ii) $\displaystyle\int_0^{\pi} x \sin^2 x \, dx$,

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x \, dx}{\sin x + \cos x}$.

(1987-CE-A MATH 2 #04) (6 marks)

4. Using the substitution $x = \sin\theta$, evaluate $\int_0^{\frac{1}{2}} \frac{2x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$.

(1987-CE-A MATH 2 #08) (20 marks)

8. (a) Using the substitution $u = \tan x$, find

$$\int \tan^{n-2}x \sec^2 x\,\mathrm{d}x ,$$

where n is an integer and $n \geq 2$.

(b) (i) By writing $\tan^n x$ as $\tan^{n-2}x \tan^2 x$, show that

$$\int_0^{\frac{\pi}{4}} \tan^n x\,\mathrm{d}x = \frac{1}{n-1} - \int_0^{\frac{\pi}{4}} \tan^{n-2}x\,\mathrm{d}x ,$$

where n is an integer and $n \geq 2$.

(ii) Evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\,\mathrm{d}x$.

(c) Show that $\int_{-\frac{\pi}{4}}^{0} \tan^6 x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^6 x\,\mathrm{d}x$.

Hence evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^6 x\,\mathrm{d}x$.

(1988-CE-A MATH 2 #06) (6 marks)

6. Evaluate $\int_0^2 \frac{x^2 \, dx}{\sqrt{9 - x^3}}$.

(1988-CE-A MATH 2 #08) (20 marks)

8. (a) Using the substitution $u = \sin x$, evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$.

Leave the answer as a fraction.

(b) Let $y = \sin x \cos^{2n-1} x$, where n is a positive integer.

Find $\frac{dy}{dx}$.

Hence show that

$$2n \int \cos^{2n} x \, dx - (2n-1) \int \cos^{2n-2} x \, dx = \sin x \cos^{2n-1} x + C ,$$

where C is a constant.

(c) (i) Using (b), show that

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos^{2n-2} x \, dx ,$$

where n is a positive integer.

(ii) Evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$ in terms of π.

(d) Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$ in terms of π.

(1989-CE-A MATH 2 #03) (5 marks) (Modified)

3. Evaluate $\displaystyle\int_0^2 \frac{8x^3}{\sqrt{2x^2+1}}\,\mathrm{d}x$.

(1989-CE-A MATH 2 #09) (16 marks)

9. Let n be an integer greater than 1 .

(a) Using the substitution $x = \tan\theta$, evaluate $\displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x^2}$.

(b) By differentiating $\dfrac{x}{(1+x^2)^{n-1}}$ with respect to x , show that

$$\int \frac{x^2}{(1+x^2)^n}\,\mathrm{d}x = \frac{1}{2(n-1)}\left[\int \frac{\mathrm{d}x}{(1+x^2)^{n-1}} - \frac{x}{(1+x^2)^{n-1}}\right] .$$

(c) Using the identity $\dfrac{1}{(1+x^2)^n} \equiv \dfrac{1}{(1+x^2)^{n-1}} - \dfrac{x^2}{(1+x^2)^n}$, show that

$$\int \frac{\mathrm{d}x}{(1+x^2)^n} = \frac{2n-3}{2n-2}\int \frac{\mathrm{d}x}{(1+x^2)^{n-1}} + \frac{1}{2(n-1)} \cdot \frac{x}{(1+x^2)^{n-1}} .$$

(d) Using the above results or otherwise, evaluate

(i) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}$,

(ii) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^3}$.

(1990-CE-A MATH 2 #09) (16 marks)

9. (a) (i) Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

(ii) Using the substitution $x = \pi - y$,

evaluate $\int_0^{\pi} x\cos^2 x \, dx$.

(b) Show that

(i) $\int_{\pi}^{2\pi} x\cos^2 x \, dx = \pi \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x\cos^2 x \, dx$.

(ii) $\int_0^{2\pi} x\cos^2 x \, dx = \pi^2$.

(c) Using the result of (b) (ii),

evaluate $\int_0^{\sqrt{2\pi}} x^3\cos^2 x^2 \, dx$.

(1990-AL-P MATH 2 #03) (5 marks)

3. Suppose $f(x)$ and $g(x)$ are real-valued continuous functions on $[0,a]$ satisfying the conditions that $f(x) = f(a-x)$ and $g(x) + g(a-x) = K$ where K is a constant.

Show that $\int_0^{a} f(x)\,g(x)\,dx = \frac{K}{2}\int_0^{a} f(x)\,dx$. Hence, or otherwise, evaluate $\int_0^{\pi} x \sin x \, \cos^4 x \, dx$.

(1991-CE-A MATH 2 #02) (5 marks)

2. Evaluate $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$.

(1991-CE-A MATH 2 #12) (16 marks)

12. Let m , n be positive integers.

(a) Given that $y = (1+x)^{m+1}(1-x)^n$. Find $\frac{dy}{dx}$.

Hence show that

$$(m+1)\int (1+x)^m(1-x)^n \, dx = (1+x)^{m+1}(1-x)^n + n\int (1+x)^{m+1}(1-x)^{n-1} \, dx \ .$$

(b) Using the result of (a), show that

$$\int_{-1}^{1} (1+x)^m (1-x)^n \, dx = \frac{n}{m+1}\int_{-1}^{1} (1+x)^{m+1} (1-x)^{n-1} \, dx \ .$$

(c) Without using a binomial expansion, evaluate

$$\int_{-1}^{1} (1+x)^8 \, dx \ .$$

(d) Using the substitution $x = \tan\theta$, show that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 2\theta (1+\tan\theta)^4}{\cos^6\theta} \, d\theta = \int_{-1}^{1} (1+x)^6 (1-x)^2 \, dx \ .$$

Hence, using the results of (b) and (c), evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 2\theta (1+\tan\theta)^4}{\cos^6\theta} \, d\theta \ .$$

(1992-CE-A MATH 2 #08) (16 marks)

8. (a) Let $y = \dfrac{\sin x}{2+\cos x}$.

Show that $\dfrac{dy}{dx} = \dfrac{2}{2+\cos x} - \dfrac{3}{(2+\cos x)^2}$.

(b) Using the substitution $t = \sqrt{3}\tan\theta$, evaluate

$$\int_0^1 \frac{dt}{t^2+3} .$$

(c) Using the substitution $t = \tan\dfrac{x}{2}$ and the result of (b), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} .$$

(d) Using the results of (a) and (c), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(2+\cos x)^2} .$$

(1993-CE-A MATH 2 #09) (16 marks)

9. Let m , n be integers such that $m > 1$ and $n \geq 0$.

(a) Find $\dfrac{d}{dx}(\sin^{m-1}x\cos^{n+1}x)$.

(b) Using the result of (a), show that

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2}x\cos^n x\,dx .$$

(c) Using the result of (b) and the substitution $x = \dfrac{\pi}{2} - y$, show that

$$\int_0^{\frac{\pi}{2}} \sin^n x\cos^m x\,dx = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^n x\cos^{m-2}x\,dx .$$

(d) Using the results of (b) and (c), evaluate

$$\int_0^{\frac{\pi}{2}} \sin^4 x\cos^6 x\,dx .$$

(1994-CE-A MATH 2 #10) (16 marks)

10. (a) Using the substitution $x = \tan\theta$, evaluate

$$\int_0^1 \frac{\mathrm{d}x}{1+x^2} \ .$$

(b) Given $-\pi < x < \pi$ and $t = \tan\frac{x}{2}$. By expressing $\sin x$ and $\cos x$ in terms of t , show that

$$3 + 2\sin x + \cos x = \frac{2(2+2t+t^2)}{1+t^2} \ .$$

Hence show that

$$\int \frac{\mathrm{d}x}{3+2\sin x+\cos x} = \int \frac{\mathrm{d}t}{1+(1+t)^2} \ .$$

(c) Using (b), evaluate

$$\int_{\frac{-\pi}{2}}^{0} \frac{\mathrm{d}x}{3+2\sin x+\cos x} \ .$$

(d) Using the result of (c), evaluate

$$\int_{\frac{-\pi}{2}}^{0} \frac{(2\sin x+\cos x)\,\mathrm{d}x}{3+2\sin x+\cos x} \ .$$

(1995-CE-A MATH 2 #08) (16 marks)

8. Let n be an integer greater than 1 .

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n-1}(1-x^2)^{\frac{3}{2}}\right] = (n-1)x^{n-2}\sqrt{1-x^2} - (n+2)x^n\sqrt{1-x^2} \ .$$

(b) Using (a), show that

$$\int_0^1 x^n\sqrt{1-x^2}\,\mathrm{d}x = \frac{n-1}{n+2}\int_0^1 x^{n-2}\sqrt{1-x^2}\,\mathrm{d}x \ .$$

(c) Using the substitution $x = \sin\theta$, evaluate

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \ .$$

(d) Using (b) and (c), evaluate the following integrals:

(i) $\displaystyle\int_0^1 x^4\sqrt{1-x^2}\,\mathrm{d}x$,

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\,\mathrm{d}\theta$.

(1996-AL-P MATH 2 #03) (6 marks)

3. (a) Suppose $\mathrm{f}(x)$ is continuous on $[0,a]$. Show that $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\,\mathrm{d}x$.

Furthermore, if $\mathrm{f}(x)+\mathrm{f}(a-x)=K$ for all $x \in [0,a]$, where K is a constant, prove that

(i) $K = 2\,\mathrm{f}\left(\frac{a}{2}\right)$;

(ii) $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = a\,\mathrm{f}\left(\frac{a}{2}\right)$.

(b) Hence, or otherwise, evaluate $\int_0^{2\pi} \frac{1}{e^{\sin x}+1}\,\mathrm{d}x$.

(1996-CE-A MATH 2 #09) (16 marks)

9. (a) Evaluate $\int_0^{\pi} \sin^5 x\,\mathrm{d}x$.

(Hint: Let $t = \cos x$.)

(b) Using the substitution $u = \pi - x$ and the result of (a), evaluate

$$\int_0^{\pi} x\sin^5 x\,\mathrm{d}x \ .$$

(c) By differentiating $y = x^2\sin^5 x$ with respect to x and using the result of (b), evaluate

$$I_1 = \int_0^{\pi} x^2\sin^4 x\cos x\,\mathrm{d}x \ .$$

(d) Let $I_2 = \int_0^{\pi} x^2\sin^4 x\cos|x|\,\mathrm{d}x$.

State, with a reason, whether I_2 is smaller than, equal to or larger than I_1 in (c).

(1997-AL-P MATH 2 #04) (6 marks)

4. Show that $\left(\sin 2x + \sin 4x + \ldots \sin 2nx\right) \sin x = \sin nx \sin(n+1)x$.

Hence or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 6x \sin 7x}{\sin x}\, \mathrm{d}x$.

(1997-CE-A MATH 2 #11) (16 marks)

11. (a) Using the substitution $u = \cot\theta$, find

$$\int \cot^n\theta \operatorname{cosec}^2\theta \, \mathrm{d}\theta \ ,$$

where n is a non-negative integer.

(b) By writing $\cot^{n+2}\theta$ as $\cot^n\theta\cot^2\theta$, show that

$$\int \cot^{n+2}\theta \, \mathrm{d}\theta = -\frac{\cot^{n+1}\theta}{n+1} - \int \cot^n\theta \, \mathrm{d}\theta \ ,$$

where n is a non-negative integer.

(c) Using (b), or otherwise, show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot^2\theta \, \mathrm{d}\theta = 1 - \frac{\sqrt{3}}{3} - \frac{\pi}{12} \ .$$

(d) Using the substitution $x = \sec\theta$, evaluate

$$\int_{\sqrt{2}}^{2} \frac{\mathrm{d}x}{x\sqrt{(x^2-1)^5}} \ .$$

(1998-CE-A MATH 2 #06) (6 marks)

6. Using the substitution $u = \sin\theta$, evaluate

$$\int_0^{\frac{\pi}{2}} \cos^5\theta \sin^2\theta \, d\theta \ .$$

(1998-CE-A MATH 2 #09) (16 marks)

9. (a) Let a be a positive number.

(i) Show that $\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dx$.

(ii) If $f(x) = f(-x)$ for $-a \le x \le a$, show that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \ .$$

(b) Using the substitution $t = \frac{\sqrt{3}}{3}\tan\theta$, show that

$$\int_0^1 \frac{dt}{1+3t^2} = \frac{\sqrt{3}\pi}{9} \ .$$

(c) Given $I_1 = \int_0^1 \frac{1-t^2}{1+3t^2}\,dt$ and $I_2 = \int_0^1 \frac{t^2}{1+3t^2}\,dt$.

(i) Without evaluating I_1 and I_2 ,

(1) show that $I_1 + 4I_2 = 1$, and

(2) using the result of (b), evaluate $I_1 + I_2$.

(ii) Using the result of (c) (i), or otherwise, evaluate I_2 .

(d) Evaluate $\int_{-1}^{1} \frac{1+t^2}{1+3t^2}\,dt$.

(1999-AL-P MATH 2 #02) (6 marks)

2. (a) Let f be a continuous function. Show that $\int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(\pi - x)\,dx$.

(b) Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

(1999-CE-A MATH 2 #01) (3 marks)

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$.

(1999-CE-A MATH 2 #12) (16 marks)

12. (a) Prove, by mathematical induction, that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta} ,$$

where $\sin\theta \neq 0$, for all positive integers n .

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that

$$\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x} ,$$

where $\cos x \neq 0$.

(c) Using (a) and (b), evaluate

$$\int_{0.1}^{0.5} \left(\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} \right)^2 \mathrm{d}x ,$$

giving your answer correct to two significant figures.

(d) Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x + 3\sin 3x + 5\sin 5x + 7\sin 7x + \cdots + 1999\sin 1999x)\, \mathrm{d}x .$$

(2002-CE-A MATH #04) (4 marks)

4. Find $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x$.

(Hint: Let $x = \sin\theta$.)

(2004-AL-P MATH 2 #04) (Part)

4. Using the substitution $u = \frac{1}{x}$, prove that $\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2} \mathrm{d}x = 0$.

(2006-AL-P MATH 2 #03) (7 marks)

3. For any positive integers m and n , define $I_{m,n} = \int_0^{\frac{\pi}{4}} \frac{\sin^m\theta}{\cos^n\theta} \mathrm{d}\theta$.

(a) Prove that $I_{m+2,n+2} = \frac{1}{n+1}\left(\frac{1}{\sqrt{2}}\right)^{m-n} - \frac{m+1}{n+1} I_{m,n}$.

(b) Using the substitution $u = \cos\theta$, evaluate $I_{3,1}$.

(c) Using the results of (a) and (b), evaluate $I_{7,5}$.

(SP-DSE-MATH-EP(M2) #13) (14 marks)

13. (a) Let $a > 0$ and $f(x)$ be a continuous function.

Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Hence, prove that $\int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x) + f(a-x)]\,dx$.

(b) Show that $\int_0^1 \frac{dx}{x^2 - x + 1} = \frac{2\sqrt{3}\pi}{9}$.

(c) Using (a) and (b), or otherwise, evaluate $\int_0^1 \frac{dx}{(x^2 - x + 1)(e^{2x-1} + 1)}$.

(PP-DSE-MATH-EP(M2) #13) (10 marks)

13. (a) Let $f(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} f(x-p)\,dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p) + q]\,dx$, where q is a constant.

(b) Prove that $\frac{\sqrt{3} + \tan\left(x - \frac{\pi}{6}\right)}{\sqrt{3} - \tan\left(x - \frac{\pi}{6}\right)} = \frac{1 + \sqrt{3}\tan x}{2}$.

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1 + \sqrt{3}\tan x)\,dx$.

(2012-DSE-MATH-EP(M2) #13) (13 marks)

13. (a) (i) Suppose $\tan u = \dfrac{-1+\cos\dfrac{2\pi}{5}}{\sin\dfrac{2\pi}{5}}$, where $\dfrac{-\pi}{2} < u < \dfrac{\pi}{2}$. Show that $u = \dfrac{-\pi}{5}$.

(ii) Suppose $\tan v = \dfrac{1+\cos\dfrac{2\pi}{5}}{\sin\dfrac{2\pi}{5}}$. Find v , where $\dfrac{-\pi}{2} < v < \dfrac{\pi}{2}$.

(b) (i) Express $x^2 + 2x\cos\dfrac{2\pi}{5} + 1$ in the form $(x+a)^2 + b^2$, where a and b are constants.

(ii) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\dfrac{2\pi}{5}}{x^2 + 2x\cos\dfrac{2\pi}{5} + 1}\,\mathrm{d}x$.

(c) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\dfrac{7\pi}{5}}{x^2 + 2x\cos\dfrac{7\pi}{5} + 1}\,\mathrm{d}x$.

(2013-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) Let $0 < \theta < \dfrac{\pi}{2}$. By finding $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\ln(\sec\theta + \tan\theta)$, or otherwise, show that

$$\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) + C$$, where C is any constant.

(b) (i) Using (a) and a suitable substitution, show that $\displaystyle\int \frac{\mathrm{d}u}{\sqrt{u^2-1}} = \ln(u + \sqrt{u^2-1}) + C$ for $u > 1$.

(ii) Using (b)(i), show that $\displaystyle\int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}\,\mathrm{d}x = \ln\left(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6}\right)$.

(c) Let $t = \tan\phi$. Show that $\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = \dfrac{1}{1+t^2}$.

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan\phi}{\sqrt{1+2\cos^2\phi}}\,\mathrm{d}\phi$.

(2015-DSE-MATH-EP(M2) #03) (7 marks)

3. (a) Find $\int \frac{1}{e^{2u}}\,du$.

(b) Using integration by substitution, evaluate $\int_1^9 \frac{1}{\sqrt{x}e^{2\sqrt{x}}}\,dx$.

(2016-DSE-MATH-EP(M2) #10) (12 marks)

10. (a) Let $f(x)$ be a continuous function defined on the interval $(0, a)$, where a is a positive constant. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(b) Prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$.

(c) Using (b), prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi \ln 2}{8}$.

(d) Using integration by parts, evaluate $\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,dx$.

(2017-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) Using $\tan^{-1}\sqrt{2} - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$, evaluate $\int_0^1 \frac{1}{x^2+2x+3}\,dx$.

(b) (i) Let $0 \le \theta \le \frac{\pi}{4}$. Prove that $\frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$ and $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$.

(ii) Using the substitution $t = \tan\theta$, evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$.

(c) Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$.

(d) Evaluate $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta + 9}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$.

(2019-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) Using integration by parts, find $\int e^x \sin \pi x\,dx$.

(b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x\,dx$.

(2019-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{2+\cos 2x} = \frac{\sec^2 x}{2+\sec^2 x}$.

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x}\,\mathrm{d}x$.

(c) Let $\mathrm{f}(x)$ be a continuous function defined on $\mathbf{R}$ such that $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all $x \in \mathbf{R}$.

Prove that $\int_{-a}^{a} \mathrm{f}(x)\ln(1+e^x)\,\mathrm{d}x = \int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x$ for any $a \in \mathbf{R}$.

(d) Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,\mathrm{d}x$.

(2020-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) Using integration by substitution, prove that

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right)\mathrm{d}x = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin x\right)\,\mathrm{d}x \ .$$

(b) Using (a), evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\,\mathrm{d}x$.

(c) (i) Using $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$, or otherwise, prove that $\cot\frac{\pi}{12} = 2+\sqrt{3}$.

(ii) Using integration by parts, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x\csc^2 x}{\cot x - 1}\,\mathrm{d}x = \frac{\pi}{8}\ln\left(2+\sqrt{3}\right)$.

(2021-DSE-MATH-EP(M2) #09) (12 marks)

9. (a) Let $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

(i) Find $\frac{\mathrm{d}}{\mathrm{d}\theta}\ln(\sec\theta + \tan\theta)$.

(ii) Using the result of (a) (i), find $\int \sec\theta\,\mathrm{d}\theta$. Hence, find $\int \sec^3\theta\,\mathrm{d}\theta$.

(b) Let $\mathrm{g}(x)$ and $\mathrm{h}(x)$ be continuous functions defined on $\mathbf{R}$ such that $\mathrm{g}(x)+\mathrm{g}(-x) = 1$ and $\mathrm{h}(x) = \mathrm{h}(-x)$ for all $x \in \mathbf{R}$.

Using integration by substitution, prove that $\int_{-a}^{a} \mathrm{g}(x)\,\mathrm{h}(x)\,\mathrm{d}x = \int_0^a \mathrm{h}(x)\,\mathrm{d}x$ for any $a \in \mathbf{R}$.

(c) Evaluate $\int_{-1}^{1} \frac{3^x x^2}{(3^x+3^{-x})\sqrt{x^2+1}}\,\mathrm{d}x$.

ANSWERS

(1979-CE-A MATH 2 #07) (20 marks)

7. (a) $\frac{17}{6}$

(b) $\frac{dy}{dx} = \frac{-2x-y}{x+2y}$

$(x+2y)^3\frac{d^2y}{dx^2} = -6x^2-6xy-6y^2$

(1980-CE-A MATH 2 #12) (20 marks)

12. (b) $\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx = \frac{\pi}{2}$

(c) $\frac{\pi^2}{4}$

(1981-CE-A MATH 2 #03) (6 marks) (Modified)

3. 36

(1981-CE-A MATH 2 #08) (20 marks) (Modified)

8. (a) $\frac{2}{15}$

(b) (iii) $\frac{\pi\sqrt{3}}{9} - \frac{3}{4}$

(1982-CE-A MATH 2 #05) (6 marks) (Modified)

5. $\frac{1}{2}\left(\frac{\pi}{2}-\frac{1}{2}\right)$

(1983-CE-A MATH 2 #03) (5 marks) (Modified)

3. $\frac{58}{135}$

(1983-CE-A MATH 2 #11) (20 marks)

11. (c) $\frac{\pi}{4}-1$

(1983-CE-A MATH 2 #12) (20 marks)

12. (b) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$

The required value $= \frac{\pi}{6}$

(1984-CE-A MATH 2 #05) (8 marks) (Modified)

5. $\int \tan^2\theta\sec^2\theta\,d\theta = \frac{1}{3}\tan^3\theta + \text{constant}$

$\int_0^{\frac{\pi}{3}} \tan^4\theta\,d\theta = \frac{\pi}{3}$

(1984-CE-A MATH 2 #07) (20 marks)

7. (a) $\frac{9}{8}$

(b) (i) $\frac{-1}{3\sin^3\phi} + \text{constant}$

(ii) $8-2\sqrt{2}$

(1985-CE-A MATH 2 #03) (5 marks) (Modified)

3. 1

(1985-CE-A MATH 2 #08) (20 marks)

8. (a) $\frac{2}{35}$

(d) $\frac{4}{35}$

(1986-CE-A MATH 2 #08) (20 marks)

8. (b) (i) 0

(ii) $\frac{\pi^2}{4}$

(iii) $\frac{\pi}{4}$

(1987-CE-A MATH 2 #04) (6 marks)

4. $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$

(1987-CE-A MATH 2 #08) (20 marks)

8. (a) $\frac{\tan^{n-1}x}{n-1} + C$

(b) (ii) $\left(\frac{13}{15}-\frac{\pi}{4}\right)$

(c) $2\left(\frac{13}{15}-\frac{\pi}{4}\right)$

(1988-CE-A MATH 2 #06) (6 marks)

6. $\frac{4}{3}$

(1988-CE-A MATH 2 #08) (20 marks)

8. (a) $\frac{16}{35}$

(b) $\frac{dy}{dx} = \cos^{2n} x - (2n - 1)\cos^{2n-2} x \sin^2 x$

(c) (ii) $\frac{5\pi}{32}$

(d) $\frac{5\pi}{32}$

(1989-CE-A MATH 2 #03) (5 marks) (Modified)

3. $\frac{40}{3}$

(1989-CE-A MATH 2 #09) (16 marks)

9. (a) $\frac{\pi}{4}$

(d) (i) $\frac{1}{8}(\pi + 2)$

(ii) $\frac{1}{32}(3\pi + 8)$

(1990-CE-A MATH 2 #09) (16 marks)

9. (a) (i) $\frac{\pi}{2}$

(ii) $\frac{\pi^2}{4}$

(c) $\frac{\pi^2}{2}$

(1990-AL-P MATH 2 #03) (5 marks)

3. $\frac{\pi}{5}$

(1991-CE-A MATH 2 #02) (5 marks)

2. $\frac{\pi}{2} + 1$

(1991-CE-A MATH 2 #12) (16 marks)

12. (a)

$\frac{dy}{dx} = (m + 1)(1 + x)^n(1 - x)^n - n(1 + x)^{n+1}(1 - x)^{n-1}$

(c) $\frac{512}{9}$

(d) $\frac{128}{63}$

(1992-CE-A MATH 2 #08) (16 marks)

8. (b) $\frac{\sqrt{3}\pi}{18}$

(c) $\frac{\sqrt{3}\pi}{9}$

(d) $\frac{2\sqrt{3}\pi}{27} - \frac{1}{6}$

(1993-CE-A MATH 2 #09) (16 marks)

9. (a) $(m - 1)\sin^{m-2} x \cos^{n+2} x - (n + 1)\sin^m x \cos^n x$

(d) $\frac{3\pi}{512}$

(1994-CE-A MATH 2 #10) (16 marks)

10. (a) $\frac{\pi}{4}$

(c) $\frac{\pi}{4}$

(d) $-\frac{\pi}{4}$

(1995-CE-A MATH 2 #08) (16 marks)

8. (c) $\frac{\pi}{4}$

(d) (i) $\frac{\pi}{32}$

(ii) $\frac{5\pi}{256}$

(1996-AL-P MATH 2 #03) (6 marks)

3. (b) π

(1996-CE-A MATH 2 #09) (16 marks)

9. (a) $\frac{16}{15}$

(b) $\frac{8\pi}{15}$

(c) $\frac{-16\pi}{75}$

(d) I_2 is equal to I_1 because $|x| = x$

(1997-AL-P MATH 2 #04) (6 marks)

4. $\frac{1}{10}$

(1997-CE-A MATH 2 #11) (16 marks)

11. (a) $-\frac{\cot^{n+1}\theta}{n+1} + C$

(d) $\frac{-2}{3} + \frac{8\sqrt{3}}{27} + \frac{\pi}{12}$

(1998-CE-A MATH 2 #06) (6 marks)

6. $\frac{8}{105}$

(1998-CE-A MATH 2 #09) (16 marks)

9. (c) (i) (2) $\frac{\sqrt{3}\pi}{9}$

(ii) $\frac{1}{3} - \frac{\sqrt{3}\pi}{27}$

(d) $\frac{2}{3} + \frac{4\sqrt{3}\pi}{27}$

(1999-AL-P MATH 2 #02) (6 marks)

2. (b) $\frac{\pi^2}{4}$

(1999-CE-A MATH 2 #01) (3 marks)

1. $\frac{\pi}{4}$

(1999-CE-A MATH 2 #12) (16 marks)

12. (c) 0.046

(d) $\frac{1}{2}$

(2002-CE-A MATH #04) (4 marks)

4. $\frac{\pi}{6}$

(2006-AL-P MATH 2 #03) (7 marks)

3. (b) $\frac{1}{2}\ln 2 - \frac{1}{4}$

(c) $\frac{3}{2}\ln 2 - 1$

(SP-DSE-MATH-EP(M2) #13) (14 marks)

13. (c) $\frac{\sqrt{3}\pi}{9}$

(PP-DSE-MATH-EP(M2) #13) (10 marks)

13. (a) $2pq$

(c) $\frac{\pi \ln 2}{3}$

(2012-DSE-MATH-EP(M2) #13) (13 marks)

13. (a) (ii) $v = \frac{3\pi}{10}$

(b) (i) $\left(x + \cos\frac{2\pi}{5}\right)^2 + \sin^2\frac{2\pi}{5}$

(ii) $\frac{\pi}{2}$

(c) $-\frac{\pi}{2}$

(2013-DSE-MATH-EP(M2) #11) (12 marks)

11. (c) $\frac{1}{2}\ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$

(2015-DSE-MATH-EP(M2) #03) (7 marks)

3. (a) $\frac{-1}{2}e^{-2u}$+constant

(b) $\frac{1}{e^2} - \frac{1}{e^6}$

(2016-DSE-MATH-EP(M2) #10) (12 marks)

10. (d) $\frac{\pi \ln 2}{8}$

(2017-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(b) (ii) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(d) $\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(2019-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) $\frac{e^x \sin \pi x-\pi e^x \cos \pi x}{1+\pi^2}+\text{constant}$

(b) $\frac{\pi\left(1+e^3\right)}{1+\pi^2}$

(2019-DSE-MATH-EP(M2) #10) (13 marks)

10. (b) $\frac{\sqrt{3}\pi}{18}$

(d) $\frac{\pi}{16}-\frac{\sqrt{3}\pi}{36}$

(2020-DSE-MATH-EP(M2) #10) (13 marks)

10. (b) $\frac{\pi \ln 2}{24}$

(2020-DSE-MATH-EP(M2) #09) (12 marks)

9. (a) (i) $\sec\theta$

(ii)

$$\int \sec\theta \, d\theta=\ln(\sec\theta+\tan\theta)+\text{ constant}$$

$$\int \sec^3\theta \, d\theta=\frac{1}{2}(\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta))+\text{ constant}$$

(c) $\frac{1}{2}\left(\sqrt{2}-\ln(\sqrt{2}+1)\right)$

6. Applications of Definite Integration

6. (a) Areas of Plane Figures

(1986-CE-A MATH 2 #11) (20 marks)

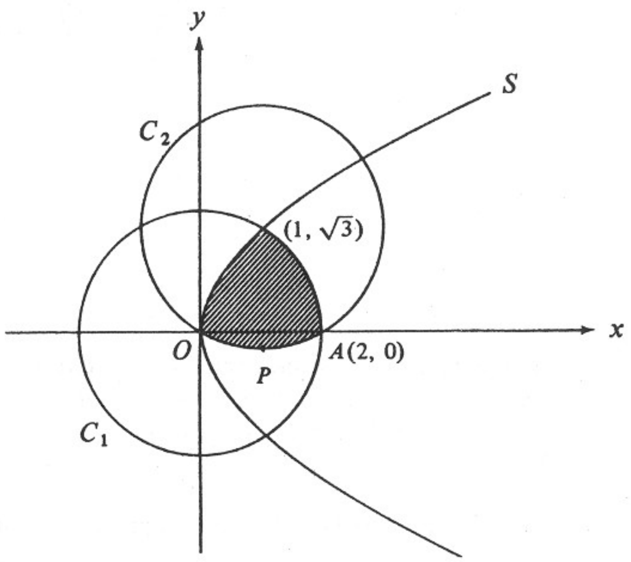
11. (a) (i) Using the substitution $x = 2\sin\theta$, evaluate

$$\int_1^2 \sqrt{4-x^2}\,\mathrm{d}x$$

(ii) Express $3+2x-x^2$ in the form $a^2-(x-b)^2$ where a and b are constants.

Using the substitution $x-b = a\sin\theta$, evaluate $\int_0^1 \sqrt{3+2x-x^2}\,\mathrm{d}x$.

(b)



In Figure 1, the shaded region is bounded by the two circles

C_1 : $x^2+y^2=4$,

C_2 : $(x-1)^2+(y-\sqrt{3})^2=4$

and the parabola

S : $y^2=3x$.

(i) $P(x, y)$ is a point on the minor arc OA of C_2 . Express y in terms of x .

(ii) Find the area of the shaded region.

(1991-CE-A MATH 2 #05) (7 marks)

5. The slope at any point (x, y) of a curve C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = 4-2x$ and C passes through the point $(1, 0)$.

(a) Find an equation of C .

(b) Find the area of the finite region bounded by C and the x-axis.

(1992-CE-A MATH 2 #06) (6 marks)

6.

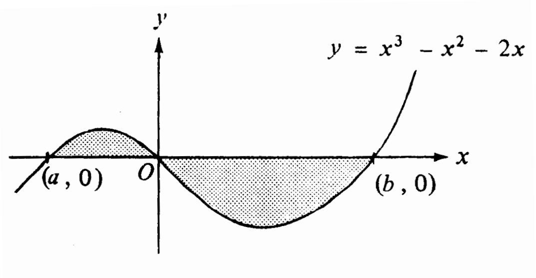


Figure 1

The curve $y = x^3 - x^2 - 2x$ cuts the x-axis at the origin and the points $(a, 0)$ and $(b, 0)$, as shown in Figure 1.

(a) Find the values of a and b .

(b) Find the total area of the shaded parts.

(1993-CE-A MATH 2 #05) (6 marks)

5.

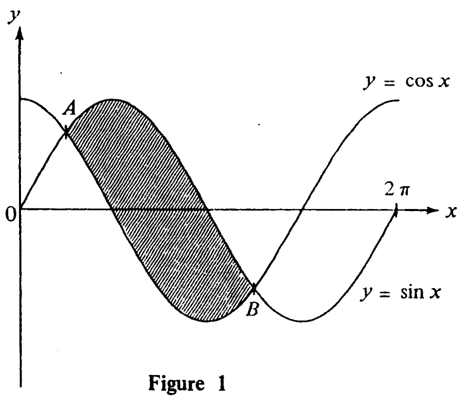


Figure 1

Figure 1 shows the curves of $y = \sin x$ and $y = \cos x$, where $0 \le x \le 2\pi$, intersecting at points A and B .

(a) Find the coordinates of A and B .

(b) Find the area of the shaded region as shown in Figure 1.

(1994-CE-A MATH 2 #07) (6 marks)

7.

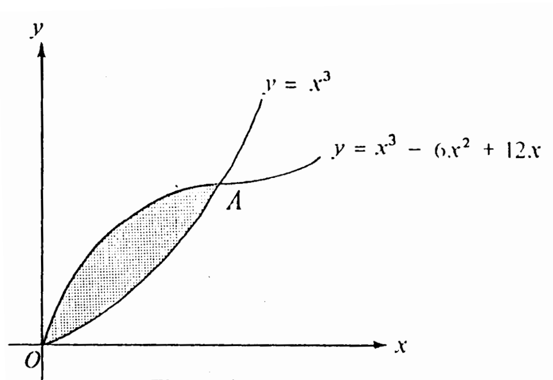


Figure 1

Figure 1 shows two curves $y = x^3$ and $y = x^3 - 6x^2 + 12x$ intersecting at the origin and a point A .

(a) Find the coordinates of A .

(b) Find the area of the shaded region in Figure 1.

(1995-CE-A MATH 2 #05) (6 marks)

5.

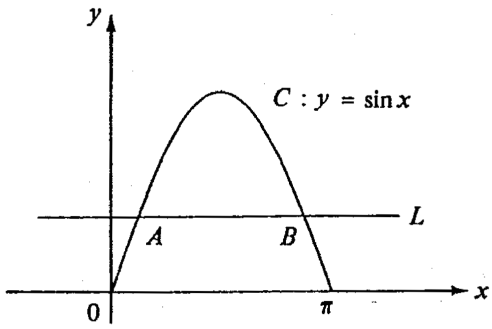


Figure 1

Figure 1 shows the curve $C : y = \sin x$ for $0 \leq x \leq \pi$.

(a) Find the area of the finite region bounded by the curve C and the x-axis.

(b) A horizontal line L cuts C at two points A and B . A is the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

(i) Write down the coordinates of B .

(ii) Find the area of the finite region bounded by C and L .

(1996-CE-A MATH 2 #05) (6 marks)

5.

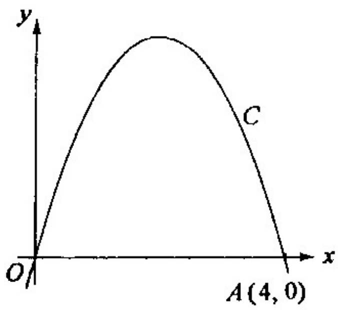


Figure 1(a)

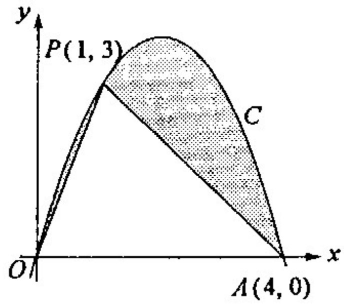


Figure 1(b)

The curve $C : y = 4x - x^2$ cuts the x-axis at the origin O and the point $A(4, 0)$ as shown in Figure 1(a).

(a) Find the area of the region bounded by C and the line segment OA .

(b) In Figure 1(b), the shaded region is enclosed by the curve C and the line segments OP and PA , where P is the point $(1, 3)$. Using (a), find the total area of the shaded region.

(1998-CE-A MATH 2 #08) (Modified)

8.

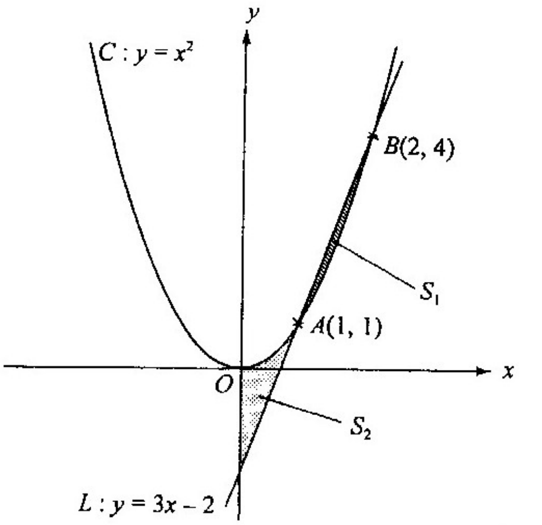


Figure 1

In Figure 1, the line $L : y = 3x - 2$ and the curve $C : y = x^2$ interest at two points $A(1, 1)$ and $B(2, 4)$. Denote the area of the region bounded by C and line segment AB by S_1 . Also, denote the area bounded by C , L and the y-axis by S_2 . Find $S_1 + S_2$.

(1999-CE-A MATH 2 #04) (5 marks)

1.

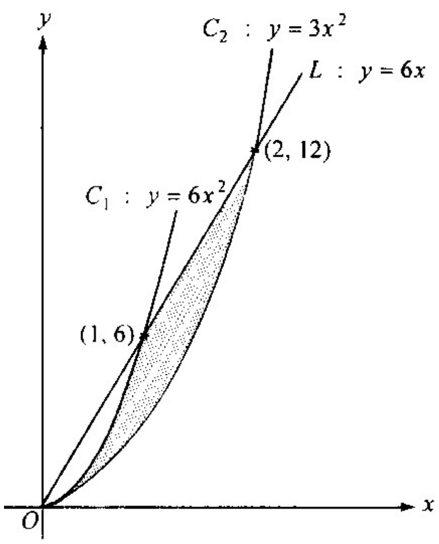


Figure 2

In Figure 2, the line $L : y = 6x$ and the curves $C_1 : y = 6x^2$ and $C_2 : y = 3x^2$ all pass through the origin, L also intersects C_1 and C_2 at the points $(1, 6)$ and $(2, 12)$ respectively. Find the area of the shaded region.

(2000-AS-M & S #03) (5 marks)

3.

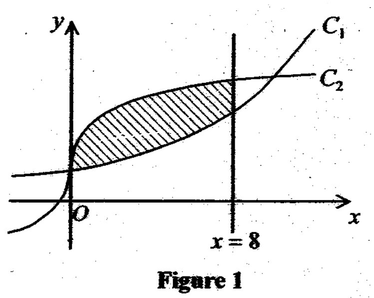


Figure 1

Figure 1 shows the graph of the two curves

$C_1 : y = e^{\frac{x}{8}}$ and

$C_2 : y = 1 + x^{\frac{1}{3}}$.

Find the area of the shaded region.

(2000-CE-A MATH 2 #08) (16 marks)

8. (a) Find $\int \cos 3x \cos x \, dx$.

(b) Show that $\dfrac{\sin 5x - \sin x}{\sin x} = 4 \cos 3x \cos x$.

Hence, or otherwise, find $\int \dfrac{\sin 5x}{\sin x} dx$.

(c) Using a suitable substitution, show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x} dx$$

(d)

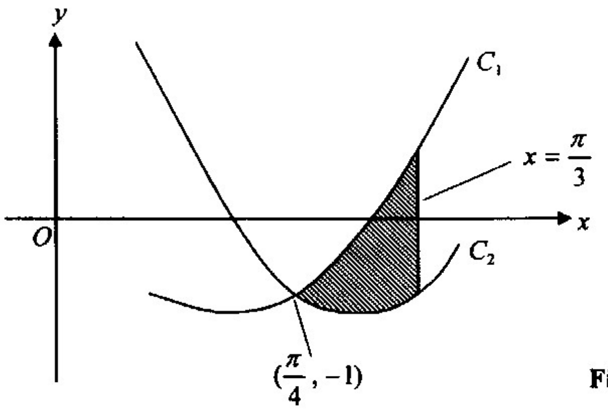


Figure 4

In Figure 4, the curves $C_1 : y = \dfrac{\cos 5x}{\cos x}$ and $C_2 : y = \dfrac{\sin 5x}{\sin x}$ intersect at the point $\left(\dfrac{\pi}{4}, -1\right)$. Find the area of the shaded region bounded by C_1 , C_2 and the line $x = \dfrac{\pi}{3}$.

(2002-CE-A MATH #06) (5 marks)

6.

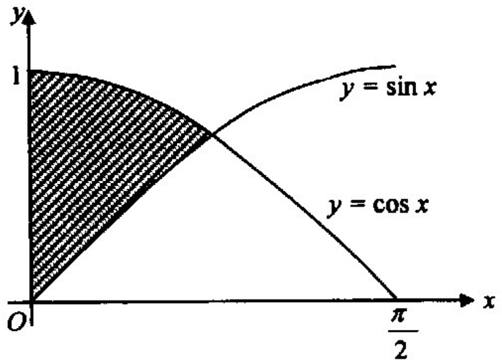


Figure 1

Figure 1 shows the curves $y = \sin x$ and $y = \cos x$. Find the area of the shaded region.

(2003-CE-A MATH #09) (5 marks)

9.

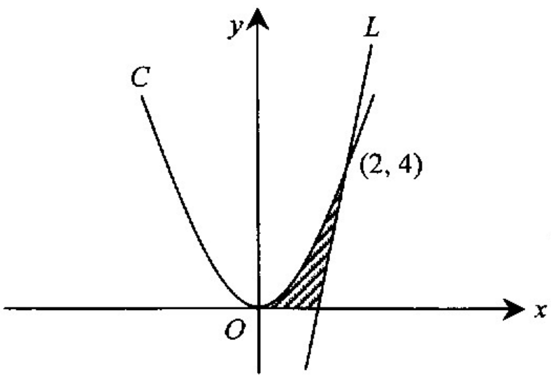


Figure 2

In Figure 2, the curve $C : y = x^2$ and line $L : y = 4x - 4$ intersect at the point $(2, 4)$. Find the area of the shaded region bounded by C , L and the x-axis.

(2005-CE-A MATH #13) (6 marks)

13. (a) Find $\int \sin \pi x \, \mathrm{d}x$.

(b)

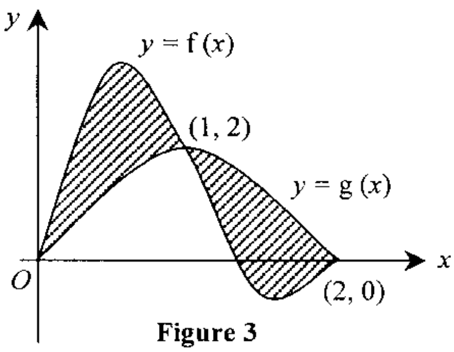


Figure 3

Figure 3 shows two curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ intersecting at three points $(0, 0)$, $(1, 2)$ and $(2, 0)$ for $0 \le x \le 2$. It is given that $\mathrm{f}(x) - \mathrm{g}(x) = 2 \sin \pi x$. Find the area of the shaded region as shown in Figure 3.

(2007-CE-A MATH #06) (5 marks)

6.

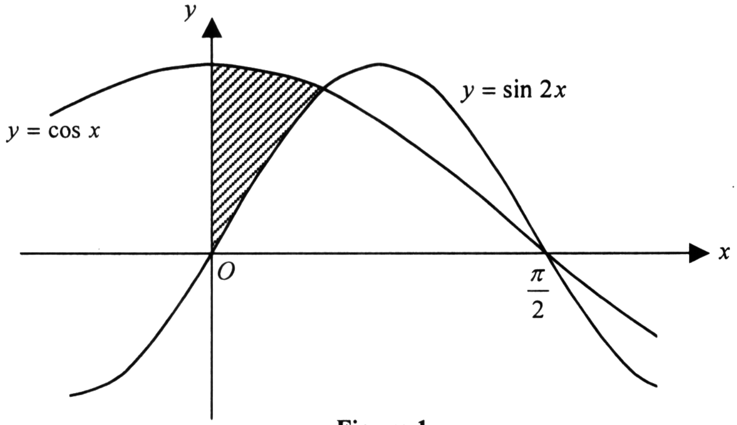


Figure 1

Figure 1 shows the graphs of $y = \sin 2x$ and $y = \cos x$. Find the area of the shaded region.

(2008-CE-A MATH #10) (5 marks)

10.

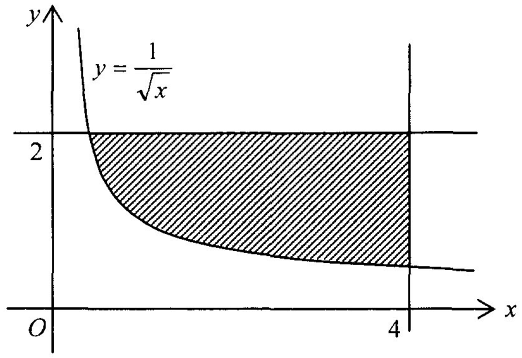


Figure 2

Find the area of the shaded region bounded by the curve $y = \dfrac{1}{\sqrt{x}}$ and the straight lines $y = 2$, $x = 4$.

(2009-CE-A MATH #10) (5 marks)

10. (a) Solve $8\cos x = \sec^2 x$ for $0 < x < \dfrac{\pi}{2}$.

(b)

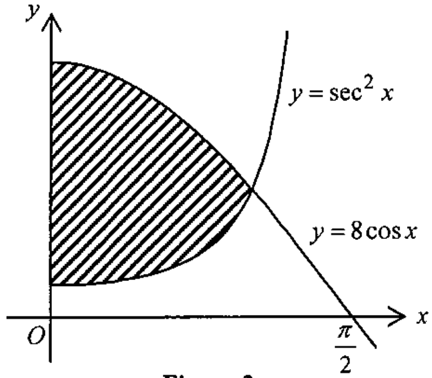


Figure 2

Figure 2 shows the graphs of $y = 8\cos x$ and $y = \sec^2 x$. Find the area of the shaded region.

(2010-CE-A MATH #03) (4 marks)

3. Let $I = \int_{-2}^{1} x(1-x)(x+2)\,\mathrm{d}x$.

(a) Find the value of I .

(b)

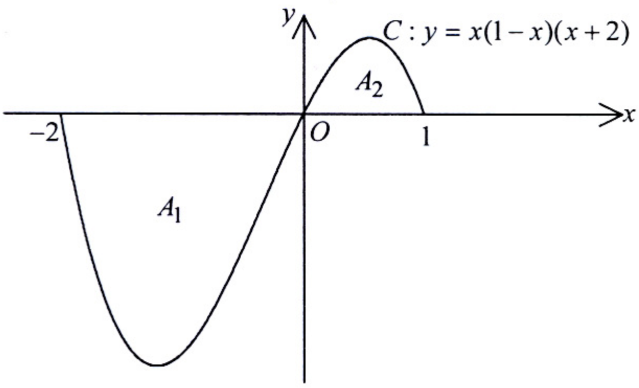


Figure 1

Figure 1 shows the graph of $C: y = x(1-x)(x+2)$ for $-2 \le x \le 1$. Let A_1 denote the area of the region bounded by C and the x-axis when $x < 0$, and A_2 denote the area of the region bounded by C and the x-axis when $x \ge 0$. Without finding the values of A_1 and A_2 , express I in terms of A_1 and A_2 .

(2011-CE-A MATH #11) (7 marks)

11.

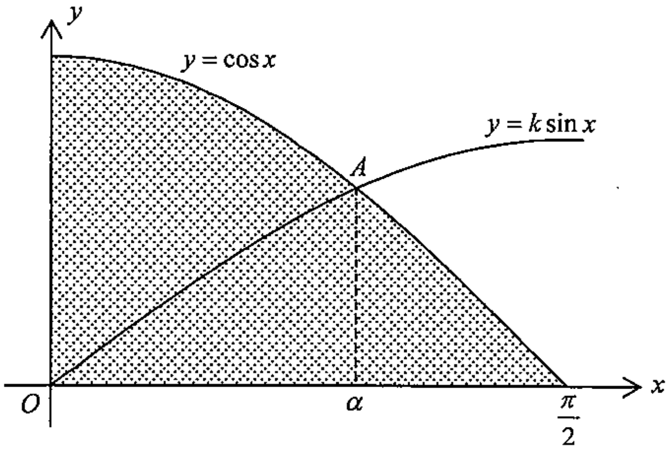


Figure 1

In Figure 1, the curves $y = \cos x$ and $y = k \sin x$, where $k > 0$, intersect at the point A . It is given that the x-coordinate of A is α , where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $\tan \alpha = \frac{1}{k}$.

(b) If the curve $y = k \sin x$ bisects the shaded region bounded by the curves $y = \cos x$, x-axis and y-axis, find the value of k .

(2012-DSE-MATH-EP(M2) #14) (12 marks)

14. Consider the curve $\Gamma : y = kx^p$, where $k > 0$, $p > 0$. In Figure 7, the tangent to Γ at $A\left(a, ka^p\right)$ cuts the x-axis at $B(-a, 0)$, where $a > 0$.

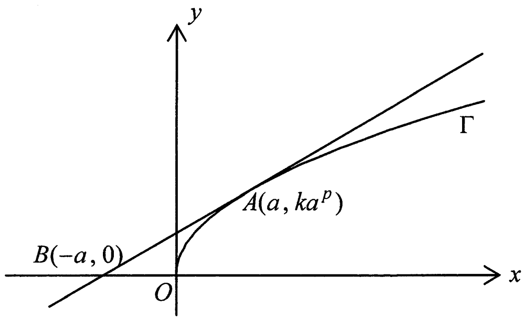


Figure 7

(a) Show that $p = \dfrac{1}{2}$.

(b) Suppose that $a = 1$. As shown in Figure 8, the circle C , with radius 2 and centre on the y-axis, touches Γ at point A .

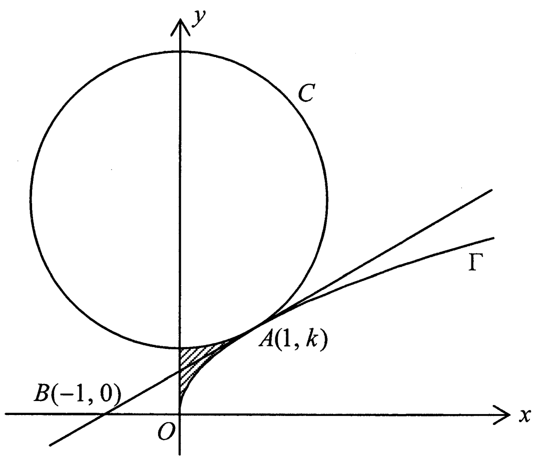


Figure 8

(i) Show that $k = \dfrac{2\sqrt{3}}{3}$.

(ii) Find the area of the shaded region bounded by Γ , C and the y-axis.

(2014-DSE-MATH-EP(M2) #06) (6 marks)

6. (a) Find $\int xe^{-x}\,dx$.

(b)

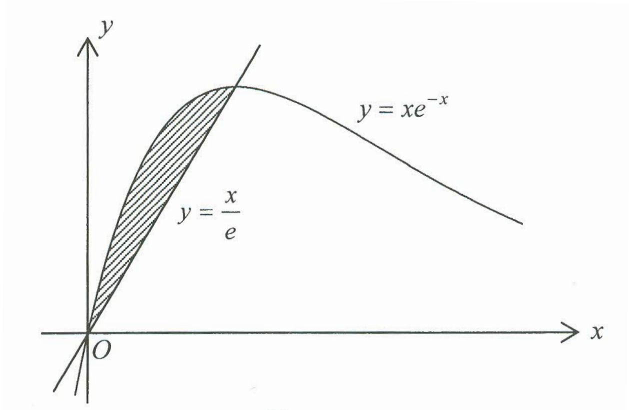


Figure 1

Figure 1 shows the shaded region bounded by curve $y = xe^{-x}$ and the straight line $y = \frac{x}{e}$. Find the area of the shaded region.

(2017-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) Using integration by parts, find $\int x^2 e^{-x}\,dx$.

(b) Find the area of the region bounded by the graph of $y = x^2 e^{-x}$, the x-axis and the straight line $x = 6$.

(2018-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) Using integration by parts, find $\int u\left(5^u\right)du$.

(b) Define $f(x) = x\left(5^{2x}\right)$ for all real numbers x. Find the area of the region bounded by the graph of $y = f(x)$, the straight line $x = 1$ and the x-axis.

(2019-DSE-MATH-EP(M2) #09) (12 marks)

9. Consider the curve $\Gamma : y = \frac{1}{3}\sqrt{12 - x^2}$, where $0 < x < 2\sqrt{3}$. Denote the tangent to Γ at $x = 3$ by L .

(a) Find the equation of L .

(b) Let C be the curve $y = \sqrt{4 - x^2}$, where $0 < x < 2$. It is given that L is a tangent to C . Find

(i) the point(s) of contact of L and C ;

(ii) the point(s) of intersection of C and Γ ;

(iii) the area of the region bounded by L , C and Γ .

ANSWERS

(1986-CE-A MATH 2 #11) (20 marks)

11. (a) (i) $2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$

(ii) $3+2x-x^2=2^2-(x-1)^2$

$\int_0^1 \sqrt{3+2x-x^2}\,dx = \frac{\pi}{3}+\frac{\sqrt{3}}{2}$

(b) (i) $y=-\sqrt{4-(x-1)^2}+\sqrt{3}$

(ii) $\left(\frac{4\pi}{3}-\frac{5\sqrt{3}}{6}\right)$

(1991-CE-A MATH 2 #05) (7 marks)

5. (a) $y=-x^2+4x-3$

(b) $\frac{4}{3}$

(1992-CE-A MATH 2 #06) (6 marks)

6. (a) $a=-1$, $b=2$

(b) $\frac{37}{12}$

(1993-CE-A MATH 2 #05) (6 marks)

5. (a) $A=\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$, $B=\left(\frac{5\pi}{4},\frac{-\sqrt{2}}{2}\right)$

(b) $2\sqrt{2}$

(1994-CE-A MATH 2 #07) (6 marks)

7. (a) $A=(2,8)$

(b) 8

(1995-CE-A MATH 2 #05) (6 marks)

5. (a) 2

(b) (i) $B=\left(\frac{5\pi}{6},\frac{1}{2}\right)$

(ii) $\sqrt{3}-\frac{\pi}{3}$

(1996-CE-A MATH 2 #05) (6 marks)

5. (a) $\frac{32}{3}$

(b) $\frac{14}{3}$

(1998-CE-A MATH 2 #08) (6 marks)

8. 1

(1999-CE-A MATH 2 #04) (5 marks)

1. $\frac{\pi}{4}$

(2000-AS-M & S #03) (5 marks)

3. $28-8e$

(2000-CE-A MATH 2 #08) (16 marks)

8. (a) $\frac{1}{8}\sin 4x+\frac{1}{4}\sin 2x+\text{constant}$

(b) $\int \frac{\sin 5x}{\sin x}dx = x+\frac{1}{2}\sin 4x+\sin 2x+\text{constant}$

(d) $2-\sqrt{3}$

(2002-CE-A MATH #06) (5 marks)

6. $\sqrt{2}-1$

(2003-CE-A MATH #09) (5 marks)

9. $\frac{2}{3}$

(2005-CE-A MATH #13) (6 marks)

13. (a) $-\frac{1}{\pi}\cos \pi x+\text{constant}$

(b) $\frac{8}{\pi}$

(2007-CE-A MATH #06) (5 marks)

6. $\frac{1}{4}$

(2008-CE-A MATH #10) (5 marks)

10. $\frac{9}{2}$

(2009-CE-A MATH #10) (5 marks)

10. (a) $x = \frac{\pi}{3}$

(b) $3\sqrt{3}$

(2010-CE-A MATH #03) (4 marks)

3. (a) $\frac{-9}{4}$

(b) $A_2 - A_1$

(2011-CE-A MATH #11) (7 marks)

11. (b) $k = \frac{3}{4}$

(2012-DSE-MATH-EP(M2) #14) (12 marks)

14. (b) (ii) $\frac{13\sqrt{3}}{18} - \frac{\pi}{3}$

(2014-DSE-MATH-EP(M2) #06) (6 marks)

6. (a) $-xe^{-x} - e^{-x}$ + constant

(b) $1 - \frac{5}{2e}$

(2017-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) $-e^{-x}\left(x^2 + 2x + 2\right)$ + constant

(b) $2 - \frac{50}{e^6}$

(2018-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) $\frac{5^u(u \ln 5 - 1)}{(\ln 5)^2}$ + constant

(b) $\frac{25 \ln 5 - 12}{2(\ln 5)^2}$

(2019-DSE-MATH-EP(M2) #09) (12 marks)

9. (a) $x + \sqrt{3}y - 4 = 0$

(b) (i) $\left(1,\sqrt{3}\right)$

(ii) $\left(\sqrt{3},1\right)$

(iii) $\frac{4}{\sqrt{3}} - \frac{2\pi}{3}$

6. (b) Curve Sketching and Area

(2001-AS-M & S #10) (15 marks)

10. Let $f(x) = \dfrac{5x + 45}{x + 3}$ for $x \neq -3$ and $g(x) = ka^{-\frac{1}{9}x}$ where k and a are positive constants. Figure 3 shows a sketch of $y = g(x)$. It is known that $f(0) = g(0)$ and $f(9) = g(9)$.

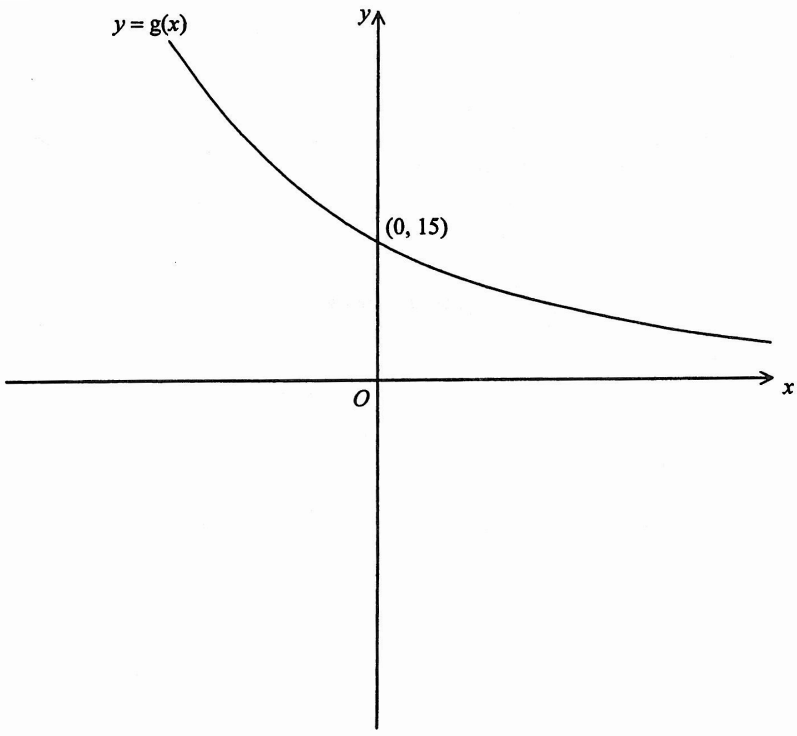


Figure 3

(a) Determine the values of k and a .

(b) Find the equations of the horizontal and vertical asymptotes of $y = f(x)$.

(c) Sketch $y = f(x)$ and its asymptotes on Figure 3. Indicate the points where the curve cuts the axes and $y = g(x)$.

(d) Let A be the area bounded by the curve $y = f(x)$, the x-axis, the y-axis and the line $x = 9$.

(i) Find the value of A.

(ii) If the area bounded by the curve $y = g(x)$, the x-axis, the lines $x = \alpha$ and $x = \alpha + 9$ is also A , find the value of α . (Give your answer correct to 4 decimal places.)

(2002-AS-M & S #10) (15 marks)

10. Let $f(x)=\dfrac{ax+b}{cx+1}$ and $g(x)=-(x-3)(x+1)^3$, where a , b and c are constants. It is known that $f(0)=g(0)$, $f(3)=g(3)$ and $f(-2)=g(-2)$.

(a) (i) Find the values of a , b and c .

(ii) Find the horizontal and vertical asymptotes of the graph of $y=f(x)$.

(iii) Sketch the graph of $y=f(x)$ and its asymptotes. Indicate the point(s) where the curve cuts the y-axis.

(b) (i) Find all relative extreme point(s) and point(s) of inflexion of the graph of $y=g(x)$.

(ii) On the diagram in (a) (iii), sketch the graph of $y=g(x)$. Indicate all the relative extreme point(s) and the point(s) of inflexion, the point(s) where the graph cuts the coordinates axes and where it cuts the graph of $y=f(x)$.

(c) Find the area enclosed by the graph of $y=f(x)$ and $y=g(x)$.

(2003-AS-M & S #07) (15 marks)

7. Define $f(x)=\dfrac{20-4x}{7-2x}$ for all $x\neq\dfrac{7}{2}$ and $g(x)=\dfrac{a+bx}{3+cx}$ for all $x\neq\dfrac{-3}{c}$, where a , b and c are positive constants.

Let C_1 and C_2 be the curves $y=f(x)$ and $y=g(x)$ respectively.

It is given that the x-intercept and the y-intercept of C_2 are -3 and 4 respectively. Also, it is known that C_1 and C_2 have a common horizontal asymptote.

(a) Find the equations of the vertical asymptote(s) and horizontal asymptote(s) to C_1 .

(b) Find the values of a , b and c .

(c) Sketch C_1 and C_2 on the same diagram and indicate the asymptote(s), intercept(s) and the point(s) of intersection of the two curves.

(d) If the area enclosed by C_1 , C_2 and the straight line $x=\lambda$, where $0<\lambda<\dfrac{7}{2}$, is $3\ln 3$ square units, find the exact value(s) of λ .

(2004-AS-M & S #07) (15 marks)

7. Define $f(x)=\dfrac{2x+5}{2x+6}$ and $g(x)=-f(x)$ for all $x\neq-3$.

Let C_1 and C_2 be the curves $y=f(x)$ and $y=g(x)$ respectively.

(a) (i) Find the equations of the vertical asymptote(s) and horizontal asymptote(s) to C_1 .

(ii) Sketch C_1 and indicate its asymptote and its intercept(s).

(b) On the diagram sketched in (a) (ii), sketch C_2 and indicate its asymptote(s), its intercept(s) and the point(s) of intersection of the two curves.

(c) Let $\lambda>7$. If the area enclosed by C_1 , C_2 and the straight line $x=\lambda$ is $\left(2\lambda+5+\ln(\lambda-7)\right)$ square units, find the exact value(s) of λ .

(2004-CE-A MATH #17) (12 marks)

17. (a) Let $y = (x - \pi)\sin x + \cos x$.

(i) Show that $\dfrac{dy}{dx} = (x - \pi)\cos x$.

Hence find $\int (x - \pi)\cos x \, dx$.

(ii) Figure 10 shows the graph of $y = (x - \pi)\cos x$ for $0 \le x \le \dfrac{3\pi}{2}$.

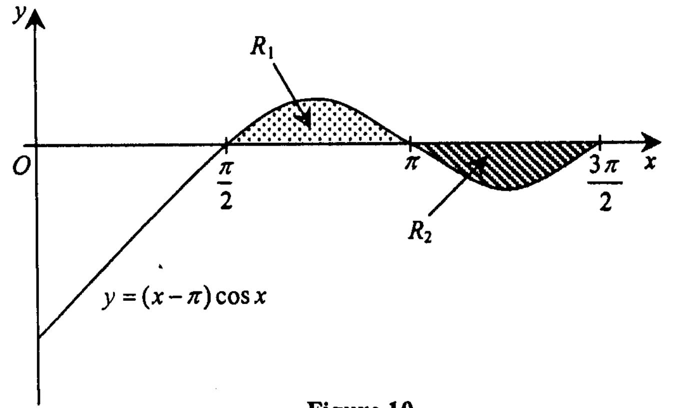


Figure 10

(1) Find the areas of the two shaded region R_1 and R_2 as shown in Figure 10.

(2) Find $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x - \pi)\cos x \, dx$.

(b)

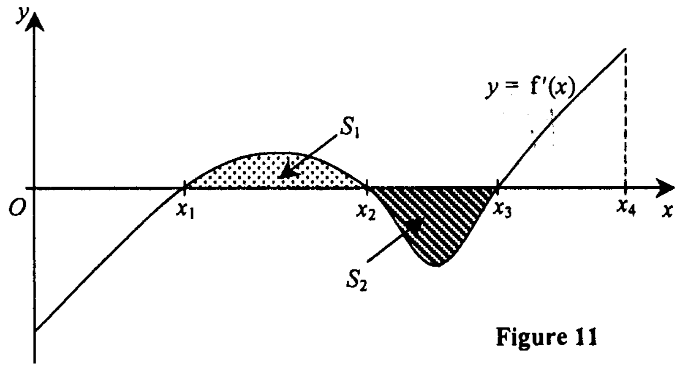


Figure 11

Let $f(x)$ be a continuous function. Figure 11 shows a sketch of the graph of $y = f'(x)$ for $0 \le x \le x_4$. It is known that the areas of the shaded regions S_1 and S_2 as shown in Figure 11 are equal.

(i) Show that $f(x_1) = f(x_3)$.

(ii) Furthermore, $f(0) = f(x_4) = 0$ and $f(x) \neq 0$ for $0 < x < x_4$. In Figure 12, draw a sketch of the graph of $y = f(x)$ for $0 \le x \le x_4$.

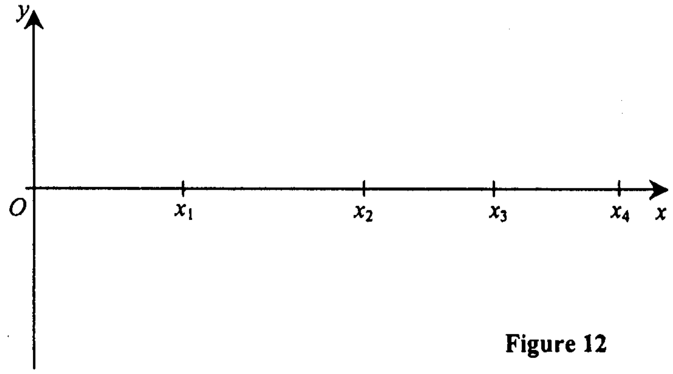


Figure 12

(2005-AS-M & S #07) (15 marks)

7. Define $f(x)=\dfrac{2x+a}{x+2}$ for all $x\neq -2$ and let $g(x)=-x^2+x+b$, where a and b are constants. Let C_1 and C_2 be the curves $y=f(x)$ and $y=g(x)$ respectively.

It is given that C_1 and C_2 have a common y-intercept and $f(3)=g(3)$.

(a) Find the values of a and b .

(b) Find the equations of the horizontal asymptote(s) and vertical asymptote(s) to C_1 .

(c) Sketch C_1 and C_2 on the same diagram and indicate the asymptote(s) to C_1 . Also indicate the intercept(s) and the points of intersection of C_1 and C_2 .

(d) Find the area enclosed by C_1 and C_2 .

(2006-AS-M & S #07) (15 marks)

7. Define $f(x)=\dfrac{a+bx}{4-x}$ for all $x\neq 4$. Let $g(x)=f(-x)$ for all $x\neq -4$. Let C_1 and C_2 be the curves $y=f(x)$ and $y=g(x)$ respectively. It is given that the y-intercept of C_1 is $-\dfrac{3}{2}$ while the x-intercept of C_2 is -2 .

(a) Find the values of a and b .

(b) (i) Find the equations of the vertical asymptote(s) and the horizontal asymptote(s) to C_1 .

(ii) Sketch C_1 and indicate its asymptote(s) and intercept(s).

(c) On the diagram sketched in (b) (ii), sketch C_2 and indicate its asymptote(s), its intercept(s) and the point(s) of intersection of the two curves.

(d) Find the area enclosed by C_1 , C_2 and the straight line $y=9$.

(2006-AL-P MATH 2 #07) (15 marks)

7. Let $f(x)=\dfrac{x^2-x-6}{x+6}$ ($x\neq -6$).

(a) Find $f'(x)$ and $f''(x)$.

(b) Solve each of the following inequalities:

(i) $f'(x)>0$

(ii) $f'(x)<0$

(iii) $f''(x)>0$

(iv) $f''(x)<0$

(c) Find the relative extreme point(s) of the graph of $y=f(x)$.

(d) Find the asymptote(s) of the graph of $y=f(x)$.

(e) Sketch the graph of $y=f(x)$.

(f) Find the area of the region bounded by the graph of $y=f(x)$ and the x-axis.

(2006-CE-A MATH #13) (7 marks)

13. Let $f(x)$ be the polynomial. Figure 2 shows a sketch of the curve $y = f'(x)$, where $-2 \le x \le 6$. The curve cuts the x-axis at the origin and $(a, 0)$, where $0 < a < 6$. It is known that the areas of the shaded regions R_1 and R_2 as shown in Figure 2 are 3 and 1 respectively.

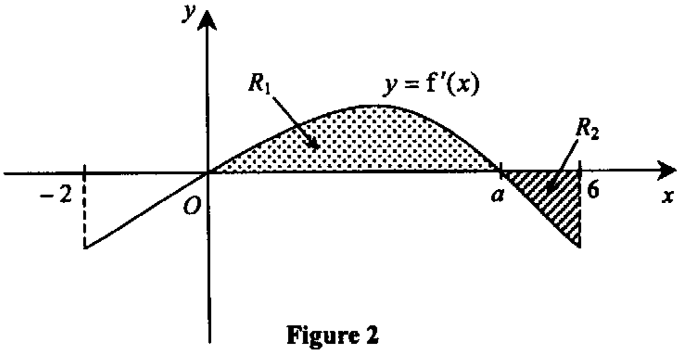


Figure 2

(a) Write down the x-coordinates of the maximum and minimum points of the curve $y = f(x)$ for $-2 < x < 6$.

(b) It is known that $f(-2) = 2$ and $f(0) = 1$.

(i) By considering $\int_0^a f'(x)\,dx$, find the value of $f(a)$.

(ii) In Figure 3, sketch the curve $y = f(x)$ for $-2 \le x \le 6$.

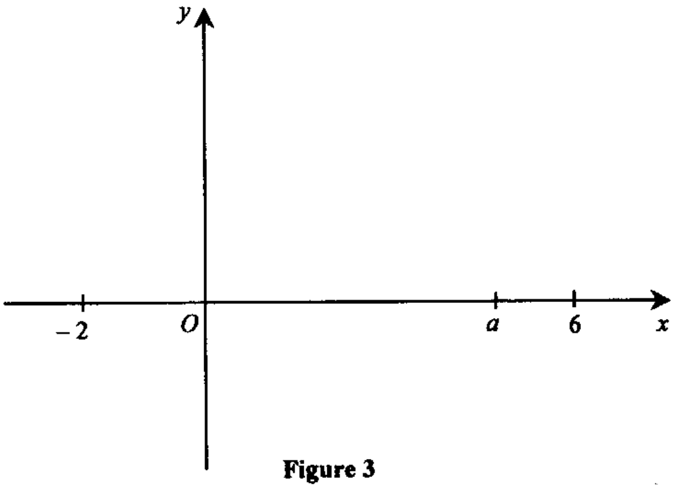


Figure 3

(2007-AS-M & S #07) (15 marks)

7. Define $f(x) = \dfrac{8x - 40}{x + 4}$ for all $x \ne -4$. Let $g(x) = \dfrac{(x+4)^2(x-5)}{8}$.

Let C_1 and C_2 be the curves $y = f(x)$ and $y = g(x)$ respectively.

(a) Sketch C_1 and indicate its asymptote(s) and its intercept(s).

(b) (i) Find the coordinates of the relative extreme point(s) and the point(s) of inflexion of C_2 .

(ii) On the diagram sketched in (a), sketch C_2 and indicate its relative extreme point(s), its intercept(s), its point(s) of inflexion and the point(s) of intersection of the two curves.

(c) Find the area enclosed by C_1 and C_2 .

(2009-AS-M & S #07) (15 marks)

7. Let C_1 be the curve $y = \dfrac{2x-1}{hx-1}$, where h is a non-zero constant and $x \neq \dfrac{1}{h}$.

It is given that C_1 has a vertical asymptote $x = 1$ and a horizontal asymptote $y = k$, where k is a constant.

(a) (i) Find the values of h and k .

(ii) Sketch the graph of C_1 . Indicate its asymptotes and intercepts.

(b) Let C_2 be the curve $y = -x^2 + px + q$, where p and q are constants. Let A be the point where C_1 cuts the x-axis. It is given that C_2 intersects C_1 at A , and the tangents to both curves at A are perpendicular to each other.

(i) Find the values of p and q .

(ii) On the diagram sketched in (a) (ii), sketch the graph of C_2 .

(iii) Find the area of the region bounded by the curves C_1 , C_2 and the y-axis.

(2010-AS-M & S #07) (15 marks)

7. Define $f(x) = \dfrac{x+4}{x+a}$ where a is a constant and $x \neq -a$. Let C_1 be the curve $y = f(x)$ with the vertical asymptote $x = 2$.

(a) (i) Find the value of a .

(ii) Find the equation(s) of the horizontal asymptote(s) of C_1 .

(iii) Find $f'(x)$ and hence find the range of x for which $f(x)$ is decreasing.

(iv) Sketch the curve of C_1 . Indicate its asymptote(s) and intercept(s).

(b) Let C_2 be the curve $y = g(x)$, where $g(x) = f(x-2) + 2$.

(i) Using the result in (a) (iv), or otherwise, sketch the curve C_2 . Indicate its asymptote(s) and intercept(s).

(ii) Let C_3 be the curve $y = [g(x)]^2$. Find the area of the region bounded by the curve C_3 and the axes.

(2011-AS-M & S #07) (15 marks)

7. Let C be the curve $y = \dfrac{ax+b}{cx+d}$ for all $x \neq \dfrac{-d}{c}$, where a , b , c , d are constants, $a \neq 0$ and $c \neq 0$. It is given that C has a vertical asymptote $2x - 1 = 0$ and a horizontal asymptote $y + 2 = 0$, and C passes through the origin. Let D be the curve $y = \dfrac{cx+d}{ax+b}$ for all $x \neq \dfrac{-b}{a}$.

(a) Show that $d = \dfrac{-c}{2}$.

Hence find the equation of C .

(b) Find the coordinates of all the intersecting points of curves C and D .

(c) Sketch the curves C and D on the same diagram, indicating their asymptotes, intercepts and their points of intersection.

(d) Find the exact value of the area of the region bounded by the curves C , D and the positive x-axis.

(SP-DSE-MATH-EP(M2) #12) (14 marks)

12. Let $\mathrm{f}(x)=\dfrac{4}{x-1}-\dfrac{4}{x+1}-1$, where $x \neq \pm 1$.

(a) (i) Find the x- and y- intercept(s) of the graph of $y=\mathrm{f}(x)$.

(ii) Find $\mathrm{f}'(x)$ and prove that

$$\mathrm{f}''(x)=\frac{16\left(3x^2+1\right)}{(x-1)^3(x+1)^3}$$

for $x \neq \pm 1$.

(iii) For the graph of $y=\mathrm{f}(x)$, find all the extreme points and show that there are no points of inflexion.

(b) Find all the asymptote(s) of the graph of $y=\mathrm{f}(x)$.

(c) Sketch the graph of $y=\mathrm{f}(x)$.

(d) Let S be the area bounded by the graph of $y=\mathrm{f}(x)$, the straight line $x=3$, $x=a$ ($a>3$) and $y=-1$. Find S in terms of a . Deduce that $S<4\ln 2$.

(2012-AS-M & S #07) (15 marks)

7. Let $\mathrm{f}(x)=\dfrac{ax+b}{c-x}$ for all $x \neq c$, where a , b and c are constants, and $\mathrm{g}(x)=\mathrm{f}(-x)$ for all $x \neq -c$. Let C_1 and C_2 be the curves of $y=\mathrm{f}(x)$ and $y=\mathrm{g}(x)$ respectively. It is given that the vertical asymptotes of C_2 is $x=-3$, the y-intercept of C_1 is $\dfrac{4}{3}$ and the x-intercept of C_2 is 2.

(a) Find the values of a , b and c .

(b) (i) Find the equations of the vertical and horizontal asymptotes of C_1 .

(ii) Sketch the graphs of C_1 and C_2 on the same diagram. Indicate the asymptotes, intercepts and their intersecting point(s).

(c) If the area of the region bounded by C_1 , C_2 and the line $x=-k$ (where $0<k<3$) is $10\ln\dfrac{3}{2}$, find the value of k .

(2013-AS-M & S #07) (15 marks)

7. Let $k \neq 0$ be a constant.

Define $\mathrm{f}(x)=\dfrac{x}{kx-1}$ for all $x \neq \dfrac{1}{k}$, and $\mathrm{g}(x)=\mathrm{f}\left(\dfrac{1}{x}\right)$ for all $x \neq 0$ and $x \neq k$.

Let C_1 be the curve $y=\mathrm{f}(x)$ and C_2 be the curve $y=\mathrm{g}(x)$. It is given that C_2 has a vertical asymptote $x=2$.

(a) Find the value of k.

(b) Find the points of intersection of the curves C_1 and C_2 .

(c) Sketch the curves C_1 and C_2 on the same diagram, indicating their asymptotes, intercepts and points of intersection.

(d) Find the exact value of the area enclosed by C_1 , C_2 and the y-axis.

(2015-DSE-MATH-EP(M2) #09) (13 marks)

9. Define $f(x) = \dfrac{x^2 + 12}{x - 2}$ for all $x \neq 2$.

(a) Find $f'(x)$.

(b) Prove that the maximum value and the minimum value of $f(x)$ are -4 and 12 respectively.

(c) Find the asymptote(s) of the graph of $y = f(x)$.

(d) Find the area of the region bounded by the graph of $y = f(x)$ and the horizontal line $y = 14$.

(2016-DSE-MATH-EP(M2) #09) (13 marks)

9. Let a and b be constants. Define $f(x) = x^3 + ax^2 + bx + 5$ for all real numbers x . Denote the curve $y = f(x)$ by C . It is given that $P(-1, 10)$ is a turning point of C .

(a) Find a and b .

(b) Is P a maximum point of C ? Explain your answer.

(c) Find the minimum value(s) of $f(x)$.

(d) Find the point(s) of inflexion of C .

(e) Let L be the tangent to C at P . Find the area of the region bounded by C and L .

(2020-DSE-MATH-EP(M2) #09) (12 marks)

9. Let $f(x) = \dfrac{(x + 4)^3}{(x - 4)^2}$ for all real numbers $x \neq 4$. Denote the graph of $y = f(x)$ by H .

(a) Find the asymptote(s) of H .

(b) Find $f''(x)$.

(c) Someone claims that there are two turning points of H . Do you agree? Explain your answer.

(d) Find the point(s) of inflexion of H .

(e) Find the area of the region bounded by H , the x-axis and the y-axis.

ANSWERS

(2001-AS-M & S #10) (15 marks)

10. (a) $k = 15$, $a = 2$

(b) Horizontal asymptotes: $y = 5$

Vertical asymptotes: $x = -3$

(d) (i) $45 + 30\ln 4$

(ii) $\alpha \approx 1.5253$

(2002-AS-M & S #10) (15 marks)

10. (a) (i) $a = -1$, $b = 3$, $c = 1$

(ii) Horizontal asymptotes: $y = -1$

Vertical asymptotes: $x = -1$

(b) (i) Maximum Point: $(2, 27)$

Points of inflexion: $(-1, 0)$, $(1, 16)$

(c) $\dfrac{267}{5} - 8\ln 2$

(2003-AS-M & S #07) (15 marks)

7. (a) Horizontal asymptotes: $y = 2$

Vertical asymptotes: $x = \dfrac{7}{2}$

(b) $a = 12$, $b = 4$, $c = 2$

(d) $\lambda = 1 + \dfrac{5\sqrt{6}}{6}$

(2004-AS-M & S #07) (15 marks)

7. (a) (i) Vertical Asymptote: $x = -3$

Horizontal Asymptote: $y = 1$

(c) $\lambda = 2 + \dfrac{1}{2}\sqrt{102}$

(2004-CE-A MATH #17) (12 marks)

17. (a) (i) $(x - \pi)\sin x + \cos x +$ constant

(ii) (1) Area of $R_1 = \dfrac{\pi}{2} - 1$

Area of $R_2 = \dfrac{\pi}{2} - 1$

(2) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x - \pi)\cos x \, dx = 0$

(2005-AS-M & S #07) (15 marks)

7. (a) $a = 24$, $b = 12$

(b) Horizontal asymptotes: $y = 2$

Vertical asymptotes: $x = -2$

(d) $\dfrac{51}{2} - 20\ln\left(\dfrac{5}{2}\right)$

(2006-AS-M & S #07) (15 marks)

7. (a) $a = -6$, $b = 3$

(b) (i) Horizontal asymptotes: $y = -3$

Vertical asymptotes: $x = 4$

(d) $84 - 36\ln 2$

(2006-AL-P MATH 2 #07) (15 marks)

7. (a) $f'(x) = \dfrac{x(x + 12)}{(x + 6)^2}$

$f''(x) = \dfrac{72}{(x + 6)^3}$

(b) (i) $x < -12$ or $x > 0$

(ii) $-12 < x < -6$ or $-6 < x < 0$

(iii) $x > -6$

(iv) $x < -6$

(c) Maximum point: $(-12, -25)$

Minimum point: $(0, -1)$

(d) Vertical Asymptote: $x = -6$

Oblique Asymptote: $y = x - 7$

(f) $\dfrac{65}{2} - 72\ln\dfrac{3}{2}$

(2006-CE-A MATH #13) (7 marks)

13. (a) Maximum: $x = a$

Minimum: $x = 0$

(b) (i) $f(a) = 4$

(2007-AS-M & S #07) (15 marks)

7. (b) (i) Minimum point: $\left(2, \frac{-27}{2}\right)$

Maximum point: $(-4, 0)$

Point of Inflexion: $\left(-1, \frac{-27}{4}\right)$

(c) $\frac{2955}{32} - 144 \ln\left(\frac{3}{2}\right)$

(2009-AS-M & S #07) (15 marks)

7. (a) (i) $h = 1$, $k = 2$

(b) (i) $p = \frac{5}{4}$, $q = \frac{-3}{8}$

(iii) $\frac{103}{96} - \ln 2$

(2010-AS-M & S #07) (15 marks)

7. (a) (i) $a = -2$

(ii) $y = 1$

(iii) $f'(x) = \frac{-6}{(x-2)^2}$

(b) (ii) $27 - 36 \ln 2$

(2011-AS-M & S #07) (15 marks)

7. (a) $y = \frac{-4x}{2x - 1}$

(b) $\left(\frac{1}{6}, 1\right)$ and $\left(\frac{-1}{2}, -1\right)$

(d) $\frac{5}{4} \ln 3 - \ln 2 - \frac{1}{2}$

(SP-DSE-MATH-EP(M2) #12) (14 marks)

12. (a) (i) x-intercepts are -3 and 3

y-intercepts are -9

(ii) $f'(x) = \frac{-16x}{(x-1)^2(x+1)^2}$

(iii) Maximum point $(0, -9)$

(b) Vertical Asymptote: $x = -1$ and $x = 1$

Horizontal Asymptote: $y = -1$

(d) $S = 4 \ln\left(\frac{a-1}{a+1}\right) + 4 \ln 2$

(2012-AS-M & S #07) (15 marks)

7. (a) $a = 2$, $b = 4$, $c = 3$

(b) (i) Vertical Asymptote: $x = 3$

Horizontal Asymptote: $y = -2$

(c) $\sqrt{3}$

(2013-AS-M & S #07) (15 marks)

7. (a) $k = 2$

(b) $(1, 1)$ and $\left(-1, \frac{1}{3}\right)$

(d) $\frac{5}{4} \ln 3 - \ln 2 - \frac{1}{2}$

(2015-DSE-MATH-EP(M2) #09) (13 marks)

9. (a) $f'(x) = \frac{x^2 - 4x - 12}{(x-2)^2}$

(c) Vertical Asymptote: $x = 2$

Oblique Asymptote: $y = x + 2$

(d) $30 - 32 \ln 2$

(2016-DSE-MATH-EP(M2) #09) (13 marks)

9. (a) $a = -3$ and $b = -9$

(c) -22

(d) $(1, -6)$

(e) 108

(2020-DSE-MATH-EP(M2) #09) (12 marks)

9. (a) Vertical Asymptote: $x = 4$

Oblique Asymptote: $y = x + 20$

(b) $\frac{384(x+4)}{(x-4)^4}$

(c) ... Disagree

(d) $(-4, 0)$

(e) $136 - 192 \ln 2$

6. (c) Volume of Solids and Rates of Change

(1984-CE-A MATH 2 #10) (20 marks)

10. (a) Use the substitution $x = a \sin \phi$ to show that $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{2}$.

(b) Figure 1 shows two semicircles APB and AQB with a common centre $C(0, b)$ and equal radii a . AB is parallel to the x-axis.

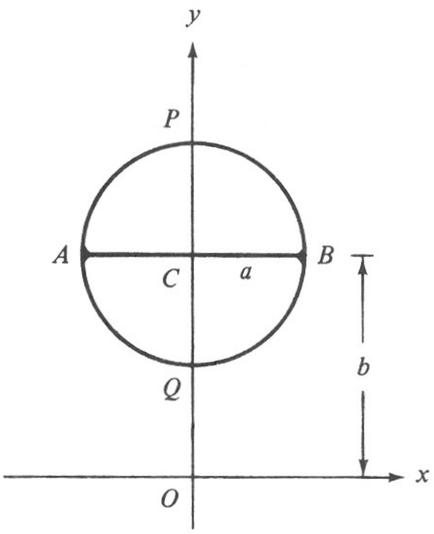


Figure 1

(i) Show that the equation APB is

$$y = b + \sqrt{a^2 - x^2}$$

and that of AQB is

$$y = b - \sqrt{a^2 - x^2} \ .$$

(ii) The region bounded by the two semicircles is revolved about the x-axis to generate a solid (called an anchor-ring). Use the result in (a) to prove that the volume of the anchor-ring is $2\pi^2 a^2 b$.

(c) A sweet has the form of an anchor-ring with $a = 2$ mm and $b = 8$ mm . Write down its volume in terms of π . The sweet is now dropped into water and it dissolves with a rate of change of volume given by

$$\frac{dV}{dt} = -32\pi^2(2 - t) \text{ mm3/h}$$

where V is the volume in mm3, t is the time in hours.

Find V in terms of t and hence find the time required to dissolve the whole sweet completely.

(1991-CE-A MATH 2 #11) (16 marks)

11.

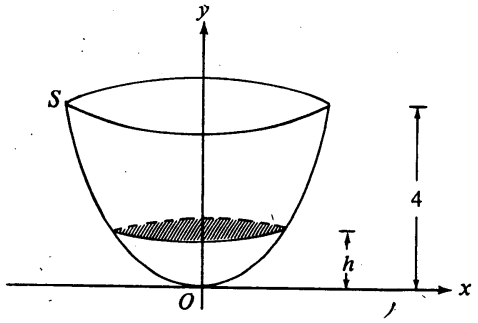


Figure 3(a)

An object S is in the shape of the solid of revolution of the region bounded by the curve $x^2 = 4y$ and the line $y = 4$ revolved about the y-axis, as shown in Figure 3(a).

(a) Find the volume of S .

(b) It is given that if S is cut by a plane parallel to its top surface at a distance h from O (see Figure 3(a)), the mass of the part of S below the plane is given by

$$\int_0^h \pi\left(16y - 3y^2\right) dy \text{ , where } 0 < h \leq 4 \text{ .}$$

(i) Find the mass of S .

(ii) If the plane mentioned above cuts S into two parts of equal volumes, find

(1) the value of h ,

(2) the ratio of the mass of the lower part to that of the upper part.

(c)

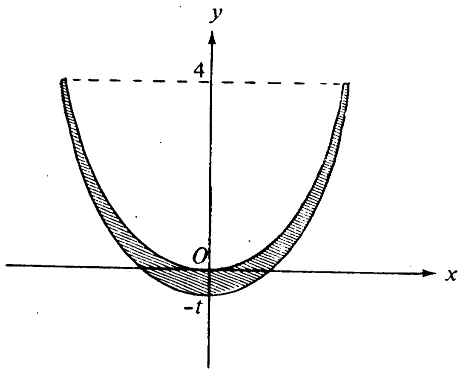


Figure 3(b)

In Figure 3(b), the shaded region is bounded by the curves $x^2 = 4y$, $x^2 = 4(y + t)$ and the line $y = 4$, where t is a small positive number. The solid of revolution of the shaded area revolved about the y-axis represents a layer of paint coated on the curved surface of S . Show that the volume of paint is approximately equal to $16\pi t$.

(1992-CE-A MATH 2 #11) (16 marks)

11. (a)

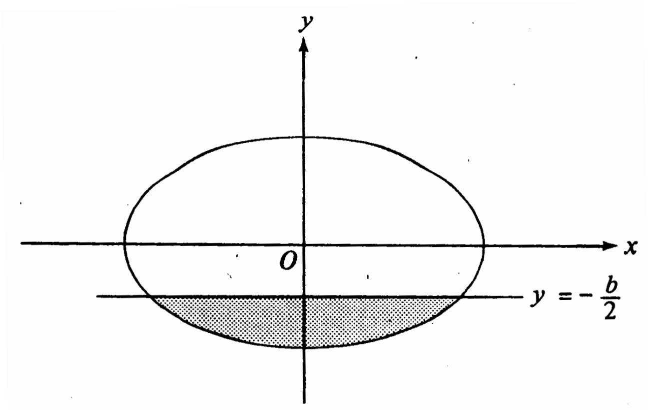


Figure 3 (a)

The shaded region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $y=\frac{-b}{2}$, as shown in Figure 3 (a), is revolved about the y-axis. Show that the volume of the solid of revolution is $\frac{5\pi a^2 b}{24}$.

(b)

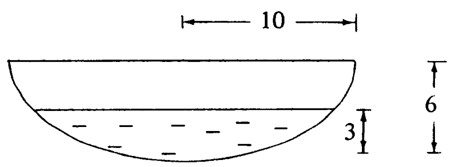


Figure 3 (b)

A bowl is generated by revolving the lower half of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the y-axis. The depth of the bowl is 6 units and the radius of its rim is 10 units. The bowl contains water to a depth of 3 units. (See Figure 3 (b).)

(i) Find the area of the water surface.

(ii) Using the result of (a), find the volume of water.

(iii) The water in the bowl is heated. At time t seconds after heating, the volume of water decreases at a rate of $\frac{\pi}{100}(25+2t)$ cubic units per second.

(1) Find the volume of water remaining in the bowl after t seconds.

(2) Calculate the time required to dry up the water in the bowl.

(1993-CE-A MATH 2 #12) (16 marks)

12.

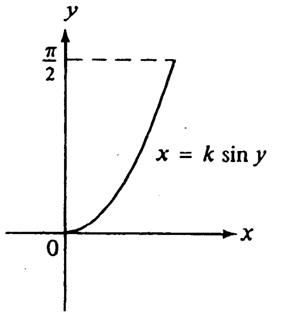


Figure 4 (a)

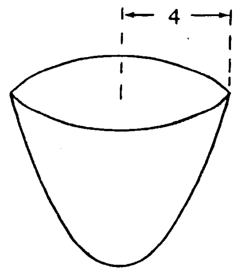


Figure 4 (b)

Figure 4 (a) shows the curve $x = k \sin y$, where $k > 0$ and $0 \leq y \leq \frac{\pi}{2}$. A bowl is formed by revolving the curve about the y-axis.

(a) Show that the capacity of the bowl is $\frac{1}{4}k^2\pi^2$ cubic units.

(b) Given that the radius of the rim of the bowl is 4 units. (See Figure 4 (b).) The bowl is full of water.

(i) Find the volume of water.

(ii) The water is now pumped out of the bowl at a rate of $(\pi + 2t)$ cubic units per minute, where t is the time in minutes after pumping starts. Find the time taken to pump out half of the water and the time taken to pump out the remaining water in the bowl. Give both answers in terms of π .

(1994-CE-A MATH 2 #13) (16 marks)

13.

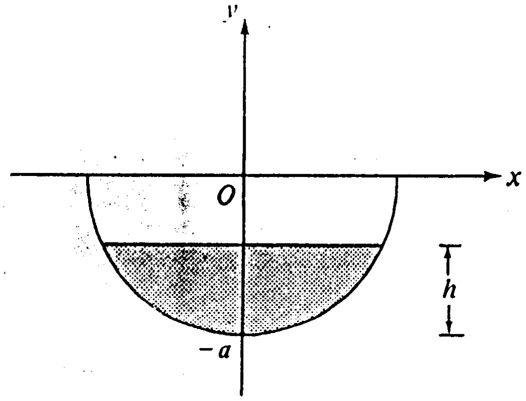


Figure 4

A bowl is generated by revolving the lower half of the circle $x^2 + y^2 = a^2$ about the y-axis. At time t , the bowl contains water to a depth of h , where $0 \le h \le a$ (see Figure 4). Let V be the volume of water in the bowl and A be the area of the water surface.

(a) Show that

(i) $V = \pi h^2 \left(a - \frac{1}{3} h \right)$,

(ii) $A = \pi h(2a - h)$.

(b) Suppose the water evaporates at a rate proportional to the area of the water surface at that instant,

i.e. $\frac{\mathrm{d}v}{\mathrm{d}t} = -kA$,

where k is a positive constant.

(i) Show that $\frac{\mathrm{d}h}{\mathrm{d}t}$ is a constant.

(ii) It is given that the depth of water is $\frac{3}{4}a$ at $t = 0$ and the water will completely evaporate at $t = 30$.

(1) Show that, for $0 \le t \le 30$,

$$h = \frac{a}{40}(30 - t) \ .$$

(2) Find the volume of water in the bowl at $t = 10$.

(1995-CE-A MATH 2 #12) (16 marks)

12. (a) Using the substitution $y - k = \cos\theta$, show that

$$\int_{k-1}^{k+1} \sqrt{1-(y-k)^2}\,dy = \frac{\pi}{2} .$$

Hence show that

(i) $$\int_0^2 \left[2+\sqrt{1-(y-1)^2}\right]^2 dy = \frac{28}{3} + 2\pi ,$$

(ii) $$\int_2^4 \left[2-\sqrt{1-(y-3)^2}\right]^2 dy = \frac{28}{3} - 2\pi .$$

(b)

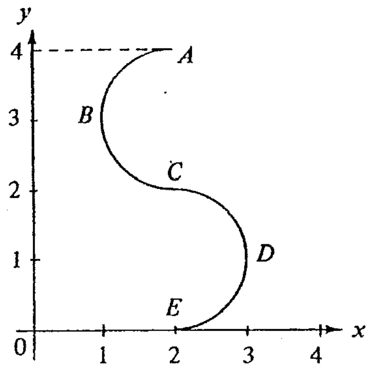


Figure 5(a)

Figure 5(a) shows two semicircles ABC and CDE centred at $(2, 3)$ and $(2, 1)$ respectively. Their radii are both equal to 1.

(i) Show that the equation of the semicircle ABC is

$$x = 2 - \sqrt{1-(y-3)^2} ,$$

and that of the semicircle CDE is

$$x = 2 + \sqrt{1-(y-1)^2} .$$

(ii) A pot is formed by revolving the curve $ABCDE$ and the line segment OE about the y-axis , where O is the origin. Using (a), find the capacity of the pot.

(c)

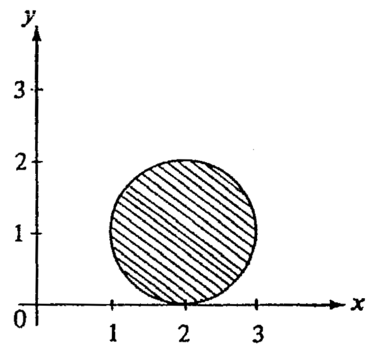


Figure 5(b)

The shaded region enclosed by the circle $(x-2)^2 + (y-1)^2 = 1$, as shown in Figure 5(b), is revolved about the y-axis to form a solid. Using (a) and (b), or otherwise, find the volume of the solid.

(1996-CE-A MATH 2 #11) (16 marks)

11.

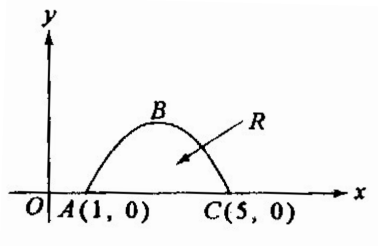


Figure 4 (a)

In Figure 4(a), the region R is enclosed by the parabola $y = 4 - (x - 3)^2$ and the line segment AC , where A and C are the points $(1, 0)$ and $(5, 0)$ respectively. B is the vertex of the parabola.

(a) Write down the coordinates of B .

(b) (i) Show that the equation of the curve AB is

$$x = 3 - \sqrt{4 - y} \ .$$

(ii) Write down the equation of the curve BC .

(c)

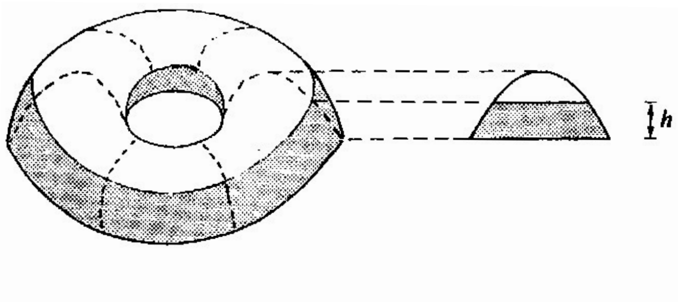


Figure 4(b)

A jelly ring is in the shape of the solid of revolution of the region R in Figure 4(a) about the y-axis. Furthermore, the jelly ring contains two layers. Let h be the height of the lower layer. (See Figure 4(b).)

(i) Show that the volume of the lower layer of the jelly ring is

$$8\pi \left[8 - (4 - h)^{\frac{3}{2}} \right] \ .$$

(ii) If the two layers have equal volumes, find the value of h correct to 3 significant figures.

(d)

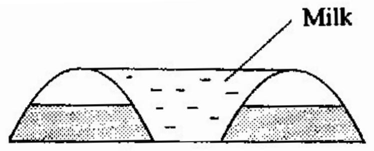


Figure 4(c)

If milk is poured into the centre of the jelly ring in (c) until it is completely filled (see Figure 4(c)), find the volume of milk required.

(1997-CE-A MATH 2 #10) (16 marks)

10.

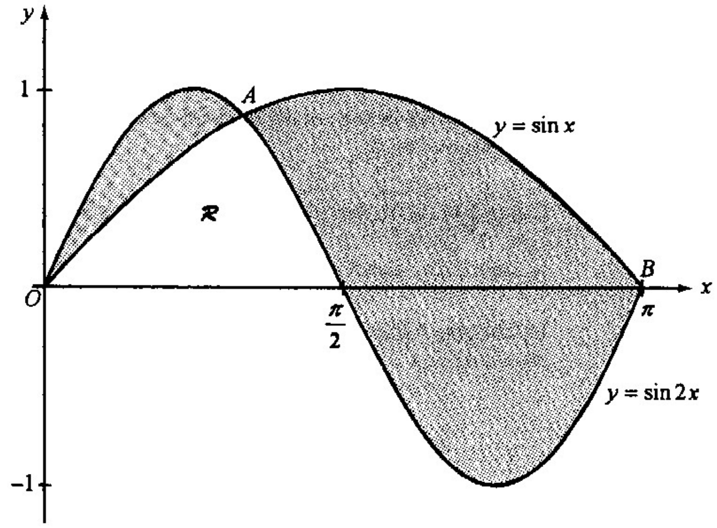


Figure 2(a)

Figure 2(a) shows the curves $y = \sin x$ and $y = \sin 2x$ for $0 \le x \le \pi$. The two curves intersect at the origin O , point A and point $B(\pi, 0)$.

(a) Show that the coordinates of A are $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$.

(b) Find the area of the shaded region in Figure 2(a).

(c) R is the region bounded by the two curves and the x-axis from $x = 0$ to $\frac{\pi}{2}$. (See Figure 2(a).)

If the region R is revolved about the x-axis, find the volume of the solid of revolution generated.

(d) Figure 2(b) shows the curve $y = \sin x$ for $0 \le x \le \pi$.

In Figure 2(b), sketch the curve $y = |\sin x|$ for $0 \le x \le \pi$.

In the same figure, shade the region whose area is represented by the expression

$$\int_0^{\pi} \left| |\sin 2x| - \sin x \right| \mathrm{d}x \ .$$

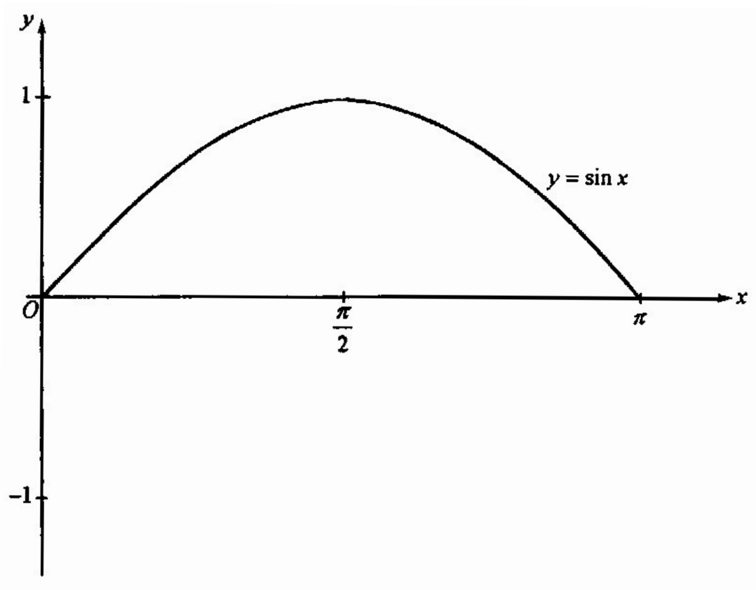


Figure 2(b)

(1998-CE-A MATH 2 #12) (16 marks)

12. (a)

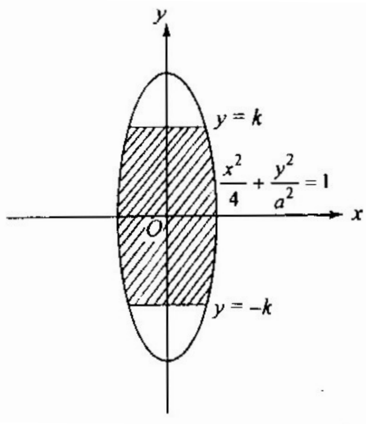


Figure 4(a)

In Figure 4(a), the shaded region enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{a^2}=1$, the lines $y=k$ and $y=-k$, where $0 \le k \le a$, is revolved about the y-axis. Show that the volume of the solid of revolution is $8k\left(1-\frac{k^2}{3a^2}\right)\pi$.

(b)

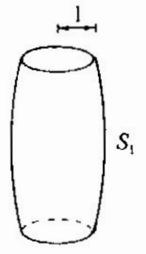


Figure 4(b)

A solid S_1 is in the shape of the solid of revolution described in (a). Furthermore, the radii of the plane circular faces of the solid are both equal to 1. (See Figure 4(b).)

(i) Show that the height of S_1 is equal to $\sqrt{3}a$.

(ii) Find the volume of S_1 in terms of a and π.

(c)

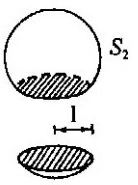


Figure 4(c)

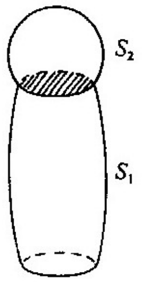


Figure 4(d)

A solid sphere of radius 2 is cut along a plane into two unequal portions as shown in Figure 4 (c). The radius of the plane circular face is equal to 1. The larger portion S_2 is joined to S_1 in (b) to form a toy as shown in Figure 4 (d).

(i) Show that the height of the toy is $2+(a+1)\sqrt{3}$.

(ii) Using (b), or otherwise, find the volume of the toy in terms of a and π.

(1999-CE-A MATH 2 #13) (16 marks)

13.

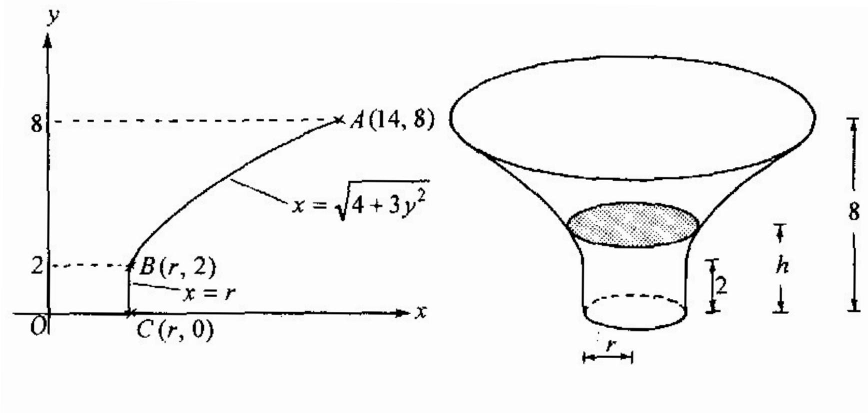


Figure 6(a) Figure 6(b)

A curve passes through three points $A(14, 8)$, $B(r, 2)$ and $C(r, 0)$ as shown in Figure 6(a). The curve consists of two parts. The equation of the part joining A and B is $x = \sqrt{4 + 3y^2}$ and the part joining B and C is the vertical line $x = r$.

(a) Find the value of r .

(b) A pot, 8 units in height, is formed by revolving the curve and the line segment OC about the y-axis, where O is the origin. (See Figure 6(b).) If the pot contains water to a depth of h units, where $h > 2$, show that the volume of water V in the pot is $\left(h^3 + 4h + 16\right)\pi$ cubic units.

(c) Initially, the pot in (b) contains water to a depth greater than 3 units. The water is now pumped out at a constant rate of 2π cubic units per second. Find the rate of change of the depth of the water in the pot with respect to time when

(i) the depth of the water is 3 units, and

(ii) the depth of the water is 1 unit.

(2000-CE-A MATH 2 #11) (16 marks)

11. (a) In Figure 7(a), the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the x-axis, the y-axis and the line $y = -h$, where $h > 0$. If the shaded region is revolved about the y-axis, show that the volume of the solid generated is $\left(r^2 h - \frac{1}{3}h^3\right)\pi$ cubic units.

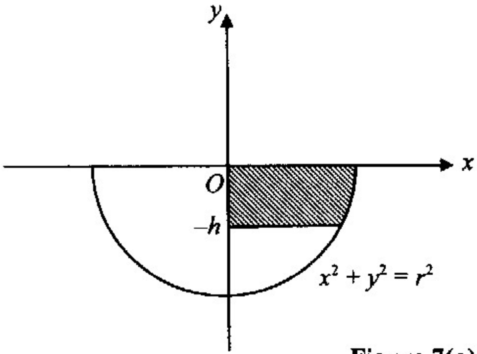


Figure 7(a)

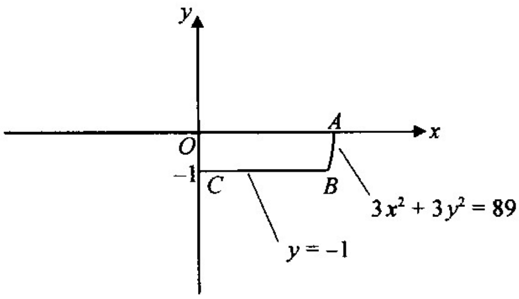


Figure 7(b)

(b) In Figure 7(b), A and C are points on the x-axis and y-axis respectively, AB is an arc of the circle $3x^2 + 3y^2 = 89$ and BC is a segment of the line $y = -1$. A mould is formed by revolving AB and BC about the y-axis. Using (a), or otherwise, show that the capacity of the mould is $\frac{88\pi}{3}$ cubic units.

(c)

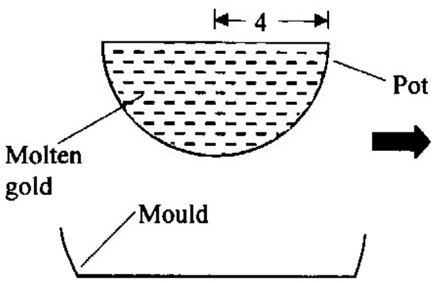


Figure 7(c)

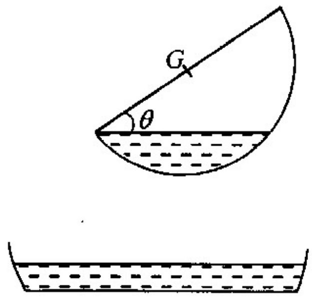


Figure 7(d)

A hemispherical pot of inner radius 4 units is completely filled with molten gold. (See Figure 7(c).) The molten gold is then poured into the mould mentioned in (b) by steadily tilting the pot. Suppose the pot is tilted through an angle θ and G is the centre of the rim of the pot. (See Figure 7(d).)

(i) Find, in terms of θ ,

(1) the distance between G and the surface of the molten gold remaining in the pot.

(2) the volume of gold poured into the mould.

(ii) When the mould is completely filled with molten gold, show that

$$8\sin^3\theta - 24\sin\theta + 11 = 0 \ .$$

Hence find the value of θ .

(2001-CE-A MATH #16) (12 marks)

16.

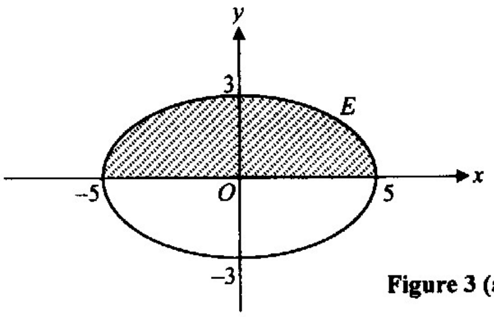


Figure 3 (a)

Figure 3 (a) shows the ellipse $E : \frac{x^2}{25} + \frac{y^2}{9} = 1$. The shaded region in the first and second quadrants is bounded by E and the x-axis.

(a) Using integration and the substitution $x = 5\sin\theta$, find the area of the shaded region.

(b) A piece of chocolate is in the shape of the solid of revolution formed by revolving the shaded region in Figure 3 (a) about the x-axis. Using the integration, find the volume of the chocolate.

(c)

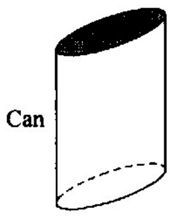


Figure 3 (b)

Four pieces of the chocolate mentioned in (b) are packed in a can which is in the shape of a right non-circular cylinder (see Figure 3 (b)). The chocolates are placed one above the other. Their axes of revolution are parallel to the base of the can and in the same vertical plane. For economic reasons, the can is in a shape such that it can just hold the chocolates. Find the capacity of the can.

(2002-CE-A MATH #16) (12 marks)

16. (a)

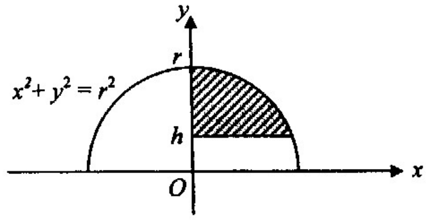


Figure 8

In Figure 8, the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the y-axis and the line $y = h$, where $0 \le h \le r$. The shaded region is revolved about the y-axis. Show that the volume of the solid generated is $\frac{\pi}{3}\left(2r^3 - 3r^2h + h^3\right)$.

(b)

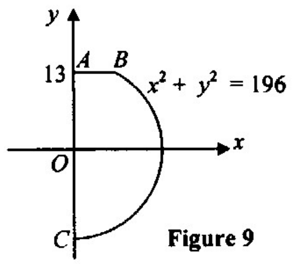


Figure 9

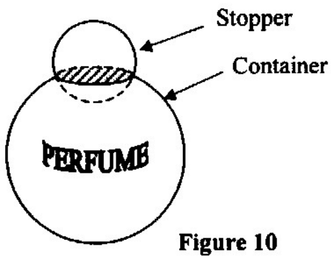


Figure 10

In Figure 9, A and C are points on the y-axis, BC is an arc of the circle $x^2 + y = 196$ and AB is a segment of the line $y = 13$. A pot is formed by revolving BC about the y-axis.

(i) Find the capacity of the pot.

(ii) Figure 10 shows a perfume bottle. The container is in the shape of the pot described above and the stopper is a solid sphere of radius 6 . Find the capacity of the perfume bottle.

(2003-CE-A MATH #19) (12 marks)

19. A water tank is formed by revolving the curve $y = x^2$, where $0 \le x \le 2$, about the y-axis (see Figure 6). Starting from time $t = 0$, water is pumped into the tank at a constant rate of 2π cubic units per minute. Let the volume and the depth of water (measured from the lowest point of the tank) in the tank at time t (in minutes) be V cubic units and h units respectively.

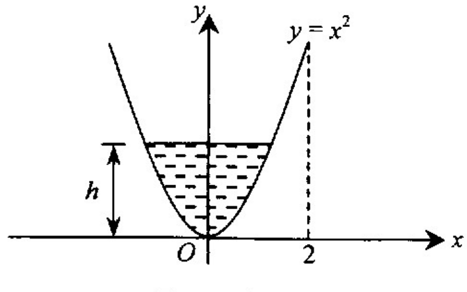


Figure 6

(a) Express V in terms of h.
Hence show that it takes 4 minutes to fill up the tank.

(b) Show that $\dfrac{dh}{dt} = \dfrac{2}{h}$.

(c) Which of the sketches in Figure 7 best describes the relation between h and t? Explain your answer.

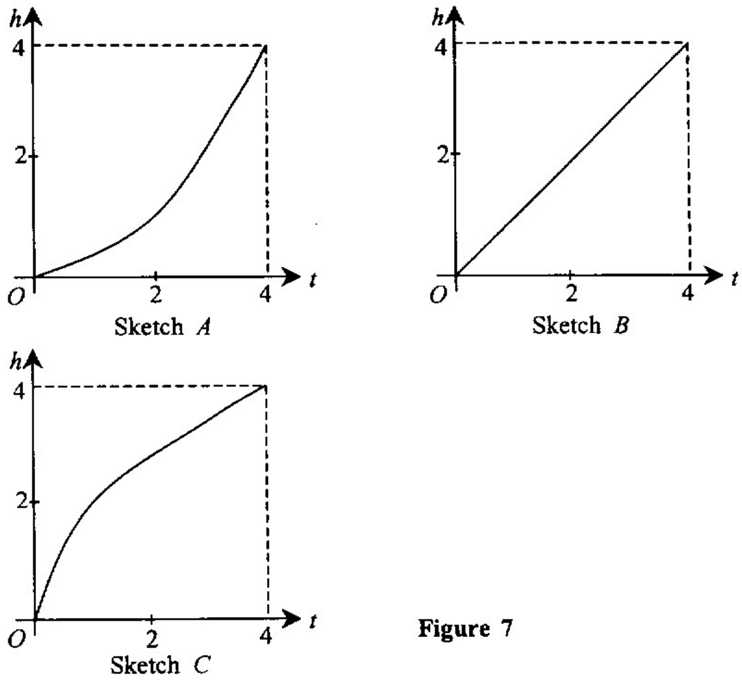


Figure 7

(d)

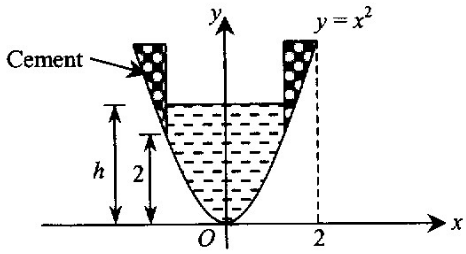


Figure 8

An engineer decides to modify the tank by laying cement on the upper part of its interior wall, so that the interior of the tank becomes cylindrical in shape for $y \ge 2$ as shown in Figure 8. Water is pumped into the empty new tank at a constant rate of 2π cubic units per minute until it is full. Sketch the graph of h against t for the new tank in the same sketch you chose in (c).

(2004-CE-A MATH #04) (4 marks)

4.

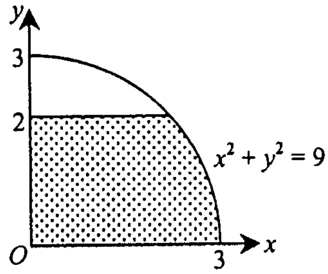


Figure 1

In Figure 1, the shaded region is bounded by the circle $x^2 + y^2 = 9$, the x-axis, the y-axis and the line $y = 2$. Find the volume of the solid generated by revolving the region about the y-axis.

(2005-CE-A MATH #16) (12 marks)

16.

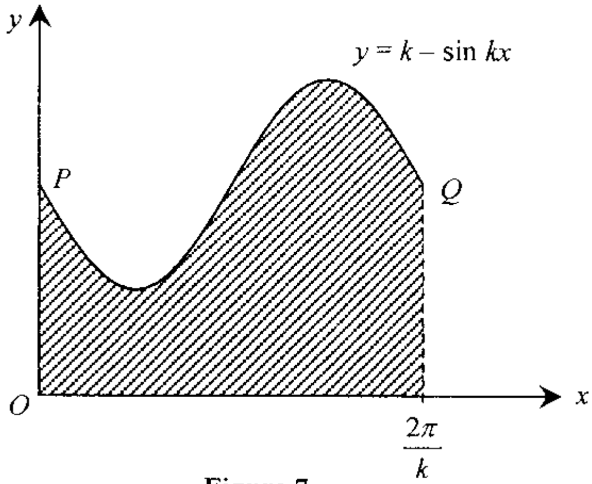


Figure 7

Figure 7 shows the curve $y = k - \sin kx$ for $0 \le x \le \frac{2\pi}{k}$, where $k > 1$. P and Q are the end points of the curve.

(a) (i) Find the coordinates of points P and Q .

(ii) Without using integration, find the area of the shaded region as shown in Figure 7.

(b) A solid is formed by revolving the shaded region in Figure 7 about the x-axis. Let V be the volume of the solid.

(i) Show that $V = \pi^2 \left(2k + \frac{1}{k}\right)$.

(ii) Suppose that $2 \le k \le 3$. Find the greatest value of V as k varies.

(2006-CE-A MATH #16) (12 marks)

16. Let C be the curve $y = \frac{1}{3}x^2 - \frac{4}{3}x + 1$. C_1 is a part of C with $0 \le x \le 1$ and C_2 is a part of C with $3 \le x \le 4$.

(a) (i) Show that the equation fo C_1 is $x = 2 - \sqrt{3y + 1}$.

(ii) Write down the equation of C_2 in the form $x = f(y)$.

(b)

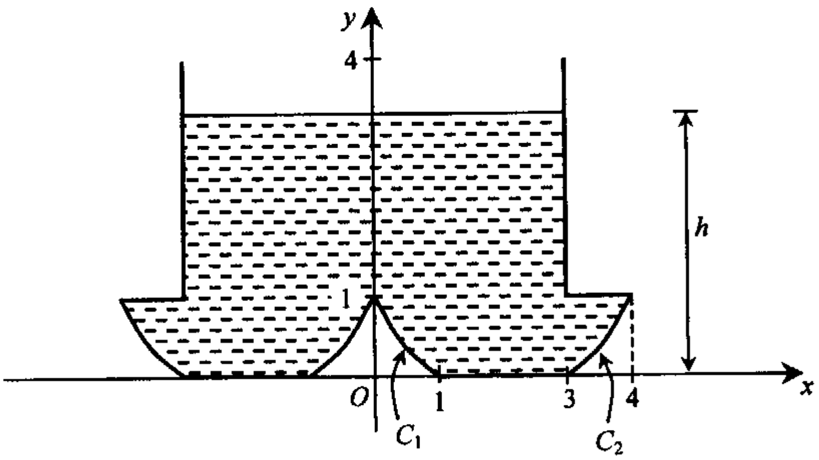


Figure 6

A container is formed by revolving C_1 , the line segment $y = 0$ (for $1 \le x \le 3$), C_2 , the line segment $y = 1$ (for $3 \le x \le 4$) and the line segment $x = 3$ (for $1 \le y \le 4$) about the y-axis (see Figure 6). Starting from time $t = 0$, water is poured into the container at a constant rate of 8π cubic units per minute. Let the volume and depth of water in the container at time t minute be V cubic units and h units respectively.

(i) Consider $0 < h < 1$.

(1) Show that $V = \frac{16\pi}{9}\left[(3h + 1)^{\frac{3}{2}} - 1\right]$.

(2) Find $\frac{dh}{dt}$ in term of h .

(ii) Consider $1 < h < 4$. Find $\frac{dh}{dt}$.

(iii) It is known that $h = 1$ at $t = t_1$ and $h = 4$ at $t = t_2$. Sketch a graph to show how h varies with t for $0 \le t \le t_2$. (You are not required to find the values of t_1 and t_2 .)

(2007-AL-P MATH 2 #05) (7 marks)

5. (a) Find $\int \left(\frac{(x - 2)(x - 5)}{x}\right)^2 dx$.

(b) Let D be the region bounded by the curve $y = \frac{x(x - 3)}{x + 2}$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(2007-CE-A MATH #18) (12 marks)

18. (a) It is given that the curve $y = 2\sqrt{x} - x$ has a horizontal tangent at $x = r$. Show that $r = 1$.

(b)

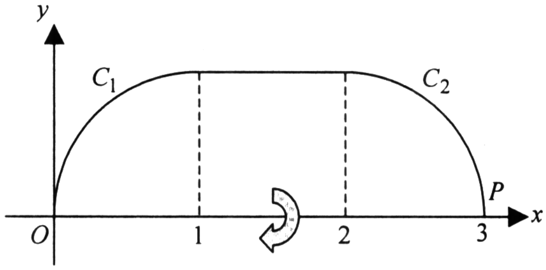


Figure 11

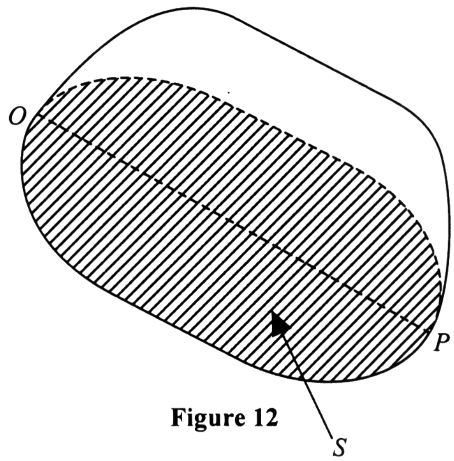


Figure 12

Let O be the origin and P be the point $(3, 0)$. Figure 11 shows a region bounded by:

(1) the curve $C_1 : y = 2\sqrt{x} - x$ (for $0 \leq x \leq 1$),

(2) the line segment $y = 1$ (for $1 \leq x \leq 2$),

(3) the curve $C_2 : y = 2\sqrt{3 - x} - (3 - x)$ (for $2 \leq x \leq 3$), and

(4) OP.

Figure 12 shows a solid formed by revolving the region about the x-axis by $180°$.

(i) The base of the solid is denoted by S in Figure 12. Find the area of S.

(ii) Show the the volume of the solid is $\frac{37}{30}\pi$.

(iii)

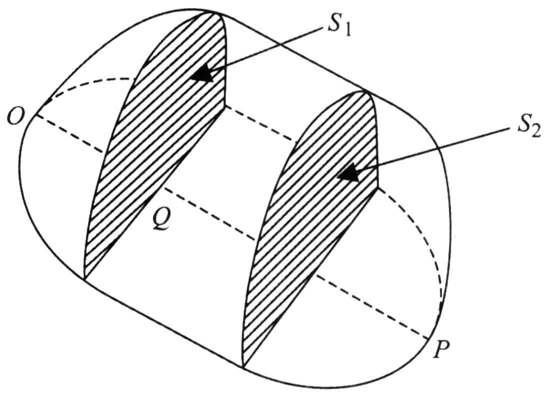


Figure 13

Mrs. Chan has baked a cake which is in the shape of the solid in Figure 12. She cuts the cake into three parts of equal volumes for her three children. The cross-sections formed, S_1 and S_2 , are perpendicular to OP (see Figure 13). Let the intersection of OP and S_1 be Q . Find $OQ : OP$.

(2008-AL-P MATH 2 #05) (6 marks)

5. (a) Find $\int y\sqrt{1-y^2}\,dy$.

(b) Let D be the region bounded by the curve $x = y^{\frac{1}{2}}\left(1-y^2\right)^{\frac{1}{4}}$ and the y-axis. Find the volume of the solid of revolution generated by revolving D about the y-axis.

(2008-CE-A MATH #17) (12 marks)

17. (a) Let $f(x) = x \sin x$.

(i) Show that $\int_0^{\pi} \left[f(x) + f(\pi - x)\right] dx = 2\pi$.

(ii)

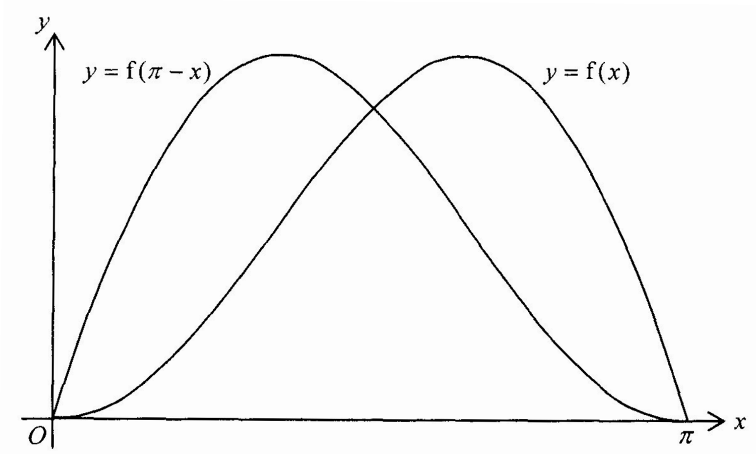


Figure 6

Figure 6 shows the graphs of $y = f(x)$ and $y = f(\pi - x)$ for $0 \le x \le \pi$. Using (a)(i), and by considering the symmetry of the graphs of $y = f(x)$ and $y = f(\pi - x)$, write down the value of $\int_0^{\pi} x \sin x \, dx$.

(b) (i) Find $\frac{d}{dx}\left(x^2 \sin x\right)$, and hence evaluate $\int_0^{\pi} x^2 \cos x \, dx$.

(ii) R denotes the region bounded between $y = x \sin \frac{x}{2}$ and the x-axis for $0 \le x \le \pi$. Find the volume of the solid formed by revolving R about the x-axis.

(2009-CE-A MATH #15) (12 marks)

15. (a)

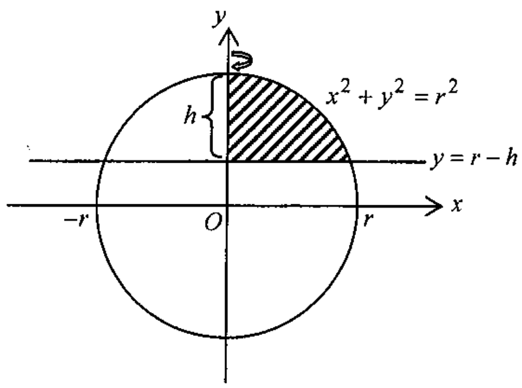


Figure 5

In Figure 5, the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the y-axis and the straight line $y = r - h$, where $0 \le h \le 2r$. Show that the volume of the solid generated by revolving the shaded region about the y-axis is $\pi r h^2 - \dfrac{\pi h^3}{3}$.

(b)

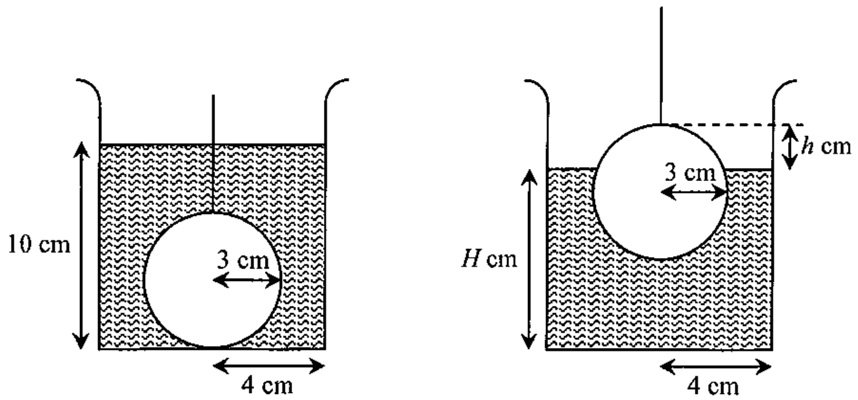


Figure 6 **Figure 7**

A metal sphere of radius 3 cm , with a thin string attached, is placed inside a circular cylindrical container of base radius 4 cm . Water is poured into the container until the depth of the water is 10 cm (see Figure 6). The sphere is then being pulled vertically out of the water. Let H cm and h cm be the depth of the water and the distance between the top of the sphere and the water surface respectively (see Figure 7).

(i) Prove that $H = \dfrac{1}{48}\left(h^3 - 9h^2 + 480\right)$.

(ii) The sphere is being pulled at a constant speed of $\dfrac{1}{4}$ cm s^{-1} . At the instant when $h = 3$, find the rate of change of

(1) the depth of the water,

(2) the distance between the top of the sphere and the water surface.

(2010-AL-P MATH 2 #05) (7 marks)

5. (a) Find $\int (5 - x)\sqrt{x - 1}\,\mathrm{d}x$.

(b) Let D be the region bounded by the curve $y = (5 - x)^{\frac{1}{2}}(x - 1)^{\frac{1}{4}}$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(2010-CE-A MATH #16) (12 marks)

16. (a)

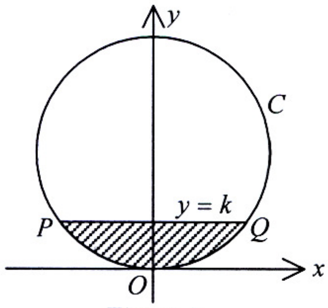


Figure 7

Figure 7 shows the shaded region bounded by the circle $C: x^2 + y^2 - 2ay = 0$ and the line $PQ: y = k$ where $0 \le k \le 2a$ and $a > 0$. Show that the volume of the solid by revolving the shaded region about the y-axis is $\pi\left(ak^2 - \dfrac{k^3}{3}\right)$.

(b)

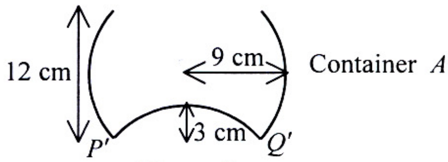


Figure 8

Figure 8 shows a container A in the shape of part of a sphere of radius 9 cm with the bottom part reflected about the horizontal plane passing through P' and Q'. The height of the container is 12 cm and the height of the bottom part of the container is 3 cm.

(i) Find the capacity of container A.

(ii)

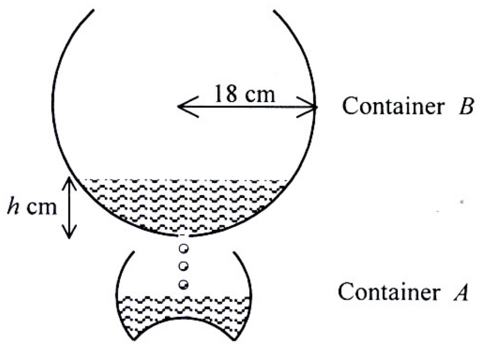


Figure 9

In Figure 9, container B is in the shape of part of the sphere of radius 18 cm. Initially, container A is empty and container B contains water of volume 909π cm³. Water is leaking through a hole at the bottom of container B into container A. Let h cm be the depth of water in container B. Consider the moment when container A is just full.

(1) Show that $h^3 - 54h^2 + 459 = 0$.

(2) If the volume of water in container A is increasing at the rate 45π cm³ s^{-1}, find the rate of decrease of water level in container B.

(2011-AL-P MATH 2 #05) (7 marks)

5. (a) Using the substitution $x = 5 + 2\sin\theta$, find $\int \sqrt{(x-3)(7-x)}\,\mathrm{d}x$.

(b) Let D be the region bounded by the curve $y = \left[(x-3)(7-x)\right]^{\frac{1}{4}}$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(2011-CE-A MATH #14) (12 marks)

14. (a)

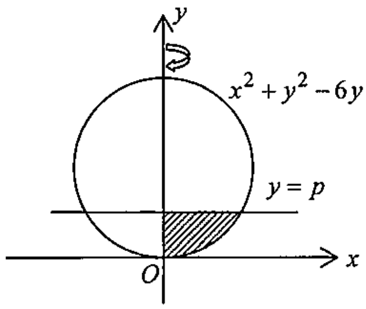


Figure 5

In Figure 5, a shaded region is bounded by the circle $x^2 + y^2 - 6y = 0$, the y-axis and the straight line $y = p$, where $0 \le p \le 6$. Show that the volume of the solid generated by revolving the shaded region about the y-axis is $\dfrac{\pi p^2(9-p)}{3}$.

(b)

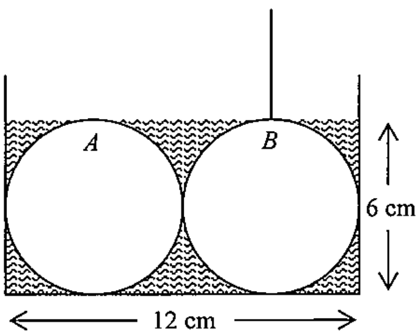


Figure 6

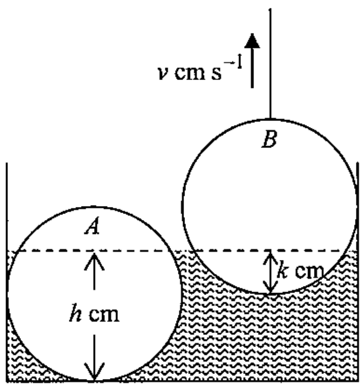


Figure 7

Two metal spheres, A and B , both of diameters 6 cm are placed inside a circular cylindrical container of base diameter 12 cm . The sphere B is attached by a wire. Water is poured into the container until the depth is 6 cm (see Figure 6).

(i) Find the volume of the water.

(ii) By considering the volume of water in Figure 7, or otherwise, prove that

$$k^3 - 9k^2 + h^3 - 9h^2 + 108h - 432 = 0 \ .$$

(iii) Suppose sphere B is being pulled at a uniform speed v cm s^{-1} and the depth of water is decreasing at a rate of 5 cm s^{-1} at the instant when $h = 5$. Find the value of v .

(PP-DSE-MATH-EP(M2) #14) (10 marks)

14. (a)

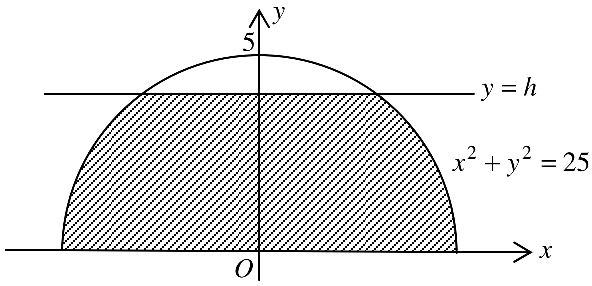


Figure 3

In Figure 3, the shaded region enclosed by the circle $x^2 + y^2 = 25$, the x-axis and the straight line $y = h$ (where $0 \le h \le 5$) is revolved about the y-axis. Show that the volume of the solid of revolution is $\left(25h - \frac{h^3}{3}\right)\pi$.

(b)

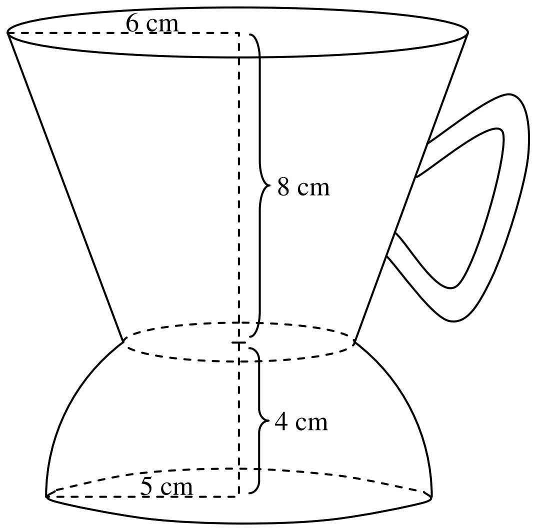


Figure 4

In Figure 4, an empty coffee cup consists of two portions. The lower portion is in the shape of the solid described in (a) with height 4 cm . The upper portion is a frustum of a circular cone. The height of the frustum is 8 cm . The radius of the top of the cup is 6 cm . Hot coffee is poured into the cup to a depth h cm at a rate of 8 cm³ s^{-1} ,where $0 \le h \le 12$. Let V cm³ be the volume of coffee in the cup.

(i) Find the rate of increase of depth of coffee when the depth is 3 cm .

(ii) Show that $V = \frac{164\pi}{3} + \frac{3\pi}{64}(h+4)^3$ for $4 \le h \le 12$.

(iii) After the cup is fully filled, suddenly it cracks at the bottom. The coffee leaks at a rate of 2 cm³ s^{-1}. Find the rate of decrease of the depth of coffee after 15 seconds of leaking, giving your answer correct to 3 significant figures.

(2012-AL-P MATH 2 #05) (7 marks)

5. (a) Find the derivative of $\sqrt{x}e^{2\sqrt{x}}$.

(b) Let D be the region bounded by the curve $y = e^{\sqrt{x}}$, the straight line $x = 4$, the straight line $x = 9$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(2012-DSE-MATH-EP(M2) #09) (4 marks)

9. (a) Using integration by parts, find $\int x \sin x \, dx$.

(b)

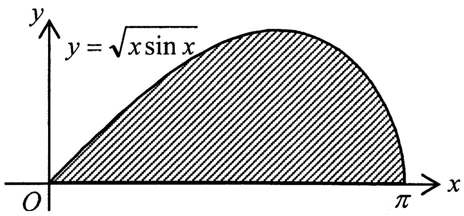


Figure 4

Figure 4 shows the shaded region bounded by the curve $y = \sqrt{x \sin x}$ for $0 \le x \le \pi$ and the x-axis. Find the volume of the solid generated by revolving the region about the x-axis.

(2013-AL-P MATH 2 #05) (6 marks)

5. (a) Find $\int \frac{(x-12)^2(x-6)}{x} dx$.

(b) Define $f(x) = (x-6)\sqrt{\frac{x}{x+6}}$ for all $x \ge 0$.

Let D be the region bounded by the curve $y = f(x)$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(2013-DSE-MATH-EP(M2) #05) (6 marks)

6.

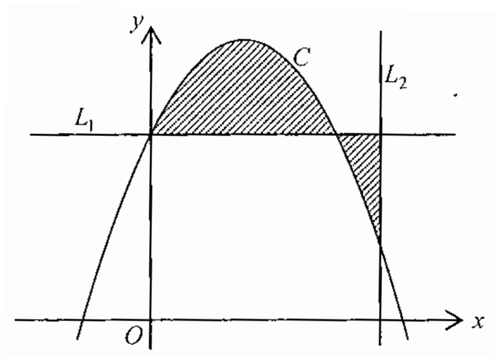


Figure 1

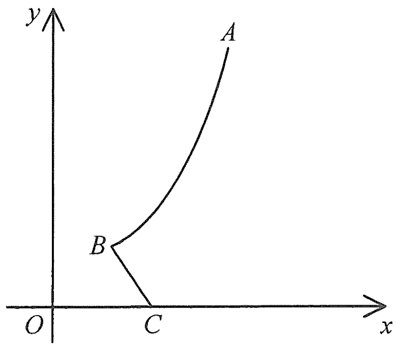
Figure 1 shows the shaded region with boundaries $C : y = \frac{-x^2}{2} + 2x + 4$, $L_1 : y = 4$ and $L_2 : x = 5$. It is given that C intersects L_1 at $(0,4)$ and $(4,4)$.

(a) Find the area of the shaded region.

(b) Find the volume of solid of revolution when the shaded region is revolved about L_1 .

(2015-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) In the figure, the curve Γ consists of the curve AB, the line segments BC and CO, where O is the origin, B lies in the first quadrant and C lies on the x-axis. The equations of AB and BC are $x^2 - 4y + 8 = 0$ and $3x + y - 9 = 0$ respectively.



(i) Find the coordinates of B.

(ii) Let h be the y-coordinate of A, where $h > 3$. A cup is formed by revolving Γ about the y-axis. Prove that the capacity of the cup is $\pi\left(2h^2 - 8h + 25\right)$.

(b) A cup described in (a)(ii) is placed on a horizontal table. The radii of the base and the lip of the cup are 3 cm and 6 cm respectively.

(i) Find the capacity of the cup.

(ii) Water is poured into the cup at a constant rate of 24π cm^3/s. Find the rate of change of the depth of water when the volume of water in the cup is 35π cm^3.

(2016-DSE-MATH-EP(M2) #07) (8 marks)

7. (a) Using integration by substitution, find $\int (1 + \sqrt{t+1})^2 \, dt$.

(b) Consider the curve $\Gamma : y = 4x^2 - 4x$, where $1 \le x \le 4$. Let R be the region bounded by Γ, the straight line $y = 48$ and the two axes. Find the volume of the solid of revolution generated by revolving R about the y-axis.

(2017-DSE-MATH-EP(M2) #09) (13 marks)

9. Define $f(x) = \dfrac{x^2 - 5x}{x + 4}$ for all $x \neq -4$. Denote the graph of $y = f(x)$ by G.

(a) Find the asymptote(s) of G.

(b) Find $f'(x)$.

(c) Find the maximum point(s) and the minimum point(s) of G.

(d) Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(2018-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) (i) Prove that $\int \sin^4 x \, dx = \frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x \, dx$.

(ii) Evaluate $\int_0^{\pi} \sin^4 x \, dx$.

(b) (i) Let $f(x)$ be a continuous function such that $f(\beta - x) = f(x)$ for all real numbers x , where β is a constant. Prove that $\int_0^{\beta} x f(x) \, dx = \frac{\beta}{2}\int_0^{\beta} f(x) \, dx$.

(ii) Evaluate $\int_0^{\pi} x \sin^4 x \, dx$.

(c) Consider the curve $G : y = \sqrt{x} \sin^2 x$, where $\pi \le x \le 2\pi$. Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(2019-DSE-MATH-EP(M2) #04) (6 marks)

4. Define $g(x) = \frac{\ln x}{\sqrt{x}}$ for all $x \in (0, 99)$. Denote the graph of $y = g(x)$ by G .

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G , the x-axis and the vertical line passing through the maximum point of G . Find the volume of the solid of revolution generated by revolving R about the x-axis.

(2020-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) Find $\int \sin^2 \theta \, d\theta$.

(b) Define $f(x) = 4x(1 - x^2)^{\frac{1}{4}}$ for all $x \in [0, 1]$. Denote the graph of $y = f(x)$ by G . Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(2021-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) Using integration by parts, find $\int (\ln x)^2 dx$.

(c) Consider the curve $C : y = \sqrt{x} \ln\left(x^2 + 1\right)$, where $x \ge 0$. Let R be the region bounded by C , the straight line $x = 1$ and the x-axis . Find the volume of the solid of revolution generated by revolving R about the x-axis .

ANSWERS

(1984-CE-A MATH 2 #10) (20 marks)

10. (c) $V = 16\pi^2 t^2 - 64\pi^2 t + C$

$t = 2$, the required time is 2 hours

(1991-CE-A MATH 2 #11) (16 marks)

11. (a) 32π

(b) (i) 64π

(ii) (1) $2\sqrt{2}$

(2) $\left(2\sqrt{2} - 1\right) : 1$

(1992-CE-A MATH 2 #11) (16 marks)

11. (b) (i) 75π

(ii) 125π

(iii) (1) $125\pi - \dfrac{\pi}{100}\left(25t + t^2\right)$

(2) 100 seconds

(1993-CE-A MATH 2 #12) (16 marks)

12. (b) (i) $4\pi^2$

(ii) Half of the water: π min.

Remaining water: $\left(\dfrac{\sqrt{17} - 3}{2}\right)\pi$ min.

(1994-CE-A MATH 2 #13) (16 marks)

13. (b) (ii) (2) $\dfrac{5}{24}\pi a^3$

(1995-CE-A MATH 2 #12) (16 marks)

12. (b) (ii) $\dfrac{56\pi}{3}$

(c) $4\pi^2$

(1996-CE-A MATH 2 #11) (16 marks)

11. (a) $B = (3, 4)$

(b) (ii) $x = 3 + \sqrt{4 - y}$

(c) (ii) 1.48

(d) 12π

(1997-CE-A MATH 2 #10) (16 marks)

10. (b) 2.5

(c) $\dfrac{\pi}{16}\left(4\pi - 3\sqrt{3}\right)$

(1998-CE-A MATH 2 #12) (16 marks)

12. (b) (ii) $3\sqrt{3}a\pi$

(c) (ii) $\dfrac{16\pi}{3} + 3\sqrt{3}\pi$

(1999-CE-A MATH 2 #13) (16 marks)

13. (a) $r = 4$

(c) (i) $\dfrac{dh}{dt} = \dfrac{-2}{31}$ units per sec.

(ii) $\dfrac{dh}{dt} = \dfrac{-1}{8}$ units per sec.

(2000-CE-A MATH 2 #11) (16 marks)

11. (c) (i) (1) $4\sin\theta$

(2) $\pi\left(64\sin\theta - \dfrac{64}{3}\sin^3\theta\right)$

(ii) $\theta = \dfrac{\pi}{6}$

(2001-CE-A MATH #16) (12 marks)

16. (a) $\dfrac{15\pi}{2}$

(b) 60π

(c) 360π

(2002-CE-A MATH #16) (12 marks)

16. (b) (i) 3645π

(ii) 3600π

(2003-CE-A MATH #19) (12 marks)

19. (a) $V = \dfrac{1}{2}\pi h^2$

(c) Sketch C

(2004-CE-A MATH #04) (4 marks)

4. $\dfrac{46\pi}{3}$

(2005-CE-A MATH #16) (12 marks)

16. (a) (i) $P(0, k)$, $Q\left(\frac{2\pi}{k}, k\right)$

(ii) 2π

(b) (ii) $\frac{19\pi^2}{3}$

(2006-CE-A MATH #16) (12 marks)

16. (a) (ii) $x = 2 + \sqrt{3y+1}$

(b) (i) (2) $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{(3h+1)^{\frac{1}{2}}}$

(ii) $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{8}{9}$

(2007-AL-P MATH 2 #05) (7 marks)

5. (a) $\frac{x^3}{3} - 7x^2 + 69x - 140\ln|x| - \frac{100}{x}$ + constant

(b) $\pi\left[129 - 140\ln\left(\frac{5}{2}\right)\right]$

(2007-CE-A MATH #18) (12 marks)

18. (b) (i) $\frac{16}{3}$

(iii) 49 : 135

(2008-AL-P MATH 2 #05) (6 marks)

5. (a) $\frac{-1}{3}\left(1-y^2\right)^{\frac{3}{2}}$ + constant

(b) $\frac{\pi}{3}$

(2008-CE-A MATH #17) (12 marks)

17. (a) (ii) π

(b) (i) -2π

(ii) $\frac{\pi^4}{6} + \pi^2$

(2009-CE-A MATH #15) (12 marks)

15. (b) (ii) (1) $\frac{-9}{28}$ cm s^{-1}

(2) $\frac{4}{7}$ cm s^{-1}

(2010-AL-P MATH 2 #05) (7 marks)

5. (a) $\frac{8}{3}(x-1)^{\frac{3}{2}} - \frac{2}{5}(x-1)^{\frac{5}{2}}$ + constant

(b) $\frac{128\pi}{15}$

(2010-CE-A MATH #16) (12 marks)

16. (b) (i) 756π cm^3

(ii) (2) $\frac{5}{11}$ cm s^{-1}

(2011-AL-P MATH 2 #05) (7 marks)

5. (a) $2\sin^{-1}\left(\frac{x-5}{2}\right) + \frac{x-5}{2}\sqrt{(x-3)(7-x)}$ + constant

(b) $2\pi^2$

(2011-CE-A MATH #14) (12 marks)

14. (b) (i) 144π cm^3

(iii) 26

(PP-DSE-MATH-EP(M2) #14) (10 marks)

14. (b) (i) $\frac{1}{2\pi}$ cm s^{-1}

(iii) 0.0183 cm s^{-1}

(2012-AL-P MATH 2 #05) (7 marks)

5. (a) $\frac{1}{2\sqrt{x}}e^{2\sqrt{x}} + e^{2\sqrt{x}}$

(b) $\frac{5}{2}\pi e^6 - \frac{3}{2}\pi e^4$

(2012-DSE-MATH-EP(M2) #09) (4 marks)

9. (a) $-x\cos x + \sin x$ + constant

(b) π^2

(2013-AL-P MATH 2 #05) (6 marks)

5. (a) $\frac{1}{3}x^3 - 15x^2 + 288x - 864\ln|x|$ + constant

(b) $\pi(612 - 864\ln 2)$

(2013-DSE-MATH-EP(M2) #05) (6 marks)

6. (a) $\frac{13}{2}$

(b) $\frac{125\pi}{12}$

(2015-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) (i) $(2, 3)$

(b) (i) 179π cm^3

(ii) 2 cm s^{-1}

(2016-DSE-MATH-EP(M2) #07) (8 marks)

7. (a) $2t + \frac{t^2}{2} + \frac{4}{3}(t+1)^{\frac{3}{2}}$ + constant

(b) 426π

(2017-DSE-MATH-EP(M2) #09) (13 marks)

9. (a) Vertical asymptote: $x = -4$

Oblique asymptote: $y = x - 9$

(b) $f'(x) = 1 - \frac{36}{(x+4)^2}$

(c) Maximum point: $(-10, -25)$

Minimum point: $(2, -1)$

(d) $\left[\frac{2285}{3} - 1872 \ln\left(\frac{3}{2}\right)\right]\pi$

(2018-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) (ii) $\frac{3\pi}{8}$

(b) (ii) $\frac{3\pi^2}{16}$

(c) $\frac{9\pi^3}{16}$

(2019-DSE-MATH-EP(M2) #04) (6 marks)

4. (b) $\frac{8\pi}{3}$

(2020-DSE-MATH-EP(M2) #04)

4. (a) $\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$ + constant

(b) π^2

(2021-DSE-MATH-EP(M2) #07)

7. (a) $x(\ln x)^2 - 2x\ln x + 2x$+constant

(b) $\pi\left((\ln 2)^2 - 2\ln 2 + 1\right)$

3. ALGEBRA AREA

1. Matrices and Determinants

(1991-AL-P MATH 1 #01) (4 marks)

1. Factorize the determinant $\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

(1992-AL-P MATH 1 #03) (7 marks)

3. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 1 & \lambda \end{pmatrix}$.

If B^{-1} exists and $B^{-1}AB = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, find λ , a and b .

Hence find A^{100} .

(1993-AL-P MATH 1 #06) (7 marks)

6. (a) Show that if A is a 3×3 matrix such that $A^T = -A$, then $\det A = 0$.

(b) Given that

$$B = \begin{pmatrix} 1 & -2 & 74 \\ 2 & 1 & -67 \\ -74 & 67 & 1 \end{pmatrix} ,$$

use (a), or otherwise, to show $\det(I - B) = 0$.

Hence deduce that $\det(I - B^4) = 0$.

(1994-AL-P MATH 1 #01) (6 marks)

1. Let $A = \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$.

Find $P^{-1}AP$.

Find A^n , where n is a positive integer.

(1995-AL-P MATH 1 #01) (6 marks)

1. (a) Let $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$ where a , $b \in \mathbf{R}$ and $a \neq b$.

Prove that $A^n = \begin{pmatrix} a^n & \dfrac{a^n - b^n}{a - b} \\ 0 & b^n \end{pmatrix}$ for all positive integers n .

(b) Hence, or otherwise, evaluate $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95}$.

(1996-AL-P MATH 1 #01) (6 marks)

1. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

Evaluate $A^3 - 5A^2 + 8A - 4I$.

Hence, or otherwise, find A^{-1} .

(1997-AL-P MATH 1 #07) (7 marks)

7. (a) Let A be a 3×3 non-singular matrix. Show that

$$\det\left(A^{-1} - xI\right) = -\frac{x^3}{\det A} \det\left(A - x^{-1}I\right) .$$

(b) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$.

Show that 4 is a root of $\det(A - xI) = 0$ and hence find the other roots in surd form.

Solve $\det\left(A^{-1} - xI\right) = 0$.

(1998-AL-P MATH 1 #09) (15 marks)

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a , b , c , $d \in \mathbf{R}$, $a \neq 0$ and $\det A = 0$.

(a) Show that $A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$ for some $k \in \mathbf{R}$.

(b) Find P in the form of $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ such that $PA = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ for some α , $\beta \in \mathbf{R}$.

If $a + d \neq 0$, find Q in the form of $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ such that $PQP^{-1}Q = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ for some $\gamma \in \mathbf{R}$.

(c) Find S such that $S\begin{pmatrix} 3 & 7 \\ 6 & 14 \end{pmatrix}S^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ for some $\lambda \in \mathbf{R}$.

Hence, or otherwise, evaluate $\begin{pmatrix} 3 & 7 \\ 6 & 14 \end{pmatrix}^n$ where n is a positive integer.

(1999-AL-P MATH 1 #09) (15 marks)

9. (a) Let A and B be two square matrices of the same order. If $AB = BA = 0$, show that

$(A + B)^n = A^n + B^n$ for any positive integer n .

(b) Let $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where a , b are not both zero. If $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, show that $AB = BA = 0$ if and only if $p = r = 0$ and $aq + bs = 0$.

(c) Let $C = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where x , z are non-zero and distinct. Find non-zero matrices D and E such that $C = D + E$ and $DE = ED = 0$.

(d) Evaluate $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{99}$.

(2000-AL-P MATH 1 #01) (5 marks)

1. Let $M = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & b & a \\ \mu & c & -b \end{pmatrix}$ where $b^2 + ac = 1$. Show by induction that

$$M^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ n\left[\lambda(1+b)+\mu a\right] & 1 & 0 \\ n\left[\lambda c+\mu(1-b)\right] & 0 & 1 \end{pmatrix}$$ for all positive integers n .

Hence or otherwise, evaluate $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 2 \\ 1 & -4 & -3 \end{pmatrix}^{2000}$.

(2002-AL-P MATH 1 #12) (15 marks)

12. (a) Let A be a 3×3 matrix such that

$$A^3 + A^2 + A + I = 0 \ ,$$

where I is the 3×3 identity matrix.

(i) Prove that A has an inverse, and find A^{-1} in terms of A .

(ii) Prove that $A^4 = I$.

(iii) Prove that $\left(A^{-1}\right)^3 + \left(A^{-1}\right)^2 + A^{-1} + I = 0$.

(iv) Find a 3×3 invertible matrix B such that $B^3 + B^2 + B + I \neq 0$.

(b) Let $X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.

(i) Using (a)(i) or otherwise, find X^{-1} .

(ii) Let n be a positive integer. Find X^n .

(iii) Find two 3×3 matrices Y and Z , other than X , such that $Y^3 + Y^2 + Y + I = 0$, $Z^3 + Z^2 + Z + I = 0$.

(2003-AL-P MATH 1 #08) (15 marks)

8. (a) If $\det\begin{pmatrix} -2-\alpha & \sqrt{3} \\ \sqrt{3} & -\alpha \end{pmatrix} = 0$, find the two values of α .

(b) Let α_1 and α_2 be the values obtained in (a), where $\alpha_1 < \alpha_2$. Find θ_1 and θ_2 such that

$$\begin{pmatrix} -2-\alpha_1 & \sqrt{3} \\ \sqrt{3} & -\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \quad 0 \le \theta_1 < \pi ,$$

$$\begin{pmatrix} -2-\alpha_2 & \sqrt{3} \\ \sqrt{3} & -\alpha_2 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \quad 0 \le \theta_2 < \pi .$$

Let $P = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}$. Evaluate P^n , where n is a positive integer.

Prove that $P^{-1}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P$ is a matrix of the form $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

(c) Evaluate $\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}^n$, where n is a positive integer.

(2004-AL-P MATH 1 #08) (15 marks)

8. Let $A = \begin{pmatrix} \alpha-k & \alpha-\beta-k \\ k & \beta+k \end{pmatrix}$, where α , β , $k \in \mathbf{R}$ with $\alpha \neq \beta$.

Define $X = \dfrac{1}{\alpha-\beta}(A-\beta I)$ and $Y = \dfrac{1}{\beta-\alpha}(A-\alpha I)$, where I is the 2×2 identity matrix.

(a) Evaluate XY , YX , $X+Y$, X^2 and Y^2 .

(b) Prove that $A^n = \alpha^n X + \beta^n Y$ for all positive integers n .

(c) Evaluate $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}^{2004}$.

(d) If α and β are non-zero real numbers, guess an expression for A^{-1} in terms of α , β , X and Y , and verify it.

(2007-AL-P MATH 1 #05) (6 marks)

5. Let P be a non-singular 2×2 real matrix and $Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where α and β are two distinct real numbers.

Define $M = P^{-1}QP$ and denote the 2×2 identity matrix by I .

(a) Find real numbers λ and μ , in terms of α and β , such that $M^2 = \lambda M + \mu I$.

(b) Prove that $\det\left(M^2 + \alpha\beta I\right) = \alpha\beta(\alpha+\beta)^2$.

(2011-AL-P MATH 1 #08) (15 marks)

8. (a) Let $A = \begin{pmatrix} 4-b & a \\ b & 4-a \end{pmatrix}$ be a real matrix and $P = \begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}$, where $ab > 0$.

(i) Prove that P is a non-singular matrix.

(ii) Evaluate $P^{-1}AP$.

(iii) For any positive integer n, find d_1 and d_2 such that $A^n = P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}$.

(b) Let $B = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$. For any positive integer n, find $B + B^3 + B^5 + \ldots + B^{2n-1}$.

(SP-DSE-MATH-EP(M2) #10) (8 marks)

10. Let $0° < \theta < 180°$ and define $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Prove, by mathematical induction, that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

for all positive integers n.

(b) Solve $\sin 3\theta + \sin 2\theta + \sin\theta = 0$.

(c) It is given that $A^3 + A^2 + A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Find the value(s) of a.

(SP-DSE-MATH-EP(M2) #11) (12 marks)

11. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Let I and O be the 3×3 identity matrix and zero matrix respectively.

(i) Prove that $P^3 - 2P^2 - P + I = \mathbf{0}$.

(ii) Using the result of (i), or otherwise, find P^{-1}.

(b) (i) Prove that $D = P^{-1}AP$.

(ii) Prove that D and A are non-singular.

(iii) Find $\left(D^{-1}\right)^{100}$.

Hence, or otherwise, find $\left(A^{-1}\right)^{100}$.

(PP-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) It is given that $\cos(x+1) + \cos(x-1) = k\cos x$ for any real x. Find the value of k.

(b) Without using a calculator, find the value of $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$.

(PP-DSE-MATH-EP(M2) #11) (14 marks)

11. Let $A = \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}$ where α and β are distinct real numbers. Let I be the 2×2 identity matrix.

(a) Show that $A^2 = (\alpha+\beta)A - \alpha\beta I$.

(b) Using (a), or otherwise, show that $(A-\alpha I)^2 = (\beta-\alpha)(A-\alpha I)$ and $(A-\beta I)^2 = (\alpha-\beta)(A-\beta I)$.

(c) Let $X = s(A-\alpha I)$ and $Y = t(A-\beta I)$ where s and t are real numbers.

Suppose $A = X + Y$.

(i) Find s and t in terms of α and β .

(ii) For any positive integer n , prove that

$$X^n = \frac{\beta^n}{\beta-\alpha}(A-\alpha I) \text{ and } Y^n = \frac{\alpha^n}{\alpha-\beta}(A-\beta I) .$$

(iii) For any positive integer n , express A^n in the form of $pA+qI$, where p and q are real numbers.

(Note: It is known that for any 2×2 matrices H and K ,

if $HK = KH = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $(H+K)^n = H^n + K^n$ for any positive integer n.)

(2012-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) Solve the equation

$$\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0 \text{ .... (*)}$$

(b) Let x_1 , x_2 ($x_1 < x_2$) be the roots of (*). Let $P = \begin{pmatrix} a & c \\ b & 1 \end{pmatrix}$. It is given that

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = x_1\begin{pmatrix} a \\ b \end{pmatrix} , \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = x_2\begin{pmatrix} c \\ 1 \end{pmatrix} \text{ and } |P| = 1 .$$

where a , b and c are constants.

(i) Find P .

(ii) Evaluate $P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P$.

(iii) Using (b)(ii), evaluate $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^{12}$.

(2013-AL-P MATH 1 #11) (15 marks)

11. (a) Define $A = \begin{pmatrix} a & -2 \\ -2 & a+3 \end{pmatrix}$, where $a \in \mathbf{R}$.

Let λ , $\mu \in \mathbf{R}$ and $b > 0$ such that $A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $A\begin{pmatrix} b \\ 1 \end{pmatrix} = \mu \begin{pmatrix} b \\ 1 \end{pmatrix}$.

(i) Express λ in terms of a .

(ii) Prove that $b = 2$ and express μ in terms of a .

(iii) Define $M = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Denote the transpose of M by M^T .

(1) Evaluate $M^T M$.

(2) Using mathematical induction, prove that $A^n = \frac{1}{5} M D^n M^T$ for any $n \in \mathbf{N}$,

where $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

(b) Let x , $y \in \mathbf{R}$.

(i) Prove that if $(x \quad y)\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}^{2013}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, then $x = y = 0$.

(ii) Someone claims that if $(x \quad y)\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}^{2013}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, then $x = y = 0$. Do you agree? Explain your answer.

(2013-DSE-MATH-EP(M2) #08) (5 marks)

8. Let M be the matrix $\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}$, where $k \neq 0$.

(a) Find M^{-1} .

(b) If $M\begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, find the value of k .

(2013-DSE-MATH-EP(M2) #13) (13 marks)

13. For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define $\operatorname{tr}(M) = a + d$.

Let A and B be 2×2 matrices such that $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) (i) For any matrix $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $\operatorname{tr}(MN) = \operatorname{tr}(NM)$.

(ii) Show that $\operatorname{tr}(A) = 4$.

(iii) Find the value of $|A|$.

(b) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and distinct scalars λ_1 and λ_2 .

(i) Prove that $\begin{vmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p - \lambda_2 & q \\ r & s - \lambda_2 \end{vmatrix} = 0$.

(ii) Prove that λ_1 and λ_2 are the roots of the equation $\lambda^2 - \operatorname{tr}(C) \cdot \lambda + |C| = 0$.

(c) Find the two values of λ such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$.

(2014-DSE-MATH-EP(M2) #07) (7 marks)

7. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Prove, by mathematical induction, that for all positive integer n , $A^{n+1} = 2^n A$.

(b) Using the result of (a), Willy proceeds in the following way:

$A^2 = 2A$

$A^2 A^{-1} = 2AA^{-1}$

$A = 2I$

Explain why Willy arrives at a wrong conclusion.

(2014-DSE-MATH-EP(M2) #12) (11 marks)

12. Let $M = \begin{pmatrix} k-1 & k \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$, where k and p are real numbers and $p \neq -1$.

(a) (i) Find A^{-1} in terms of p .

(ii) Show that $A^{-1}MA = \begin{pmatrix} -1 & k-p \\ 0 & k \end{pmatrix}$.

(iii) Suppose $p = k$. Using (ii), find M^n in terms of k and n , where n is a positive integer.

(b) A sequence is defined by

$x_1 = 0$, $x_2 = 1$ and $x_n = x_{n-1} + 2x_{n-2}$ for $n = 3$, 4 , 5 ...

It is known that this sequence can be expressed in the matrix form $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$.

Using the result of (a)(iii), express x_n in terms of n .

(2015-DSE-MATH-EP(M2) #06) (6 marks)

6. (a) Let M be a 3×3 real matrix such that $M^T = -M$, where M^T is the transpose of M . Prove that $|M| = 0$.

(b) Let $A = \begin{pmatrix} -1 & a & b \\ -a & -1 & -8 \\ -b & 8 & -1 \end{pmatrix}$, where a and b are real numbers. Denote that 3×3 identity matrix by I .

(i) Using (a), or otherwise, prove that $|A + I| = 0$.

(ii) Someone claims that $A^3 + I$ is a singular matrix. Do you agree: Explain your answer.

(2015-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) Let λ and μ be real numbers such that $\mu - \lambda \neq 2$. Denote the 2×2 identity matrix by I .

Define $A = \dfrac{1}{\lambda - \mu + 2}(I - \mu I + M)$ and $B = \dfrac{1}{\lambda - \mu + 2}(I + \lambda I - M)$, where $M = \begin{pmatrix} \lambda & 1 \\ \lambda - \mu + 1 & \mu \end{pmatrix}$.

(i) Evaluate AB , BA and $A + B$.

(ii) Prove that $A^2 = A$ and $B^2 = B$.

(iii) Prove that $M^n = (\lambda + 1)^n A + (\mu - 1)^n B$ for all positive integers n .

(b) Using (a), or otherwise, evaluate $\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315}$.

(2016-DSE-MATH-EP(M2) #08) (8 marks)

8. Let n be a positive integer.

(a) Define $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Evaluate

(i) A^2 ,

(ii) A^n ,

(iii) $\left(A^{-1}\right)^n$.

(b) Evaluate

(i) $\sum_{k=0}^{n-1} 2^k$,

(ii) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n$.

(2017-DSE-MATH-EP(M2) #12) (12 marks)

12. Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Denote the 2×2 identity matrix by I .

(a) Using mathematical induction, prove that $A^n = 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all positive integers n .

(b) Let $B = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$.

(i) Define $P = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n = 3^n I + 3^{n-1} n \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ for any positive integer n .

(iii) Does there exist a positive integer m such that $\left|A^m - B^m\right| = 4m^2$? Explain your answer.

(2018-DSE-MATH-EP(M2) #7) (8 marks)

7. Define $M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$. Let $X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$ be a non-zero real matrix such that $MX = XM$.

(a) Express b and c in terms of a .

(b) Prove that X is a non-singular matrix.

(c) Denote the transpose of X be X^T. Express $\left(X^T\right)^{-1}$ in terms of a .

(2019-DSE-MATH-EP(M2) #02) (5 marks)

2. Let $\mathrm{P}(x) = \begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbf{R}$. It is given that the coefficient of x^3 in the expansion of $\mathrm{P}(x)$ is 160 . Find

(a) λ ,

(b) $\mathrm{P}'(0)$.

(2019-DSE-MATH-EP(M2) #11) (12 marks)

11. Let $M = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$. Denote the 2×2 identity matrix by I .

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$.

(b) Prove that $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ for all positive integers n .

(c) Does there exist a pair of 2×2 real matrices A and B such that $(M^n)^{-1} = A + \dfrac{1}{(-5)^n}B$ for all positive integers n ? Explain your answer.

(2020-DSE-MATH-EP(M2) #08) (8 marks)

8. Define $P = \begin{pmatrix} -5 & -2 \\ 15 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $M = \begin{pmatrix} 1 & a \\ b & c \end{pmatrix}$ such that $|M| = 1$ and $PM = MQ$, where a , b and c are real numbers.

(a) Find a , b and c .

(b) Define $R = \begin{pmatrix} 6 & 2 \\ -15 & -5 \end{pmatrix}$.

(i) Evaluate $M^{-1}RM$.

(ii) Using the result of (b)(i), prove that $(\alpha P + \beta R)^{99} = \alpha^{99}P + \beta^{99}R$ for any real numbers α and β .

(2021-DSE-MATH-EP(M2) #11) (12 marks)

11. Define $P = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$, where $\frac{\pi}{2} < \theta < \pi$.

(a) Let $A = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbf{R}$.

Prove that $PAP^{-1} = \begin{pmatrix} -\alpha\cos 2\theta + \beta\sin 2\theta & -\beta\cos 2\theta - \alpha\sin 2\theta \\ -\beta\cos 2\theta - \alpha\sin 2\theta & \alpha\cos 2\theta - \beta\sin 2\theta \end{pmatrix}$.

(b) Let $B = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$.

(i) Find θ such that $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda, \mu \in \mathbf{R}$.

(ii) Using the result of (b)(i), prove that $B^n = 2^{n-2}\begin{pmatrix} (-1)^n + 3 & \sqrt{3}(-1)^{n+1} + \sqrt{3} \\ \sqrt{3}(-1)^{n+1} + \sqrt{3} & 3(-1)^n + 1 \end{pmatrix}$ for any positive integer n .

(iii) Evaluate $\left(B^{-1}\right)^{555}$.

ANSWERS

(1991-AL-P MATH 1 #01) (4 marks)

1. $(a-c)(b-c)(a-b)(a+b+c)$

(1992-AL-P MATH 1 #03) (7 marks)

3. λ can be any non-zero number.

$a=1$, $b=3$

$$A^{100}=\begin{pmatrix}1 & 0\\ \dfrac{3^{100}-1}{2} & 3^{100}\end{pmatrix}$$

(1993-AL-P MATH 1 #06) (7 marks)

(1994-AL-P MATH 1 #01) (6 marks)

1. $P^{-1}AP=\begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}$

$$A^n=\frac{1}{6}\begin{pmatrix}2\cdot 7^n+4 & 8\cdot 7^n-8\\ 7^n-1 & 4\cdot 7^n+2\end{pmatrix}$$

(1995-AL-P MATH 1 #01) (6 marks)

1. (b) $\begin{pmatrix}1 & 3^{95}-1\\ 0 & 3^{95}\end{pmatrix}$

(1996-AL-P MATH 1 #01) (6 marks)

1. $A^3-5A^2+8A-4I=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

$$A^{-1}=\frac{1}{4}\begin{pmatrix}6 & 0 & 4\\ -2 & 2 & -2\\ -2 & 0 & 0\end{pmatrix}$$

(1997-AL-P MATH 1 #07) (7 marks)

7. (b) The other roots $=2\pm\sqrt{3}$

$x=\dfrac{1}{4}$ or $2\pm\sqrt{3}$

(1998-AL-P MATH 1 #09) (15 marks)

9. (b) $P=\begin{pmatrix}1 & 0\\ -k & 1\end{pmatrix}$, $Q=\begin{pmatrix}1 & \dfrac{-b}{a+kb}\\ 0 & 1\end{pmatrix}$

(c) $S=\begin{pmatrix}\dfrac{3}{17} & \dfrac{7}{17}\\ -2 & 1\end{pmatrix}$

$$A^n=\begin{pmatrix}3\cdot 17^{n-1} & 7\cdot 17^{n-1}\\ 6\cdot 17^{n-1} & 14\cdot 17^{n-1}\end{pmatrix}$$

(1999-AL-P MATH 1 #09) (15 marks)

9. (c) $D=\begin{pmatrix}x & \dfrac{xy}{x-z}\\ 0 & 0\end{pmatrix}$, $E=\begin{pmatrix}0 & \dfrac{-yz}{x-z}\\ 0 & z\end{pmatrix}$

(d) $\begin{pmatrix}2^{99} & 5\left(2^{99}-1\right)\\ 0 & 1\end{pmatrix}$

(2000-AL-P MATH 1 #01) (5 marks)

1. $\begin{pmatrix}1 & 0 & 0\\ -6000 & 1 & 0\\ 6000 & 0 & 1\end{pmatrix}$

(2002-AL-P MATH 1 #12) (15 marks)

12. (a) (i) $A^{-1}=-\left(A^2+A+I\right)$

(iv) I

(b) (i) $\begin{pmatrix}-1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$

(ii) $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 1 & 1\\ -1 & -1 & 0\\ -1 & 0 & -1\end{pmatrix}$,

$\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}$,

$\begin{pmatrix}-1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$

(iii) $-I$ or X^3

(2003-AL-P MATH 1 #08) (15 marks)

8. (a) $\alpha = 1$ or $\alpha = -3$

(b) $P^n = \begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ if n is odd.

$P^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if n is even.

$P^{-1}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$

(c)

$$\begin{pmatrix} \frac{1+(-1)^n 3^{n+1}}{4} & \frac{3^{\frac{1}{2}}\left(1+(-1)^{n+1}3^n\right)}{4} \\ \frac{3^{\frac{1}{2}}\left(1+(-1)^{n+1}3^n\right)}{4} & \frac{3\left(1+(-1)^n 3^{n-1}\right)}{4} \end{pmatrix}$$

(2004-AL-P MATH 1 #08) (15 marks)

8. (a) $XY = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $YX = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$X + Y = I$, $X^2 = X$, $Y^2 = Y$

(c) $\begin{pmatrix} \frac{2\left(7^{2004}\right)+1}{3} & \frac{2\left(7^{2004}\right)-2}{3} \\ \frac{7^{2004}-1}{3} & \frac{7^{2004}+2}{3} \end{pmatrix}$

(d) $A^{-1} = \frac{1}{\alpha}X + \frac{1}{\beta}Y$

(2007-AL-P MATH 1 #05) (6 marks)

5. (a) $\lambda = \alpha + \beta$, $\mu = -\alpha\beta$

(2011-AL-P MATH 1 #08) (15 marks)

8. (a) (ii) $\begin{pmatrix} 4 & 0 \\ 0 & 4-a-b \end{pmatrix}$

(iii) $d_1 = 4^n$, $d_2 = (4-a-b)^n$

(b)

$$\frac{1}{75}\begin{pmatrix} 4^{2n+2}-15n-16 & 4^{2n+2}+60n-16 \\ 4^{2n+1}+15n-4 & 4^{2n+1}-60n-4 \end{pmatrix}$$

(SP-DSE-MATH-EP(M2) #10) (8 marks)

10. (b) $\frac{\pi}{2}$, $\frac{2\pi}{3}$

(c) $a = -1$ or 0

(SP-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) (ii) $P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) (iii) $\left(D^{-1}\right)^{100} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2^{100}} \end{pmatrix}$

$$\left(A^{-1}\right)^{100} = \begin{pmatrix} \frac{1}{2^{100}} & 0 & 0 \\ \frac{1}{2^{100}}-1 & 1 & 0 \\ \frac{1}{2^{100}}-1 & 0 & 1 \end{pmatrix}$$

(PP-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) $k = 2\cos 1$

(b) 0

(PP-DSE-MATH-EP(M2) #11) (14 marks)

11. (c) (i) $s = \frac{\beta}{\beta - \alpha}$, $t = \frac{\alpha}{\alpha - \beta}$

(iii) $A^n = \frac{\alpha^n - \beta^n}{\alpha - \beta}A + \frac{\alpha\beta^n - \alpha^n\beta}{\alpha - \beta}I$

(2012-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) $x = -1$ or 5

(b) (i) $\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

(iii) $\begin{pmatrix} \frac{5^{12}+2}{3} & \frac{2\cdot 5^{12}-2}{3} \\ \frac{5^{12}-1}{3} & \frac{2\cdot 5^{12}+1}{3} \end{pmatrix}$

(2013-AL-P MATH 1 #11) (15 marks)

11. (a) (i) $\lambda = a + 4$

(iii) (1) $5I$

(2013-DSE-MATH-EP(M2) #08) (5 marks)

8. (a) $M^{-1} = \frac{1}{k^2}\begin{pmatrix} 0 & 0 & k \\ k & 0 & -1 \\ -k & k^2 & 1 \end{pmatrix}$

(b) 1

(2013-DSE-MATH-EP(M2) #13) (13 marks)

13. (a) (iii) 3

(c) 1 or 3

(2014-DSE-MATH-EP(M2) #07) (7 marks)

7. (b) $|A| = 0$

(2014-DSE-MATH-EP(M2) #12) (11 marks)

12. (a) (i) $A^{-1} = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}$

(iii)

$$M^n = \frac{1}{1+k}\begin{pmatrix} k^{n+1}+(-1)^n & k^{n+1}+(-1)^{n+1}k \\ k^n+(-1)^{n+1} & k^n+(-1)^n k \end{pmatrix}$$

(b) $x_n = \frac{2^{n-1}+(-1)^{n-2}}{3}$

(2015-DSE-MATH-EP(M2) #06) (6 marks)

6. (b) (ii) $|A^3 + I| = 0$, agreed.

(2015-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) (i) $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2^{630} & 6^{315} - 2^{630} \\ 0 & 6^{315} \end{pmatrix}$

(2016-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) (i) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$

(b) (i) $2^n - 1$

(ii) $\begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$

(2017-DSE-MATH-EP(M2) #12) (12 marks)

12. (b) (i) A

(2018-DSE-MATH-EP(M2) #7) (8 marks)

7. (a) $b = -2a$, $c = -3a$

(c) $\frac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$

(2019-DSE-MATH-EP(M2) #02) (5 marks)

2. (a) 2

(b) 145

(2019-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) $a = -4$, $b = 5$

(c) Yes

$A = \frac{1}{6}\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}$, $B = \frac{1}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix}$

(2020-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $a = 2$, $b = -3$, $c = -5$

(b) (i) $M^{-1}RM = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(2021-DSE-MATH-EP(M2) #11) (12 marks)

11. (b) (i) $\frac{5\pi}{6}$

(iii) $\frac{1}{2^{556}}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

2. System of Linear Equations

(1991-AL-P MATH 1 #03) (4 marks)

3. Consider the following system of linear equations:

$$\begin{cases} x + 2y + z = 1 \\ x + y + 2z = 2 \\ \quad - y + q^2 z = q \end{cases}$$

Determine all values of q for each of the following cases:

(a) The system has no solution.

(b) The system has infinitely many solutions.

(1992-AL-P MATH 1 #01) (6 marks)

1. Consider the following system of linear equations:

$$(*) \begin{cases} x + (t+3)y + 5z = 3 \\ -3x + 9y - 15z = s \\ 2x + ty + 10z = 6 \end{cases}$$

(a) If $(*)$ is consistent, find s and t .

(b) Solve $(*)$ when it is consistent.

(1993-AL-P MATH 1 #03) (6 marks)

1 Suppose the following system of linear equation is consistent:

$$(*) \begin{cases} ax + by + cz = 1 \\ bx + cy + az = 1 \\ cx + ay + bz = 1 \\ x + y + z = 3 \end{cases}, \text{ where } a , b , c \in \mathbf{R} .$$

(a) Show that $a+b+c=1$.

(b) Show that (*) has a unique solution if and only if a , b and c are not all equal.

(c) If $a=b=c$, solve $(*)$.

(1994-AL-P MATH 1 #02) (6 marks)

1. Consider the following system of linear equations:

$$(*)\begin{cases} 4x + 3y + z = \lambda x \\ 3x - 4y + 7z = \lambda y \\ x + 7y - 6z = \lambda z \end{cases}$$

Suppose λ is an integer and $(*)$ has nontrivial solutions.

Find λ and solve $(*)$.

(1994-AL-P MATH 1 #09) (15 marks)

9. (a) Consider

$$(I) : \begin{cases} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{cases} \text{ and } (II) : \begin{cases} a_{11}x + a_{12}y + a_{13} = 0 \\ a_{21}x + a_{22}y + a_{23} = 0 \\ a_{31}x + a_{32}y + a_{33} = 0 \end{cases}.$$

(i) Show that if (I) has a unique solution, then (II) has no solution.

(ii) Show that (u, v) is a solution of (II) if and only if (ut, vt, t) are solutions of (I) for all $t \in \mathbf{R}$.

(iii) If (II) has no solution and (I) has nontrivial solutions, what can you say about the solutions of (I) ?

(b) Consider

$$(III) : \begin{cases} -(3+k)x + y - z = 0 \\ -7x + (5-k)y - z = 0 \\ -6x + 6y + (k-2)z = 0 \end{cases}$$

and

$$(IV) : \begin{cases} -(3+k)x + y - 1 = 0 \\ -7x + (5-k)y - 1 = 0 \\ -6x + 6y + (k-2) = 0 \end{cases}$$

(i) Find the values of k for which (III) has non-trivial solutions.

(ii) Find the values of k for which (IV) is consistent. Solve (IV) for each of these values of k .

(iii) Solve (III) for each k such that (III) has non-trivial solutions.

(1995-AL-P MATH 1 #09) (15 marks)

9. Consider the following system of linear equations

$$(S)\ :\ \begin{cases} 2x + 2y - z = k \\ hx - 3y - z = 0 \\ -3x + hy + z = 0 \end{cases}$$

and

$$(T)\ :\ \begin{cases} 6x + 6y - 3z = 2 \\ hx - 3y - z = 0 \\ -3x + hy + z = 0 \\ -5x - 2y + 6z = h \end{cases}$$

(a) Show that (S) has a unique solution if and only if $h^2 \neq 9$. Solve (S) in this case.

(b) For each of the following cases, find the value(s) of k for which (S) is consistent, and solve (S) :

(i) $h = 3$,

(ii) $h = -3$.

(c) Find the values of h for which (T) is consistent. Solve (T) for each of these values of h .

(1996-AL-P MATH 1 #05) (6 marks)

1. (a) Solve $\begin{cases} Z + Y = a \\ Z + X = b \\ Y + X = c \end{cases}$ for X , Y and Z .

(b) If $a + b - c > 0$, $b + c - a > 0$ and $c + a - b > 0$,

solve $\begin{cases} xy + xz = a \\ xy + yz = b \\ xz + yz = c \end{cases}$ for x , y and z .

(1996-AL-P MATH 1 #09) (15 marks)

9. Consider the system of linear equations

$$(*) : \begin{cases} x + 2y - z = 3 \\ x + y + 2z = 4 \end{cases}$$

(a) Solve (*) .

(b) Find the solutions of (*) that satisfy $xy + yz + zx = 2$.

(c) Find all possible values of a and λ (a , $\lambda \in \mathbf{R}$) so that

$$\begin{cases} x + 2y - z = 3 \\ x + y + 2z = 4 \\ ax + y + z = \lambda \end{cases}$$

is solvable.

(d) Using (b), or otherwise, find all possible values of a and λ (a , $\lambda \in \mathbf{R}$) so that

$$\begin{cases} x + 2y - z = 3 \\ x + y + 2z = 4 \\ xy + yz + zx = 2 \\ ax + y + z = \lambda \end{cases}$$

has at least one solution.

(1997-AL-P MATH 1 #03) (6 marks)

3. Suppose the system of linear equations

$$(*) \begin{cases} \lambda x + ky = 0 \\ -\lambda y + z = 0 \\ x + ky + z = 0 \end{cases}$$

has nontrivial solutions.

(a) Show that λ satisfies the equation $\lambda^2 + k\lambda - k = 0$.

(b) If the quadratic equations in λ in (a) has equal roots, find k .
Solve (*) for each of these values of k .

(1997-AL-P MATH 1 #08) (15 marks)

8. Consider the following two systems of linear equations:

$$(S):\begin{cases}(a+1)x+2y-2z=0\\ x+ay+2z=0\\ 3x-y+(a-7)z=0\end{cases}$$

$$(T):\begin{cases}(a+1)x+2y-2z=6\\ x+ay+2z=5b-1\\ 3x-y+(a-7)z=1-b\end{cases}$$

(a) If (S) has infinitely many solutions, find all the values of a . Solve (S) for each of these values of a .

(b) For the smallest value of a found in (a), find the values of b so that (T) is consistent. Solve (T) for these values of a and b .

(c) Solve the system of equations

$$\begin{cases}-x+2y-2\sqrt{z}=6\\ x-2y+2\sqrt{z}=-6\\ 3x-y-9\sqrt{z}=2\\ 3x-4y-z=-11\end{cases}$$

(1998-AL-P MATH 1 #01) (6 marks)

1. Consider the system of linear equations

$$(*)\begin{cases}2x+y+2z=0\\ x+(k+1)z=0\\ kx-y+4z=0\end{cases}$$

Suppose $(*)$ has infinitely many solutions.

(a) Find k .

(b) Solve $(*)$.

(1998-AL-P MATH 1 #08) (15 marks)

8. Consider the system of linear equations in

$$(E) : \begin{cases} ax + y + bz = 1 \\ x + ay + bz = 1 \\ x + y + abz = b \end{cases}$$

(a) Show that (E) has a unique solution if and only if $a \neq -2$, $a \neq 1$ and $b \neq 0$. Solve (E) in this case.

(b) For each of the following cases, determine the value(s) of b for which (E) is consistent. Solve (E) in each case.

(i) $a = -2$,

(ii) $a = 1$.

(c) Determine whether (E) is consistent or not for $b = 0$.

(1999-AL-P MATH 1 #01) (6 marks)

1. Suppose the system of linear equations

$$(*) \begin{cases} x + y - \lambda z = 0 \\ x + \lambda y - z = 0 \\ \lambda x + y - z = 0 \end{cases}$$

has non-trivial solutions.

(a) Find all values of λ .

(b) Solve (*) for each of the values of λ obtained in (a).

(1999-AL-P MATH 1 #08) (15 marks)

8. Consider the system of linear equations

$$(E) : \begin{cases} x + \lambda y + z = \lambda \\ 3x - y + (\lambda+2)z = 7 \\ x - y + z = 3 \end{cases} \text{ where } \lambda \in \mathbf{R} .$$

(a) Show that (*E*) has a unique solution if and only if $\lambda \neq \pm 1$.

(b) Solve (*E*) for

(i) $\lambda \neq \pm 1$,

(ii) $\lambda = -1$,

(iii) $\lambda = 1$.

(c) Find the conditions on a , b , c and d so that the system of linear equations

$$\begin{cases} x + y + z = 1 \\ 3x - y + 3z = 7 \\ x - y + z = 3 \\ ax + by + cz = d \end{cases}$$

is consistent.

(2000-AL-P MATH 1 #08) (15 marks)

8. Consider the system of linear equations

$$(S) : \begin{cases} x - y - z = a \\ 2x + \lambda y - 2z = b \\ x + (2\lambda+3)y + \lambda^2 z = c \end{cases} \text{ where } \lambda \in \mathbf{R} .$$

(a) Show that (S) has a unique solution if and only if $\lambda \neq -2$. Solve (S) for $\lambda = -1$.

(b) Let $\lambda = -2$.

(i) Find the conditions on a , b and c so that (S) has infinitely many solutions.

(ii) Solve (S) when $a = -1$, $b = -2$ and $c = -3$.

(c) Consider the system of linear equations

$$(T) : \begin{cases} x - y - z + 3\mu - 5 = 0 \\ 2x - 2y - 2z + 2\mu - 2 = 0 \\ x - y + 4z - \mu - 1 = 0 \end{cases} \text{ where } \mu \in \mathbf{R} .$$

Using the results in (b), or otherwise, solve (T) .

(2001-AL-P MATH 1 #09) (15 marks)

9. Consider the system of linear equations

$$(S): \begin{cases} x + \lambda y + z = k \\ \lambda x - y + z = 1 \\ 3x + y + 2z = -1 \end{cases} \text{ where } \lambda, k \in \mathbf{R}.$$

(a) Show that (S) has a unique solution if and only if $\lambda \neq 0$ and $\lambda \neq 2$.

(b) For each of the following cases, determine the value(s) of k for which (S) is consistent. Solve (S) in each case.

(i) $\lambda \neq 0$ and $\lambda \neq 2$,

(ii) $\lambda = 0$,

(iii) $\lambda = 2$.

(c) If some solution of (x, y, z) of

$$\begin{cases} x + z = 0 \\ -y + z = 1 \\ 3x + y + 2z = -1 \end{cases}$$

satisfies $(x-p)^2 + y^2 + z^2 = 1$, find the range of values of p .

(2002-AL-P MATH 1 #08) (15 marks)

8. (a) Consider the system of linear equations in x , y , z .

$$(S): \begin{cases} ax - 2y + z = 0 \\ x - y + 2z = b \\ y + az = b \end{cases}, \text{where a }, b \in \mathbf{R}.$$

(i) Show that (S) has a unique solution if and only if $a^2 \neq 1$. Solve (S) in this case.

(ii) For each of the following cases, determine the value(s) of b for which (S) is consistent, and solve (S) for such value(s) of b .

(1) $a = 1$,

(2) $a = -1$.

(b) Consider the system of linear equations in x , y , z

$$(T): \begin{cases} ax - 2y + z = 0 \\ x - y + 2z = -1 \\ y + az = -1 \\ 5x - 2y + z = a \end{cases}, \text{where } a \in \mathbf{R}.$$

Find all the values of a for which (T) is consistent. Solve (T) for each of these values of a .

(2003-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x , y , z .

$$(E)\ :\ \begin{cases} x + ay - z = 0 \\ 2x - y + az = -2a \\ -x + 2a^2y + (a-3)z = 2a \end{cases},\ \text{where } a \in \mathbf{R}\ .$$

(i) Find the range of values of a for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) Solve (E) for

(1) $a = 1$,

(2) $a = -4$.

(b) Suppose (x, y, z) satisfy

$$\begin{cases} x + y - z = 0 \\ 2x - y + z = -2 \\ -x + 2y - 2z = 2 \end{cases}.$$

Find the least value of $24x^2 + 3y^2 + 2z$ and the corresponding values of x , y , z .

(2004-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x , y , z .

$$(E)\ :\ \begin{cases} x + (a-2)y + az = 1 \\ x + 2y + 4z = 1 \\ ax - y + 3z = b \end{cases},\ \text{where } a\ ,\ b \in \mathbf{R}\ .$$

(i) Prove that (E) has a unique solution if and only if $a \neq 2$ and $a \neq 4$. Solve (E) in this case.

(ii) For each of the following cases, determine the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b .

(1) $a = 2$,

(2) $a = 4$.

(b) If all solutions (x, y, z) of

$$\begin{cases} x + \quad + 2z = 1 \\ x + 2y + 4z = 1 \\ 2x - y + 3z = 2 \end{cases}$$

satisfy $k\left(x^2 - 3\right) > yz$, find the range of values of k .

(2005-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x , y , z .

$$(E): \begin{cases} x + ay + z = b \\ 2x + (a+3)y + (a-1)z = 0 \\ 3x + a^2y + (4a+1)z = -b \end{cases} \text{, where } a, b \in \mathbf{R} .$$

(i) Find the range of values of a for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) For each of the following cases, find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b .

(1) $a = 1$,

(2) $a = -2$.

(b) Suppose that a real solution of

$$\begin{cases} x - 2y + z = b \\ 2x + y - 3z = 0 \\ 3x + 4y - 7z = -b \end{cases}$$

satisfies $x^2 + y^2 + z^2 = b + 3$, where $b \in \mathbf{R}$. Find the range of values of b .

(2006-AL-P MATH 1 #07) (15 marks)

7. Consider the system of linear equations in x , y , z .

$$(E): \begin{cases} x + ay + z = 4 \\ x + (2-a)y + (3b-1)z = 3 \\ 2x + (a+1)y + (b+1)z = 7 \end{cases} \text{, where } a, b \in \mathbf{R} .$$

(a) Prove that (E) has a unique solution if and only if $a \neq 1$ and $b \neq 0$. Solve (E) in this case.

(b) (i) For $a = 1$, find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b .

(ii) Is there a real solution (x, y, z) of

$$\begin{cases} x + y + z = 4 \\ 2x + 2y + z = 6 \\ 4x + 4y + 3z = 14 \end{cases}$$

satisfying $x^2 - 2y^2 - z = 14$? Explain your answer.

(c) Is (E) consistent for $b = 0$? Explain your answer.

(2007-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x, y, z.

$$(E):\begin{cases} x - 3y = 1 \\ x + 5y + az = b \\ 2x + ay - z = 2 \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Find the range of values of a for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) Suppose that $a = -2$. Find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b.

(b) Is the system of linear equations

$$\begin{cases} x - 3y = 1 \\ x + 5y + z = 16 \\ 2x + y - z = 2 \\ x - y - z = 3 \end{cases}$$

consistent? Explain your answer.

(c) Solve the system of linear equations

$$\begin{cases} x - 3y = 1 \\ x + 5y - 2z = 16 \\ 2x - 2y - z = 2 \\ x - y - z = 3 \end{cases}.$$

(2008-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x, y, z.

$$(E):\begin{cases} x + (a+2)y + (a+1)z = 1 \\ x - 3y - z = b \\ 3x - 2y + (a-1)z = 1 \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Prove that (E) has a unique solution if and only if $a^2 \neq 4$. Solve (E) when (E) has a unique solution.

(ii) For each of the following cases, find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b.

(1) $a = 2$,

(2) $a = -2$.

(b) Find the greatest value of $2x^2 + 15y^2 - 10z^2$, where x, y and z are real numbers satisfying

$$\begin{cases} x + 4y + 3z = 1 \\ x - 3y - z = 0 \\ 3x - 2y + z = 1 \end{cases}.$$

(2009-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x , y , z .

$$(E) : \begin{cases} x + \lambda y + 2z = 1 \\ 5x - \lambda y + z = 5 \\ \lambda x - y + z = a \end{cases} \text{, where } \lambda \text{ , } a \in \mathbf{R} \text{ .}$$

(i) Find the range of values of λ for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) Suppose that $\lambda = -1$. Find the value(s) of a for which (E) is consistent, and solve (E) for such value(s) of a .

(b) Is the system of linear equations

$$\begin{cases} x - 2y + 2z = 1 \\ 5x + 2y + z = 5 \\ 2x + y - z = -3 \\ 4x + 3y - 3z = 2 \end{cases}$$

consistent? Explain your answer.

(c) Find the solution(s) of the system of linear equations

$$\begin{cases} x - y + 2z = 1 \\ 5x + y + z = 5 \\ x + y - z = 1 \end{cases}$$

satisfying $4x^2 + 2y - z = 28$.

(2010-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x , y , z

$$(E) : \begin{cases} x + y + z = 2 \\ ax - 4z = 2 \\ 3x + 4y + (a+4)z = b \end{cases}, \text{ where } a , b \in \mathbf{R} .$$

(i) Find the range of values of a for which (E) has a unique solution, and solve (E) when (E) has a unique solution.

(ii) Suppose that $a = 2$. Find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b .

(b) Consider the system of linear equation in x , y , z

$$(F) : \begin{cases} x + y + z = 2 \\ x + 2z = -1 \\ 3x + 4y + 2z = \lambda \\ 7x + 17y - 3z = \mu \end{cases}, \text{ where } \lambda , \mu \in \mathbf{R} .$$

Find the values of λ and μ for which (F) is consistent.

(c) Consider the system of linear equation in x , y , z

$$(G) : \begin{cases} x + y + z = 2 \\ x - 6z = 3 \\ 9x + 12y + 14z = 15 \\ 5x - 2y - 18z = 16 \end{cases}$$

Is (G) consistent? Explain your answer.

(2011-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in x , y , z

$$(S) : \begin{cases} y + (\lambda+1)z = 0 \\ \lambda x + 2y + 2z = \mu \\ x - \lambda y - 4z = \mu^2 \end{cases}, \text{ where } \lambda , \mu \in \mathbf{R} .$$

(i) Suppose that $\mu = 0$.

(1) Prove that (S) has non-trivial solutions if and only if $\lambda^3 + \lambda^2 - 2\lambda = 0$.

(2) Solve (S) when $\lambda = 1$.

(ii) Suppose that $\mu \neq 0$.

(1) Find the range of values of λ for which (S) has a unique solution.

(2) Solve (S) when (S) has a unique solution.

(3) Find λ and μ for which (S) has infinitely many solutions.

(b) Is there a real solution (x, y, z) of the system of linear equations

$$\begin{cases} y + 2z = 0 \\ x + 2y + 2z = 1 \\ x - y - 4z = 1 \end{cases}$$

satisfying $3x^3 + 2y^2 - z^2 = 1$? Explain your answer.

(SP-DSE-MATH-EP(M2) #07) (5 marks)

7. Solve the system of linear equations

$$\begin{cases} x + 7y - 6z = -4 \\ 3x - 4y + 7z = 13 \\ 4x + 3y + z = 9 \end{cases}.$$

(PP-DSE-MATH-EP(M2) #02) (4 marks)

2. Consider the following system of linear equations in x , y , z

$$\begin{cases} x - 7y + 7z = 0 \\ x - ky + 3z = 0 \\ 2x + y + kz = 0 \end{cases}$$, where k is a real number.

If the system has non-trivial solutions, find the two possible values of k .

(2012-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the following system of linear equations in x , y , z

$$(E) : \begin{cases} x + 2y - z = 3 \\ 2x + 5y + (a-1)z = 4 \\ (a+2)x + y + (2a+1)z = b \end{cases}$$, where a , $b \in \mathbf{R}$.

(i) Prove that (E) has a unique solution if and only if $a \neq -1$ and $a \neq -3$. Solve (E) when (E) has a unique solution.

(ii) Suppose that $a = -3$. Find b for which (E) is consistent, and solve (E) when (E) is consistent.

(b) Is the system of linear equations in real variables x , y , z

$$\begin{cases} x + 2y - z = 3 \\ 6x + 15y - 7z = 12 \\ 2x + 3y - 5z = -12 \\ 4x + 5y - 6z = 1 \end{cases}$$

consistent? Explain your answer.

(c) Find the least value of $3x^2 - 7y^2 + 8z^2$, where x , y and z are real numbers satisfying

$$\begin{cases} x + 2y - z = 3 \\ 2x + 5y - 4z = 4 \\ x - y + 5z = 9 \end{cases}.$$

(2012-DSE-MATH-EP(M2) #08) (5 marks)

8. (a) Solve the following system of linear equations:

$$\begin{cases} x + y + z = 0 \\ 2x - y + 5z = 6 \end{cases}.$$

(b) Using (a), or otherwise, solve the following system of linear equations:

$$\begin{cases} x + y + z = 0 \\ 2x - y + 5z = 6 \\ x - y + \lambda z = 4 \end{cases}, \text{ where } \lambda \text{ is a constant.}$$

(2013-AL-P MATH 1 #07) (15 marks)

7. (a) Consider the system of linear equations in real variables x , y , z

$$(E) : \begin{cases} 3x - 2y + z = 0 \\ (2a+7)x - 5y + az = b \\ -x + ay - z = 1 \end{cases}, \text{ where } a , b \in \mathbf{R} .$$

(i) Find the range of values of a for which (E) has a unique solution, and solve (E) when (E) has a unique solution.

(ii) Suppose that $a = 1$. Find b for which (E) is consistent, and solve (E) when (E) is consistent.

(b) Consider the system of linear equations in real variables x , y , z

$$(F) : \begin{cases} 3x - 2y + z = 0 \\ -9x + 5y - z = -3 \\ -x + y - z = 1 \\ \lambda x + \mu y - 7z = 4\lambda \end{cases}, \text{ where } \lambda , \mu \in \mathbf{R} .$$

Find λ and μ for which (F) has infinitely many solutions.

(c) If the real solution of the system of linear equations $\begin{cases} 3x - 2y + z = 0 \\ 11x - 5y + 2z = \beta \\ x - 2y + z = -1 \end{cases}$ satisfies $4x^2 - y^2 + z^2 = 1$, find β .

(2013-DSE-MATH-EP(M2) #09) (5 marks)

9. Consider the following system of linear equations in x , y and z

$$(E) : \begin{cases} x - ay + z = 2 \\ 2x + (1-2a)y + (2-b)z = a+4 \\ 3x + (1-3a)y + (3-ab)z = 4 \end{cases}, \text{ where } a \text{ and } b \text{ are real numbers.}$$

It is given that (E) has infinitely many solutions.

(a) Find the values of a and b .

(b) Solve (E) .

(2014-DSE-MATH-EP(M2) #09) (6 marks)

9. (a) Solve the system of linear equations $\begin{cases} x + y + z = 100 \\ x + 6y + 10z = 200 \end{cases}$.

(b) In a store, the prices of each of small, medium and large marbles are $\$0.5$, $\$3$ and $\$5$ respectively. Aubrey plans to spend all $\$100$ for exactly 100 marbles, which include m small marbles, n medium marbles and k large marbles.

Aubrey claims that there is only one set of combination of m, n and k. Do you agree? Explain your answer.

(2015-DSE-MATH-EP(M2) #05) (6 marks)

5. Solve the following systems of linear equations in real variables x, y and z:

(a) $\begin{cases} x + y + z = 2 \\ 2x + 3y - 3z = 4 \end{cases}$

(b) $\begin{cases} x + y + z = 2 \\ 2x + 3y - 3z = 4 \\ 3x + 2y + kz = 6 \end{cases}$

(2016-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + y - z = 3 \\ 4x + 6y + az = b \\ 5x + (1-a)y + (3a-1)z = b-1 \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Assume that (E) has a unique solution.

(1) Prove that $a \neq -2$ and $a \neq -12$.

(2) Solve (E).

(ii) Assume that $a = -2$ and (E) is consistent.

(1) Find b.

(2) Solve (E).

(b) Is there a real solution of the system of linear equations

$$\begin{cases} x + y - z = 3 \\ 2x + 3y - z = 7 \\ 5x + 3y - 7z = 13 \end{cases}$$

satisfying $x^2 + y^2 - 6z^2 > 14$? Explain your answer.

(2017-DSE-MATH-EP(M2) #05) (6 marks)

5. Consider the system of linear equations in real variables x , y , z

$$(E)\ :\ \begin{cases} x + 2y - z = 11 \\ 3x + 8y - 11z = 49 \\ 2x + 3y + hz = k \end{cases} \text{, where } h\ ,\ k \in \mathbf{R}\ .$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of h .

(ii) Express z in terms of h and k .

(b) Assume that (E) has infinitely many solutions. Solve (E) .

(2018-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) Consider the system of linear equations in real variables x , y , z

$$(E)\ :\ \begin{cases} x + ay + 4(a+1)z = 18 \\ 2x + (a-1)y + 2(a-1)z = 20 \\ x - y - 12z = b \end{cases} \text{, where } a\ ,\ b \in \mathbf{R}\ .$$

(i) Assume that (E) has a unique solution.

(1) Find the range of values of a .

(2) Solve (E) .

(ii) Assume that $a = 3$ and (E) is consistent.

(1) Find b .

(2) Solve (E) .

(b) Consider the system of linear equations in real variables x , y , z

$$(F)\ :\ \begin{cases} x + 3y + 16z = 18 \\ x + y + 2z = 10 \\ x - y - 12z = s \\ 2x - 5y - 45z = t \end{cases} \text{, where } s\ ,\ t \in \mathbf{R}\ .$$

Assume that (F) is consistent. Find s and t .

(2019-DSE-MATH-EP(M2) #06) (7 marks)

6. Consider the system of linear equations in real variables x , y , z

$$(E):\ \begin{cases} x - 2y - 2z = \beta \\ 5x + \alpha y + \alpha z = 5\beta \\ 7x + (\alpha-3)y + (2\alpha+1)z = 8\beta \end{cases} \text{, where } \alpha\ ,\ \beta \in \mathbf{R}\ .$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α .

(ii) Express y in terms of α and β .

(b) Assume that $\alpha = -4$. If (E) is inconsistent, find the range of values of β .

(2020-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) Consider the system of linear equations in real variables x , y , z

$$(E):\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = k \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h , k \in \mathbf{R} .$$

(i) Assume that (E) has a unique solution.

(1) Prove that $h \neq -3$.

(2) Solve (E) .

(ii) Assume that $h = -3$ and (E) is consistent.

(1) Prove that $k = -2$.

(2) Solve (E) .

(b) Consider the system of linear equations in real variables x , y , z

$$(F):\begin{cases} x - y - 2z = 1 \\ x - 2y + hz = -2 \\ 4x + hy - 7z = 7 \end{cases}, \text{ where } h \in \mathbf{R} .$$

Someone claims that there are at least two values of h such that (F) has a real solution (x , y , z) satisfying $3x^2 + 4y^2 - 7z^2 = 1$. Do you agree? Explain your answer.

(2021-DSE-MATH-EP(M2) #08) (8 marks)

8. Consider the system of linear equations in real variables x , y , z

$$(E):\begin{cases} x + (d-1)y + (d+3)z = 4-d \\ 2x + (d+2)y - z = 2d-5 \\ 3x + (d+4)y + 5z = 2 \end{cases}, \text{ where } d \in \mathbf{R} .$$

It is given that (E) has infinitely many solutions

(a) Find d . Hence, solve (E) .

(b) Someone claims that (E) has a real solution (x, y, z) satisfying $xy + 2xz = 3$. Is the claim correct? Explain your answer.

ANSWERS

(1991-AL-P MATH 1 #03) (4 marks)

3. (a) $q = -1$

(b) $q = 1$

(1992-AL-P MATH 1 #01) (6 marks)

1. (a) $s = -9$, t can be any real number.

(b) When $s = -9$, $t = -6$, $\{(3+3m-5n, m, n) : m, n \in \mathbf{R}\}$;

When $s = -9$, $t \neq -6$, $\{(3-5n, 0, n) : n \in \mathbf{R}\}$.

(1993-AL-P MATH 1 #03) (6 marks)

1 (c) $\{(m, n, 3-m-n) : m, n \in \mathbf{R}\}$

(1994-AL-P MATH 1 #02) (6 marks)

1. $\lambda = 0$, $\{(-t, t, t) : t \in \mathbf{R}\}$

(1994-AL-P MATH 1 #09) (15 marks)

9. (b) (i) $k = -2$, 2 or 4

(ii) $k = -2$, (IV) is inconsistent; $k = 2$, $x = -\frac{1}{4}$, $y = \frac{1}{4}$; $k = 4$, $x = -\frac{2}{9}$, $y = \frac{-5}{9}$.

(iii) $k = -2$, $\{(t, t, 0) : t \in \mathbf{R}\}$; $k = 2$, $\{(t, t, -4t) : t \in \mathbf{R}\}$; $k = 4$, $\{(2t, 5t, -9t) : t \in \mathbf{R}\}$.

(1995-AL-P MATH 1 #09) (15 marks)

9. (a) $\left\{\left(\frac{-k}{h+3}, \frac{k}{h+3}, -k\right)\right\}$

(b) (i) k can be all values, $\left\{\left(\frac{3k+5t}{12}, \frac{3k+t}{12}, t\right) : t \in \mathbf{R}\right\}$

(ii) $k = 0$, $\{(-t, t, 0) : t \in \mathbf{R}\}$

(c) For $h^2 \neq 9$, $k = \frac{2}{3}$, $h = -2$, $x = -\frac{2}{3}$, $y = \frac{2}{3}$, $z = -\frac{2}{3}$;

$k = \frac{2}{3}$, $h = -5$, $x = \frac{1}{3}$, $y = -\frac{1}{3}$, $z = -\frac{2}{3}$.

For $h = 3$, $k = \frac{2}{3}$, $t = \frac{17}{27}$, $x = \frac{17}{27}$, $y = \frac{7}{27}$, $z = \frac{10}{9}$.

(1996-AL-P MATH 1 #05) (6 marks)

1. (a) $X = \frac{1}{2}(b+c-a)$, $Y = \frac{1}{2}(a+c-b)$, $Z = \frac{1}{2}(a+b-c)$

(b) $x = \pm\sqrt{\frac{(a+b-c)(a+c-b)}{2(b+c-a)}}$, $y = \pm\sqrt{\frac{(a+b-c)(b+c-a)}{2(a+c-b)}}$, $z = \pm\sqrt{\frac{(a+c-b)(b+c-a)}{2(a+b-c)}}$

(1996-AL-P MATH 1 #09) (15 marks)

9. (a) $\{(5-5t, 3t-1, t) : t \in \mathbf{R}\}$

(b) $\left\{(0,2,1), \left(\frac{50}{17}, \frac{4}{17}, \frac{7}{17}\right)\right\}$

(c) If $a \neq \frac{4}{5}$, then $\lambda \in \mathbf{R}$; or If $a = \frac{4}{5}$, then $\lambda = 3$

(d) If $\lambda = 3$, then $a \in \mathbf{R}$; or $a = \frac{17\lambda - 11}{50}$

(1997-AL-P MATH 1 #03) (6 marks)

3. (b) When $k = -4$, $\lambda = 2$, $\{(2t, t, 2t) : t \in \mathbf{R}\}$; When $k = 0$, $\lambda = 0$, $\{(0, t, 0) : t \in \mathbf{R}\}$.

(1997-AL-P MATH 1 #08) (15 marks)

8. (a) $a = -2$, $\{(4t, 3t, t) : t \in \mathbf{R}\}$; $a = 3$, $\{(t, -t, t) : t \in \mathbf{R}\}$; $a = 5$, $\{(t, -t, 2t) : t \in \mathbf{R}\}$.

(b) $a = -2$, $b = -1$, $\{(2+4t, 4+3t, t) : t \in \mathbf{R}\}$

(c) $x = 6$, $y = 7$, $z = 1$

(1998-AL-P MATH 1 #01) (6 marks)

1. (a) $k = -4$ or 1

(b) $k = -4$, $\{(3t, -8t, t) : t \in \mathbf{R}\}$; $k = 1$, $\{(-2t, 2t, t) : t \in \mathbf{R}\}$.

(1998-AL-P MATH 1 #08) (15 marks)

8. (a) $x = \frac{-(a-b)}{(1-a)(2+a)}$, $y = \frac{-(a-b)}{(1-a)(2+a)}$, $z = \frac{2-b-ab}{(1-a)(2+a)b}$

(b) (i) $b = -2$, $\{(-1-2t, -1-2t, t) : t \in \mathbf{R}\}$

(ii) $b = 1$, $\{(1-s-t, s, t) : s, t \in \mathbf{R}\}$

(c) Inconsistent

(1999-AL-P MATH 1 #01) (6 marks)

1. (a) $\lambda = -2$ or 1

(b) $\lambda = -2$, $\{(-t, -t, t) : t \in \mathbf{R}\}$; $\lambda = 1$, $\{(t-s, s, t) : s, t \in \mathbf{R}\}$.

(1999-AL-P MATH 1 #08) (15 marks)

8. (b) (i) $x = 4$, $y = \frac{\lambda - 3}{\lambda + 1}$, $z = \frac{-4}{\lambda + 1}$

(ii) Inconsistent

(iii) $\{(2-t, -1, t) : t \in \mathbf{R}\}$

(c) $a \neq c$, $b - 2c + d = 0$

(2000-AL-P MATH 1 #08) (15 marks)

8. (a) $x = \dfrac{c+a}{2}$, $y = b - 2a$, $z = \dfrac{c-2b+3a}{2}$

(b) $\lambda = -2$, $b - 2a = 0$; $a = -1$, $b = -2$ and $c = 3$, $\left\{ \left(t - \dfrac{1}{5}, t, \dfrac{4}{5} \right) : t \in \mathbf{R} \right\}$

(c) (i) $\mu = 2$, $a = -1$, $b = -2$ and $c = 3$, $\left\{ \left(t - \dfrac{1}{5}, t, \dfrac{4}{5} \right) : t \in \mathbf{R} \right\}$

(ii) $\mu \neq 2$, inconsistent

(2001-AL-P MATH 1 #09) (15 marks)

9. (b) (i) $x = \dfrac{3(\lambda+k)}{2\lambda(\lambda-2)}$, $y = \dfrac{2\lambda k + \lambda - 3k}{2\lambda(\lambda-2)}$, $z = \dfrac{(\lambda+3)(\lambda+k)}{-2\lambda(\lambda-2)}$

(ii) $k = 0$, $\{(-t, t-1, t) : t \in \mathbf{R}\}$

(iii) $k = -2$, $\left\{ \left(-\dfrac{3t}{5}, -\dfrac{t+5}{5}, t \right) : t \in \mathbf{R} \right\}$

(c) $\dfrac{-1-\sqrt{3}}{2} \leq p \leq \dfrac{-1+\sqrt{3}}{2}$

(2002-AL-P MATH 1 #08) (15 marks)

8. (a) (i) $x = \dfrac{-2b}{a+1}$, $y = \dfrac{-b(a-1)}{a+1}$, $z = \dfrac{2b}{a+1}$

(ii) (1) b can be any number, $\{(2b-3t, b-t, t) : t \in \mathbf{R}\}$

(2) $b = 0$, $\{(-t, t, t) : t \in \mathbf{R}\}$

(b) (i) $b = -1$

When $a = 2$, $x = \dfrac{2}{3}$, $y = \dfrac{1}{3}$, $z = \dfrac{-2}{3}$; $a = -5$, $x = -\dfrac{1}{2}$, $y = \dfrac{3}{2}$, $z = \dfrac{1}{2}$.

(ii) $b = -1$

When $a = 1$, $x = \dfrac{1}{4}$, $y = \dfrac{-1}{4}$, $z = \dfrac{-3}{4}$

(2003-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $a \neq 1$, $a \neq \dfrac{-1}{2}$ and $a \neq -4$

$x = \dfrac{-2a(4a+1)}{(2a+1)(a+4)}$, $y = \dfrac{4a}{(2a+1)(a+4)}$, $z = \dfrac{-2a}{a+4}$

(ii) (1) $\left\{ \left(\dfrac{-2}{3}, \dfrac{2+3t}{3}, t \right) : t \in \mathbf{R} \right\}$

(2) Inconsistent

(b) least value = 9 , $x = \dfrac{-2}{3}$, $y = \dfrac{-1}{3}$, $z = -1$

(2004-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $x = \dfrac{b-2}{a-2}$, $y = \dfrac{b-a}{2(a-2)}$, $z = \dfrac{a-b}{2(a-2)}$

(ii) (1) $b = 2$, $\{(1 - 2t,\ - t, t) : t \in \mathbf{R}\}$

(2) b can be any value, $\left\{ \left(\dfrac{2b + 1 - 10t}{9}, \dfrac{4 - b - 13t}{9}, t \right) : t \in \mathbf{R} \right\}$

(b) $\dfrac{-1}{6} < k < 0$

(2005-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $a \neq -2$, $a \neq 1$ and $a \neq 3$

$$x = \frac{b\left(a^2 - 6a - 3\right)}{(a-1)(a-3)} \ , \ y = \frac{4b}{(a-1)(a-3)} \ , \ z = \frac{-2b}{a-1}$$

(ii) (1) $b = 0$, $\{(-2t, t, t) : t \in \mathbf{R}\}$

(2) b can be any value, $\left\{ \left(\dfrac{b + 5t}{5}, \dfrac{-2b + 5t}{5}, t \right) : t \in \mathbf{R} \right\}$

(b) $\dfrac{-15}{7} \leq b \leq \dfrac{15}{2}$

(2006-AL-P MATH 1 #07) (15 marks)

7. (a) $a \neq 1$ and $b \neq 0$

$$x = \frac{2ab - 4b + 1}{(a-1)b} \ , \ y = \frac{2b - 1}{(a-1)b} \ , \ z = \frac{1}{b}$$

(b) (i) $b = \dfrac{1}{2}$, $\{(2 - t, t, 2) : t \in \mathbf{R}\}$

(2007-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $a \neq -2$ and $a \neq -4$

$$x = \frac{a^2 + 6a + 3b + 5}{(a+2)(a+4)} \ , \ y = \frac{b - 1}{(a+2)(a+4)} \ , \ z = \frac{(a+6)(b-1)}{(a+2)(a+4)}$$

(ii) $b = 1$, $\left\{ \left(\dfrac{4 + 3t}{4}, \dfrac{t}{4}, t \right) : t \in \mathbf{R} \right\}$

(c) $t = -1$, $x = -2$, $y = -1$, $z = -4$

(2008-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $x = \dfrac{-a^2b - 3ab - a + 2}{4 - a^2}$, $y = \dfrac{2b}{a - 2}$, $z = \dfrac{3ab - a + 8b + 2}{4 - a^2}$

(ii) (1) $b = 0$, $\left\{\left(\dfrac{3 - 5t}{7}, \dfrac{1 - 4t}{7}, t\right) : t \in \mathbf{R}\right\}$

(2) $b = -2$, $\{(1 + t, 1, t) : t \in \mathbf{R}\}$

(b) $\dfrac{3}{2}$

(2009-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $\lambda \neq -1$ and $\lambda \neq 3$

$x = \dfrac{a\lambda - 2\lambda - 3}{(\lambda + 1)(\lambda - 3)}$, $y = \dfrac{3(a - \lambda)}{(\lambda + 1)(\lambda - 3)}$, $z = \dfrac{2\lambda(\lambda - a)}{(\lambda + 1)(\lambda - 3)}$

(ii) $a = -1$, $\left\{\left(\dfrac{1 - t}{2}, \dfrac{3t}{2}, t\right) : t \in \mathbf{R}\right\}$

(b) $\lambda = -2$, $a = 3$, inconsistent

(c) $\lambda = -1$, $a = -1$,

when $t = -2$, $x = 3$, $y = -6$, $z = -4$; when $t = 3$, $x = -2$, $y = 9$, $z = 6$.

(2010-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $a \neq -2$ and $a \neq 2$

$x = \dfrac{2(16 - 2b - a)}{4 - a^2}$, $y = \dfrac{-22 - 6a + 4b + ab - 2a^2}{4 - a^2}$, $z = \dfrac{-2 + 8a - ab}{4 - a^2}$

(ii) $b = 7$, $\{(1 + 2t, 1 - 3t, t) : t \in \mathbf{R}\}$

(b) $\lambda = 9$, $\{(-1 - 2t, t + 3, t) : t \in \mathbf{R}\}$, $\mu = 44$

(c) $a = \dfrac{2}{3}$, $b = 5$, inconsistent

(2011-AL-P MATH 1 #07) (15 marks)

7. (a) (i) (2) $\{(2t, -2t, t) : t \in \mathbf{R}\}$

(ii) (1) $\lambda \neq -2$, $\lambda \neq 0$ and $\lambda \neq 1$

(2) $x = \dfrac{\left(\lambda^2 + 2\lambda\mu + \lambda - 4\right)\mu}{\lambda(\lambda - 1)(\lambda + 2)}$, $y = \dfrac{(\lambda + 1)(1 - \lambda\mu)\mu}{\lambda(\lambda - 1)(\lambda + 2)}$, $z = \dfrac{(\lambda\mu - 1)\mu}{\lambda(\lambda - 1)(\lambda + 2)}$

(3) When $\lambda = 1$, $\mu = 1$; when $\lambda = -2$, $\mu = \dfrac{-1}{2}$.

(b) $\lambda = 1$, $\mu = 1$, there is no real solution.

(SP-DSE-MATH-EP(M2) #07) (5 marks)

7. $\{(3 - t, t - 1, t) : t \in \mathbf{R}\}$

(PP-DSE-MATH-EP(M2) #02) (4 marks)

2. $k = 19$ or 2

(2012-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $x = \dfrac{2ab + 11a + 3b + 6}{2(a+1)(a+3)}$, $y = \dfrac{3a - b}{2(a+3)}$, $z = \dfrac{b - 7a - 12}{2(a+1)(a+3)}$

(ii) $b = -9$, $\{(7 - 3t, 2t - 2, t) : t \in \mathbf{R}\}$

(b) $a = -\dfrac{4}{3}$, $b = -4$, inconsistent

(c) $a = -3$, $b = -9$, least value $= -56$

(2012-DSE-MATH-EP(M2) #08) (5 marks)

8. (a) $\{(2 - 2t, t - 2, t) : t \in \mathbf{R}\}$

(b) When $\lambda \neq 3$, $x = 2$, $y = -2$, $z = 0$; when $\lambda = 3$, $\{(2 - 2t, t - 2, t) : t \in \mathbf{R}\}$.

(2013-AL-P MATH 1 #07) (15 marks)

7. (a) (i) $a \neq 1$ and $a \neq 4$

$x = \dfrac{2a + 2b - ab - 5}{(a-1)(a-4)}$, $y = \dfrac{a + 2b - 7}{(a-1)(a-4)}$, $z = \dfrac{1 + 3ab - 4a - 2b}{(a-1)(a-4)}$

(ii) $b = 3$, $\{(t, 2t - 1, t - 2) : t \in \mathbf{R}\}$

(b) $a = 1$, $b = 3$, $\lambda = 3$, $\mu = 2$

(c) $\beta = 1$ or 2

(2013-DSE-MATH-EP(M2) #09) (5 marks)

9. (a) $a = 2$ or $b = 0$

(b) $\left\{(6 - t, -2, t) : t \in \mathbf{R}\right\}$

(2014-DSE-MATH-EP(M2) #09) (6 marks)

9. (a) $\left\{\left(80 + \dfrac{4t}{5}, 20 - \dfrac{9t}{5}, t\right) : t \in \mathbf{R}\right\}$

(b) $t = 0$, 5 or 10 . Disagreed.

(2015-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) $\left\{(2 - 6t, 5t, t) : t \in \mathbf{R}\right\}$

(b) $k = 8$, $\left\{(2 - 6t, 5t, t) : t \in \mathbf{R}\right\}$

$k \neq 8$, $t = 0$, $x = 2$, $y = 0$, $z = 0$

(2016-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) (i) (2) $x = \dfrac{3a^2 - ab + 50a + 6b - 24}{(a+2)(a+12)}$, $y = \dfrac{2(ab - 10a + 8)}{(a+2)(a+12)}$, $z = \dfrac{ab - 12a + 6b - 80}{(a+2)(a+12)}$

(ii) (1) $b = 14$

(2) $\{(2t + 2, 1 - t, t) : t \in \mathbf{R}\}$

(b) $a = -2$, $b = 14$, Greatest value is 14 , no real solution.

(2017-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) (i) $h \neq 2$

(ii) $z = \dfrac{k - 14}{h - 2}$

(b) $h = 2$, $k = 14$

$\left\{(-7t - 5, 4t + 8, t) : t \in \mathbf{R}\right\}$

(2018-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) (i) (1) $a \neq 3$ and $a \neq -1$

(2) $x = \dfrac{a^2b + ab + 10a - 2b - 50}{(a+1)(a-3)}$, $y = \dfrac{-3ab + 22a - 5b - 38}{(a+1)(a-3)}$, $z = \dfrac{b - 2}{2(a-3)}$

(ii) (1) $b = 2$

(2) $\left\{(5m + 6, -7m + 4, m) : m \in \mathbf{R}\right\}$

(b) $s = 2$, $t = -8$

(2019-DSE-MATH-EP(M2) #06) (7 marks)

6. (a) (i) $\alpha \neq -4$ and $\alpha \neq -10$

(ii) $y = -\dfrac{\beta}{\alpha + 4}$

(b) $\beta \neq 0$

(2020-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) (i) (2) $x = \dfrac{h^2 + 2hk + 7h + 7k + 14}{(h+3)^2}$, $y = \dfrac{2h - k + 7}{(h+3)^2}$, $z = \dfrac{hk - h + 4k - 1}{(h+3)^2}$

(ii) (2) $\{(t + 4, 3 - t, t) : t \in \mathbf{R}\}$

(b) The claim is agreed.

(2021-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $\{(3 - 32t, 13t - 1, t) : t \in \mathbf{R}\}$

(b) The claim is not correct.

3. Introduction to Vectors

(1991-CE-A MATH 1 #05) (7 marks)

5.

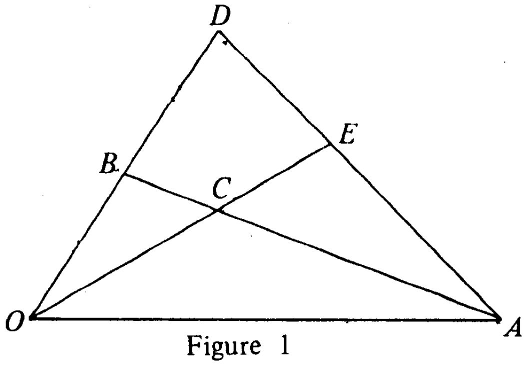


Figure 1

In Figure 1, OAD is a triangle and B is the mid-point of OD. The line OE cuts the line AB at C such that $AC : CB = 3 : 1$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) (i) Let $OC : CE = k : 1$. Express $\overrightarrow{OE}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

(ii) Let $AE : ED = m : 1$. Express $\overrightarrow{OE}$ in terms of m, $\mathbf{a}$ and $\mathbf{b}$.

Hence find k and m.

(1992-CE-A MATH 1 #01) (5 marks)

1. Given $\overrightarrow{OA} = 5\mathbf{i} - \mathbf{j}$, $\overrightarrow{OB} = -3\mathbf{i} + 5\mathbf{j}$ and APB is a straight line.

(a) Find $\overrightarrow{AB}$ and $\left|\overrightarrow{AB}\right|$.

(b) If $\left|\overrightarrow{AP}\right| = 4$, find $\overrightarrow{AP}$.

(1999-CE-A MATH 1 #10) (16 marks)

10.

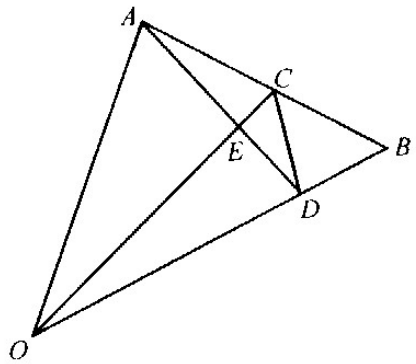


Figure 3

In Figure 3, OAB is a triangle. C and D are points on AB and OB respectively such that $AC : CB = 8 : 7$ and $OD : DB = 16 : 5$. OC and AD intersect at a point E . Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ and $\overrightarrow{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Let $\overrightarrow{OE} = r\overrightarrow{OC}$ and $\overrightarrow{AE} = k\overrightarrow{AD}$.

(i) Express $\overrightarrow{OE}$ in terms of r , $\mathbf{a}$ and $\mathbf{b}$.

(ii) Express $\overrightarrow{OE}$ in terms of k , $\mathbf{a}$ and $\mathbf{b}$.

Hence show that $r = \dfrac{6}{7}$ and $k = \dfrac{3}{5}$.

(c) It is given that $EC : ED = 1 : 2$.

(i) Using (b), or otherwise, find $EA : EO$.

(ii) Explain why $OACD$ is a cyclic quadrilateral.

(2001-CE-A MATH #14) (12 marks)

14. (a)

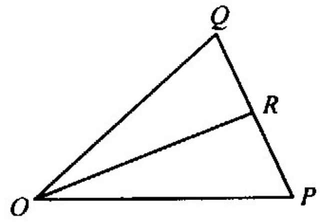


Figure 1(a)

In Figure 1 (a), OPQ is a triangle. R is a point on PQ such that $PR : RQ = r : s$. Express $\overrightarrow{OR}$ in terms of r , s , $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

Hence show that if $\overrightarrow{OR} = m\overrightarrow{OP} + n\overrightarrow{OQ}$, then $m + n = 1$.

(b)

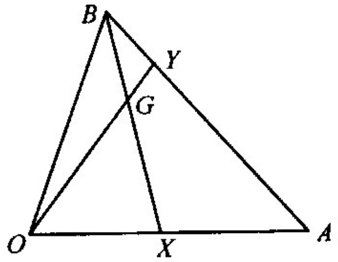


Figure 1(b)

In Figure 1 (b), OAB is a triangle. X is the mid-point of OA and Y is a point on AB . BX and OY intersect at point G where $BG : GX = 1 : 3$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Express $\overrightarrow{OG}$ in terms of **a** and **b**.

(ii) Using (a), express $\overrightarrow{OY}$ in terms of **a** and **b**.

(Hint: Put $\overrightarrow{OY} = k\overrightarrow{OG}$.)

(iii) Moreover, AG is produced to a point Z on OB . Let $\overrightarrow{OZ} = h\overrightarrow{OB}$.

(1) Find the value of h .

(2) Explain whether ZY is parallel to OA or not.

(2003-CE-A MATH #06) (5 marks)

6.

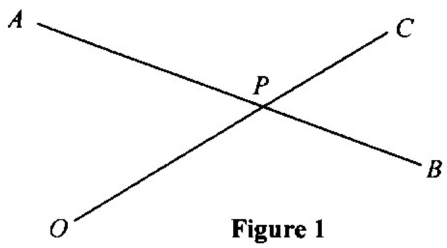


Figure 1

In Figure 1, point P divides both line segments AB and OC in the same ratio $3 : 1$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Hence show that OA is parallel to BC .

(2005-CE-A MATH #14) (12 marks)

14.

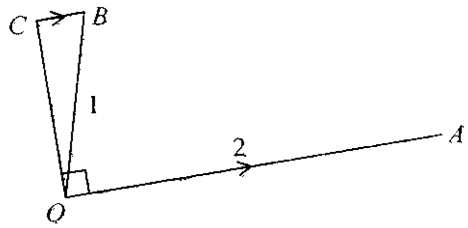


Figure 4

In Figure 4, $OA = 2$, $OB = 1$ and $\cos\angle AOB = \frac{1}{4}$. C is a point such that $CB \,//\, OA$ and $OC \perp OA$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Find $\overrightarrow{CB}$ in terms of $\mathbf{a}$.

Hence, or otherwise, show that $\mathbf{c} = \mathbf{b} - \frac{1}{8}\mathbf{a}$.

(b)

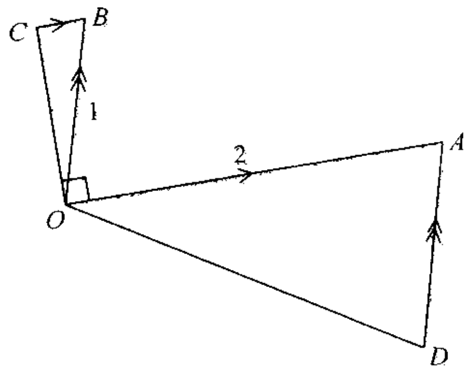


Figure 5

D is a point such that $DA \,//\, OB$ and $OD = OA$ (see Figure 5). Let $\overrightarrow{OD} = \mathbf{d}$.

(i) By finding DA , or otherwise, express $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) P is a point on the line segment CD such that $CP : PD = r : 1$. Express $\overrightarrow{OP}$ in terms of r , $\mathbf{a}$ and $\mathbf{b}$.

(iii) If M is the mid-point of AB , find the ratio in which OM divides CD .

(2008-CE-A MATH #07) (5 marks)

7. It is given that $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OB} = 5\mathbf{i} + 6\mathbf{j}$. If P is a point on AB such that $\overrightarrow{PB} = 2\overrightarrow{AP}$, find the unit vector in the direction of $\overrightarrow{OP}$.

(2011-CE-A MATH #12) (12 marks)

12.

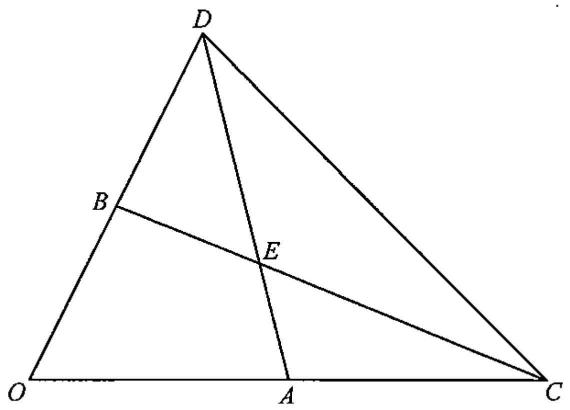


Figure 2

Figure 2 shows a triangle OCD . A and B are points on OC and OD respectively such that $OA : AC = OB : BD = 1 : h$, where $h > 0$. AD and BC intersect at E such that $AE : ED = \mu : (1 - \mu)$ and $BE : EC = \lambda : (1 - \lambda)$, where $0 < \mu < 1$ and $0 < \lambda < 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) By considering $\overrightarrow{OE}$, show that $\mu = \lambda$.

(b) F is a point on CD such that O , E and F are collinear. Show that OF is a median of ΔOCD .

(c) Using the above results, show that in a triangle, the centroid divides every median in $2 : 1$.

ANSWERS

(1991-CE-A MATH 1 #05) (7 marks)

5. (a) $\overrightarrow{OC} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

(b) (i) $\overrightarrow{OE} = \frac{k+1}{4k}\mathbf{a} + \frac{3(k+1)}{4k}\mathbf{b}$

(ii) $\overrightarrow{OE} = \frac{1}{1+m}\mathbf{a} + \frac{2m}{1+m}\mathbf{b}$

$m = \frac{3}{2}$, $k = \frac{5}{3}$

(1992-CE-A MATH 1 #01) (5 marks)

1. (a) $\overrightarrow{AB} = -8\mathbf{i} + 6\mathbf{j}$

$\left|\overrightarrow{AB}\right| = 10$

(b) $\frac{-16}{5}\mathbf{i} + \frac{12}{5}\mathbf{j}$

(1999-CE-A MATH 1 #10) (16 marks)

10. (a) $\overrightarrow{OC} = \frac{7\mathbf{a} + 8\mathbf{b}}{15}$

$\overrightarrow{AD} = \frac{16}{21}\mathbf{b} - \mathbf{a}$

(b) (i) $\frac{r}{15}(7\mathbf{a} + 8\mathbf{b})$

(ii) $(1-k)\mathbf{a} + \frac{16k}{21}\mathbf{b}$

(c) (i) 1 : 2

(2001-CE-A MATH #14) (12 marks)

14. (a) $\overrightarrow{OR} = \frac{s\overrightarrow{OP} + r\overrightarrow{OQ}}{r+s}$

(b) (i) $\overrightarrow{OG} = \frac{1}{8}\mathbf{a} + \frac{3}{4}\mathbf{b}$

(ii) $\overrightarrow{OY} = \frac{1}{7}\mathbf{a} + \frac{6}{7}\mathbf{b}$

(iii) (1) $\frac{6}{7}$

(2003-CE-A MATH #06) (5 marks)

6. (a) $\overrightarrow{OP} = \frac{\mathbf{a} + 3\mathbf{b}}{4}$

(b) $\overrightarrow{OC} = \frac{1}{3}\mathbf{a} + \mathbf{b}$

(2005-CE-A MATH #14) (12 marks)

14. (a) $\overrightarrow{CB} = \frac{1}{8}\mathbf{a}$

(b) (i) $DA = 1$, $\mathbf{d} = \mathbf{a} - \mathbf{b}$

(ii) $\frac{8r-1}{8(r+1)}\mathbf{a} + \frac{1-r}{1+r}\mathbf{b}$

(iii) 9 : 16

(2008-CE-A MATH #07) (5 marks)

7. $\frac{3\mathbf{i} + 4\mathbf{j}}{5}$

(2011-CE-A MATH #12) (12 marks)

12. (a) $\overrightarrow{OE} = (1+\mu)\mathbf{a} + \mu(1+h)\mathbf{b}$

$\overrightarrow{OE} = (1-\lambda)\mathbf{b} + \lambda(1+h)\mathbf{a}$

4. Product of Vectors

(1991-CE-A MATH 1 #08) (16 marks)

8. A , B and C are three points on a plane such that

$\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$,

$\overrightarrow{BC} = 7\mathbf{i} + \mathbf{j}$,

and $\overrightarrow{OC} = x\mathbf{i} + y\mathbf{j}$,

where O is the origin.

(a) Find $\overrightarrow{CA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$ in terms of x , y , $\mathbf{i}$ and $\mathbf{j}$.

(b) Given $\overrightarrow{AB} \cdot \overrightarrow{BC} = 4\overrightarrow{BC} \cdot \overrightarrow{CA}$.

(i) Show that $y = 30 - 7x$.

(ii) If $\left|\overrightarrow{BC}\right| = \sqrt{5}\left|\overrightarrow{CA}\right|$ and x , y are positive,

(1) find x and y ,

(2) show that CA is perpendicular to AB ,

(3) show that O lies on AB .

(1992-CE-A MATH 1 #08) (16 marks) (Modified - No figure given)

8. Given ΔOAB where $OA = 2$, $OB = 3$ and $\angle AOB = \dfrac{\pi}{3}$. D is a point on OB such that AD is perpendicular to OB . Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.

(a) Find $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{b}$.

(b) Find the length of OD .

Hence express $\overrightarrow{OD}$ in terms of $\mathbf{b}$.

(c) Let H be a point on AD such that $AH : HD = 1 : k$ and $\overrightarrow{OH}$ is perpendicular to $\overrightarrow{AB}$.

(i) Express $\overrightarrow{OH}$ in terms of k , $\mathbf{a}$ and $\mathbf{b}$.

Hence find the value of k .

(ii) OH produced meets AB at a point C . Let $AC : CB = 1 : m$ and $OH : HC = 1 : n$.

(1) Express $\overrightarrow{OC}$ in terms of m , $\mathbf{a}$ and $\mathbf{b}$.

(2) Express $\overrightarrow{OC}$ in terms of n , $\mathbf{a}$ and $\mathbf{b}$.

(3) Hence find m and n .

(1993-CE-A MATH 1 #06) (7 marks)

6. Given $\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j}$, $\overrightarrow{OB} = \mathbf{i} + \mathbf{j}$. C is a point such that $\angle ABC$ is a right angle.

(a) Find $\overrightarrow{AB}$.

(b) Find $\overrightarrow{AB} \cdot \overrightarrow{AB}$ and $\overrightarrow{AB} \cdot \overrightarrow{BC}$.

Hence find $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

(1993-CE-A MATH 1 #08) (16 marks)

8.

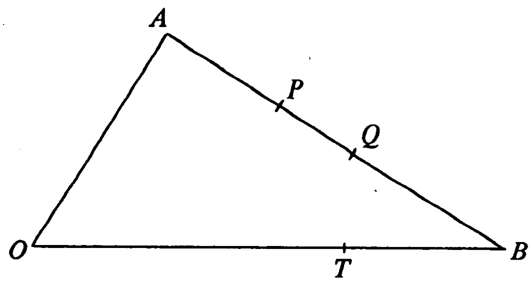


Figure 1

In Figure 1, OAB is a triangle. P , Q are two points on AB such that $AP : PB = PQ : QB = r : 1$, where $r > 0$. T is a point on OB such that $OT : TB = 1 : r$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of r , $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{OT}$ in terms of r and $\mathbf{b}$.

Hence show that $\overrightarrow{TQ} = \dfrac{a + (r^2 + r - 1)b}{(r + 1)^2}$.

(c) Find the value(s) of r such that $\overrightarrow{OA}$ is parallel to $\overrightarrow{TQ}$.

(d) Suppose $OA = 2$, $OB = 16$ and $\angle AOB = \dfrac{\pi}{3}$.

(i) Find $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{b}$.

(ii) Find the value(s) of r such that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{TQ}$.

(1994-CE-A MATH 1 #03) (6 marks)

3. P , Q and R are points on a plane such that $\overrightarrow{OP} = \mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OQ} = 3\mathbf{i} + \mathbf{j}$ and $\overrightarrow{PR} = -3\mathbf{i} - 2\mathbf{j}$, where O is the origin.

(a) Find $\overrightarrow{PQ}$ and $\left|\overrightarrow{PQ}\right|$.

(b) Find the value of $\cos\angle QPR$.

(1994-CE-A MATH 1 #10) (16 marks)

10.

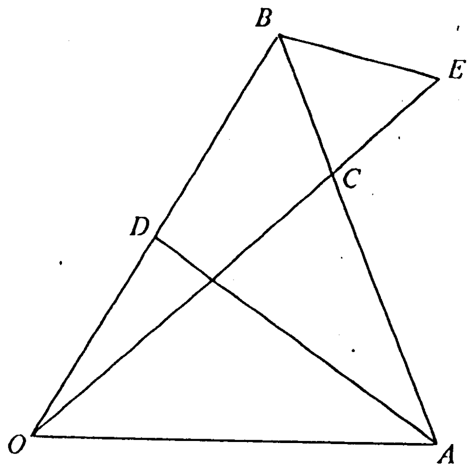


Figure 2

In Figure 2, D is the mid-point of OB and C is a point on AB such that $AC : CB = 2 : 1$. OC is produced to a point E such that $OC : CE = 1 : k$. Let $\overrightarrow{OQ} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ and $\overrightarrow{DA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Show that $\overrightarrow{BE} = \dfrac{k+1}{3}\mathbf{a} + \dfrac{2k-1}{3}\mathbf{b}$.

(c) Find the value of k such that $\overrightarrow{BE}$ is parallel to $\overrightarrow{DA}$.

(d) Given $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$, $\angle BOA = \dfrac{\pi}{3}$.

(i) Find $\mathbf{a} \cdot \mathbf{b}$.

(ii) Find the value of k such that $\overrightarrow{BE}$ is perpendicular to $\overrightarrow{OE}$.
Hence find the distance of B from OC .

(1995-CE-A MATH 1 #07) (8 marks)

7. Let $\overrightarrow{OP} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OQ} = -6\mathbf{i} + 4\mathbf{j}$. Let R be a point on PQ such that $PR : RQ = k : 1$, where $k > 0$.

(a) Express $\overrightarrow{OR}$ in terms of k , $\mathbf{i}$ and $\mathbf{j}$.

(b) Express $\overrightarrow{OP} \cdot \overrightarrow{OR}$ and $\overrightarrow{OQ} \cdot \overrightarrow{OR}$ in terms of k .

(c) Find the value of k such that OR bisects $\angle POQ$.

(1995-CE-A MATH 1 #08) (16 marks)

8.

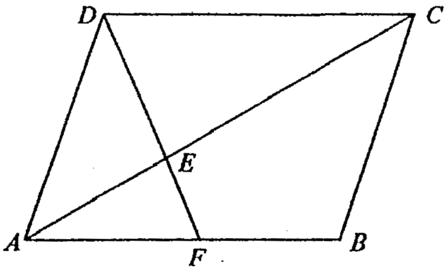


Figure 1

In Figure 1, $ABCD$ is a parallelogram and F is a point on AB . DF meets AC at a point E such that $DE : EF = \lambda : 1$, where λ is a positive number. Let $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$ and $\overrightarrow{AE} = h\overrightarrow{AC}$, $\overrightarrow{AF} = k\overrightarrow{AB}$, where h , k are positive numbers.

(a) (i) Express $\overrightarrow{AE}$ in terms of h , $\mathbf{p}$ and $\mathbf{q}$.

(ii) Express $\overrightarrow{AE}$ in terms of λ , k , $\mathbf{p}$ and $\mathbf{q}$. Hence show that $\lambda = \dfrac{1}{k}$.

(b) It is given that $|\mathbf{p}| = 3$, $|\mathbf{q}| = 2$, $\angle DAB = \dfrac{\pi}{3}$.

(i) Find $\mathbf{p} \cdot \mathbf{q}$.

(ii) Suppose DF is perpendicular to AC .

(1) By expressing $\overrightarrow{DF}$ in terms of k , $\mathbf{p}$ and $\mathbf{q}$, find the value of k .

(2) Using (a), or otherwise, find the length of AE .

(1996-CE-A MATH 1 #07) (6 marks)

7. Given $\overrightarrow{OA} = 4\mathbf{i} + 3\mathbf{j}$ and C is a point on OA such that $\left|\overrightarrow{OC}\right| = \dfrac{16}{5}$.

(a) Find the unit vector in the direction of $\overrightarrow{OA}$.

Hence find $\overrightarrow{OC}$.

(b) If $\overrightarrow{OB} = \mathbf{i} + 4\mathbf{j}$, show that BC is perpendicular to OA .

(1996-CE-A MATH 1 #10) (16 marks)

10.

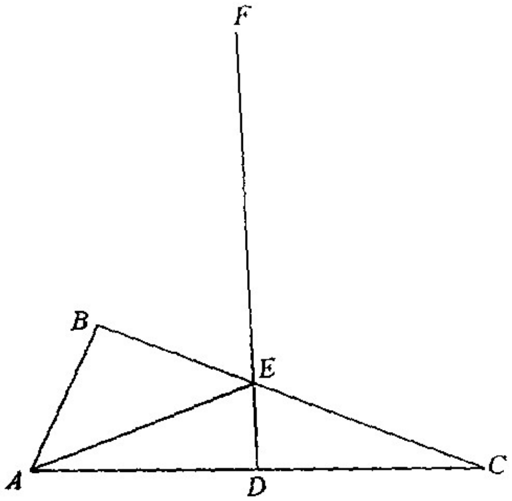


Figure 2

In Figure 2, D is the mid-point of AC and E is a point on BC such that $BE : EC = 1 : t$, where $t > 0$. DE is produced to a point F such that $DE : EF = 1 : 7$. Let $\overrightarrow{AD} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$.

(a) (i) Express $\overrightarrow{AE}$ in terms of t , $\mathbf{a}$ and $\mathbf{b}$.

(ii) Express $\overrightarrow{AE}$ in terms of $\mathbf{a}$ and $\overrightarrow{AF}$.

Hence, or otherwise, show that $\overrightarrow{AF} = \dfrac{9-7t}{1+t}\mathbf{a} + \dfrac{8t}{1+t}\mathbf{b}$.

(b) Suppose that A , B and F are collinear.

(i) Find the value of t .

(ii) It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and $\cos\angle BAC = \dfrac{1}{3}$.

(1) Find $\mathbf{a} \cdot \mathbf{b}$.

(2) Find $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and $\overrightarrow{AD} \cdot \overrightarrow{DE}$.

(3) Does the circle passing through points B , C and D also pass through point F ?
Explain your answer.

(1997-CE-A MATH 1 #07) (7 marks)

7. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors such that $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$, $|\mathbf{b}| = \sqrt{5}$ and $\cos\theta = \frac{4}{5}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(a) Find $|\mathbf{a}|$.

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

(c) If $\mathbf{b} = m\mathbf{i} + n\mathbf{j}$, find the values of m and n .

(1997-CE-A MATH 1 #09) (16 marks)

9.

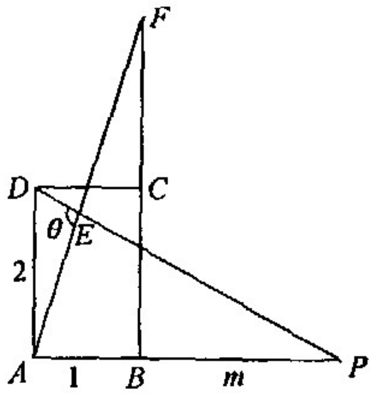


Figure 2

In Figure 2, $ABCD$ is a rectangle wilt $AB = 1$ and $AD = 2$. F is a point on BC produced with $BC = CF$. P is a variable point on AB produced such that $BP = m$. AF and DP intersect at a point E . Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$ and $\angle AED = \theta$.

(a) (i) Express $\overrightarrow{AF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Express $\overrightarrow{DP}$ in terms of m , $\mathbf{a}$ and $\mathbf{b}$.

(b) Suppose $\theta = \frac{\pi}{2}$.

(i) Show that $m = 7$.

(ii) Let $AE : EF = 1 : r$ and $DE : EP = 1 : k$.

(1) Express $\overrightarrow{AE}$ in terms of r , $\mathbf{a}$ and $\mathbf{b}$.

(2) Express $\overrightarrow{AE}$ in terms of k , $\mathbf{a}$ and $\mathbf{b}$.
Hence find the values of r and k .

(c) As m tends to infinity, θ approaches a certain value θ_1 . Find θ_1 correct to the nearest degree.

(1998-CE-A MATH 1 #05) (6 marks)

5.

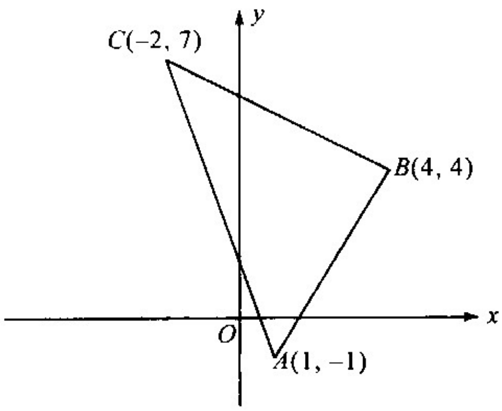


Figure 1

Figure 1 shows the points A , B and C whose position vectors are $\mathbf{i}-\mathbf{j}$, $4\mathbf{i}+4\mathbf{j}$ and $-2\mathbf{i}+7\mathbf{j}$ respectively.

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) By considering $\overrightarrow{AB}\cdot\overrightarrow{AC}$, find $\angle BAC$ to the nearest degree.

(1998-CE-A MATH 1 #09) (16 marks)

9.

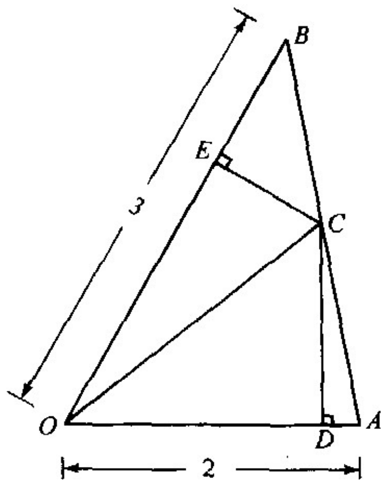


Figure 2

In Figure 2, OAB is a triangle with $OA = 2$, $OB = 3$ and $\angle AOB = \frac{\pi}{3}$. C is a point on AB such that $AC : CB = t : 1 - t$, where $0 < t < 1$. D and E are respectively the feet of perpendicular from C to OA and OB . Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) Find $\mathbf{a} \cdot \mathbf{b}$,

(ii) Express $\overrightarrow{OC}$ in terms of t , $\mathbf{a}$ and $\mathbf{b}$.

(iii) Express $\mathbf{a} \cdot \overrightarrow{OC}$ and $\mathbf{b} \cdot \overrightarrow{OC}$ in terms of t .

(b) (i) Using (a) (iii), show that $\mathbf{a} \cdot \overrightarrow{OD} = 4 - t$ and $\mathbf{b} \cdot \overrightarrow{OE} = 3 + 6t$.

(ii) If $\overrightarrow{OD} = k\mathbf{a}$ and $\overrightarrow{OE} = s\mathbf{b}$, express k and s in terms of t .

(c) Find the value of t such that $\overrightarrow{DE}$ is parallel to $\overrightarrow{AB}$.

(1999-CE-A MATH 1 #07) (6 marks)

7. Let $\mathbf{a}$, $\mathbf{b}$ be two vectors such that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $|\mathbf{b}| = 4$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$.

(a) Find $|\mathbf{a}|$.

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

(c) If the vector $(m\mathbf{a} + \mathbf{b})$ is perpendicular to $\mathbf{b}$, find the value of m .

(1999-CE-A MATH 1 #10) (16 marks)

10.

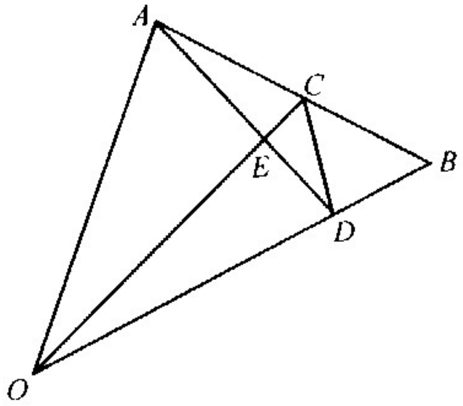


Figure 3

In Figure 3, OAB is a triangle. C and D are points on AB and OB respectively such that $AC : CB = 8 : 7$ and $OD : DB = 16 : 5$. OC and AD intersect at a point E . Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ and $\overrightarrow{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Let $\overrightarrow{OE} = r\overrightarrow{OC}$ and $\overrightarrow{AE} = k\overrightarrow{AD}$.

(i) Express $\overrightarrow{OE}$ in terms of r , $\mathbf{a}$ and $\mathbf{b}$.

(ii) Express $\overrightarrow{OE}$ in terms of k , $\mathbf{a}$ and $\mathbf{b}$.

Hence show that $r = \frac{6}{7}$ and $k = \frac{3}{5}$.

(c) It is given that $EC : ED = 1 : 2$.

(i) Using (b), or otherwise, find $EA : EO$.

(ii) Explain why $OACD$ is a cyclic quadrilateral.

(2000-CE-A MATH 1 #08) (7 marks)

8.

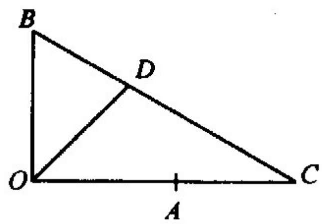


Figure 1

In Figure 1, $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$. C is a point on OA produced such that $AC = k$, where $k > 0$. D is a point on BC such that $BD : DC = 1 : 2$.

(a) Show that $\overrightarrow{OD} = \frac{1+k}{3}\mathbf{i} + \frac{2}{3}\mathbf{j}$.

(b) If $\overrightarrow{OD}$ is a unit vector, find

(i) k ,

(ii) $\angle BOD$, giving your answer correct to the nearest degree.

(2000-CE-A MATH 1 #09) (16 marks)

9.

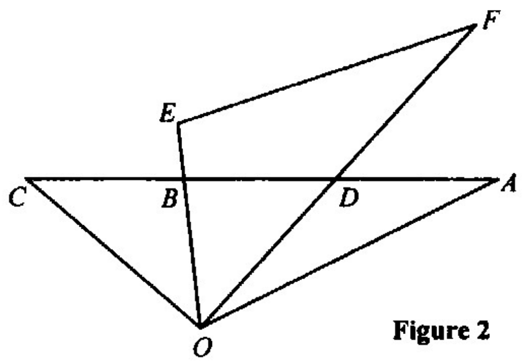


Figure 2

In Figure 2, OAC is a triangle. B and D are points on AC such that $AD = DB = BC$. F is a point on OD produced such that $OD = DF$. E is a point on OB produced such that $OE = k(OB)$, where $k > 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) Express $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Show that $\overrightarrow{OC} = \frac{-1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$.

(iii) Express $\overrightarrow{EF}$ in terms of k , $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $OA = 3$, $OB = 2$ and $\angle AOB = \frac{\pi}{3}$.

(i) Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{b}$.

(ii) Suppose that $\angle OEF = \frac{\pi}{2}$.

(1) Find the value of k .

(2) A student states that points C , E and F are collinear. Explain whether the student is correct.

(2001-AL-P MATH 1 #04) (5 marks)

4. A , B , C are the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ respectively and O is the origin.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Let $S_{\Delta XYZ}$ denote the area of the triangle with vertices X , Y and Z . Prove that

$$S_{\Delta ABC}{}^2 = S_{\Delta OAB}{}^2 + S_{\Delta OBC}{}^2 + S_{\Delta OCA}{}^2 .$$

(2001-CE-A MATH #08) (6 marks)

8. Let $\mathbf{a}$, $\mathbf{b}$ be two vectors such that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$.

(a) Find $\mathbf{a} \cdot \mathbf{b}$.

(b) Find the value of k if the vectors $(\mathbf{a} + k\mathbf{b})$ and $(\mathbf{a} - 2\mathbf{b})$ are perpendicular to each other.

(2001-CE-A MATH #14) (12 marks)

14. (a)

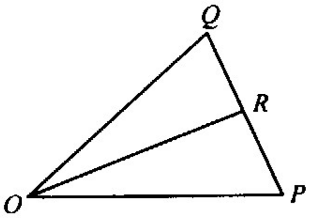


Figure 1(a)

In Figure 1 (a), OPQ is a triangle. R is a point on PQ such that $PR : RQ = r : s$.

Express $\overrightarrow{OR}$ in terms of r , s , $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Hence show that if $\overrightarrow{OR} = m\overrightarrow{OP} + n\overrightarrow{OQ}$, then $m + n = 1$.

(b)

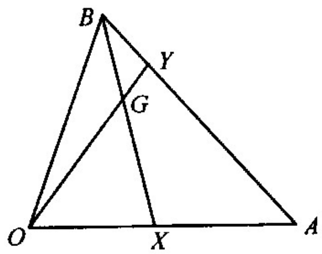


Figure 1(b)

In Figure 1 (b), OAB is a triangle. X is the mid-point of OA and Y is a point on AB . BX and OY intersect at point G where $BG : GX = 1 : 3$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Using (a), express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(Hint: Put $\overrightarrow{OY} = k\overrightarrow{OG}$.)

(iii) Moreover, AG is produced to a point Z on OB . Let $\overrightarrow{OZ} = h\overrightarrow{OB}$.

(1) Find the value of h .

(2) Explain whether ZY is parallel to OA or not.

(2002-AL-P MATH 1 #04) (6 marks)

4. Let $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ and $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = 1 + \mathbf{j}$, $\mathbf{c} = \mathbf{j} + \mathbf{k}$.

(a) Prove that $\mathbf{a}$ is not perpendicular to $\mathbf{b} \times \mathbf{c}$.

(b) Find all unit vectors which are perpendicular to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

(c) If $\theta \in [0,\pi]$ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, prove that $\frac{\pi}{4} < \theta < \frac{\pi}{3}$.

(2002-CE-A MATH #10) (6 marks)

10.

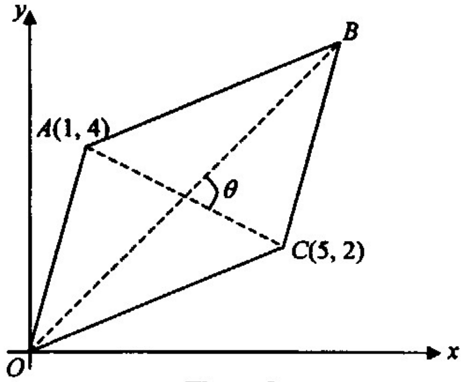


Figure 2

Figure 2 shows a parallelogram $OABC$. The position vectors of the points A and C are $\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i} + 2\mathbf{j}$ respectively.

(a) Find $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

(b) Let θ be the acute angle between OB and AC . Find θ correct to the nearest degree.

(2002-CE-A MATH #13) (12 marks)

13.

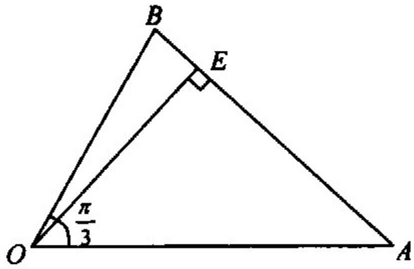


Figure 4

In Figure 4, OAB is a triangle. Point E is the foot of perpendicular from O to AB. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. It is given that $OA = 3$, $OB = 2$ and $\angle AOB = \frac{\pi}{3}$.

(a) Find $\mathbf{a} \cdot \mathbf{b}$.

(b) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
(Hint : Let $BE : EA = t : (1-t)$.)

(c) F is a variable point on OE. A student says that $\overrightarrow{BA} \cdot \overrightarrow{BF}$ is always a constant. Explain whether the student is correct or not.
If you agree with the student, find the value of that constant.
If you do not agree with the student, find two possible values of $\overrightarrow{BA} \cdot \overrightarrow{BF}$.

(2003-AL-P MATH 1 #05) (6 marks)

5. Let $\mathbf{m}$ and $\mathbf{n}$ be vectors in $\mathbf{R}^3$ and $\lambda \in \mathbf{R}$. It is given that

$$\begin{cases} \mathbf{u} = \lambda \mathbf{n} + (1-\lambda)\mathbf{m} \\ \mathbf{v} = 2(1-\lambda)\mathbf{n} - \lambda \mathbf{m} \end{cases}$$

(a) Prove that $\mathbf{u} \times \mathbf{v} = (3\lambda^2 - 4\lambda + 2)\mathbf{m} \times \mathbf{n}$.

(b) Suppose $|\mathbf{m}| = 4$, $|\mathbf{n}| = 3$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\frac{\pi}{6}$.

(i) Evaluate $|\mathbf{m} \times \mathbf{n}|$.

(ii) Find the smallest area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ as λ varies.

(2003-CE-A MATH #06) (5 marks)

6.

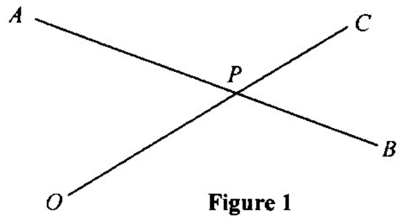


Figure 1

In Figure 1, point P divides both line segments AB and OC in the same ratio $3:1$. Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Hence show that OA is parallel to BC .

(2003-CE-A MATH #14) (12 marks)

14.

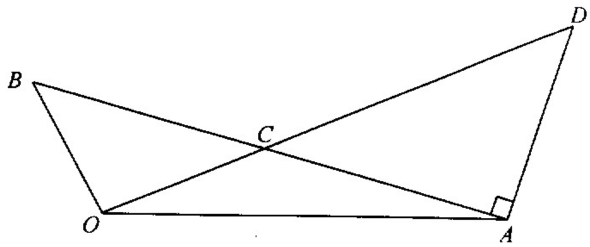


Figure 3

In Figure 3, OAB is a triangle such that $OA=3$, $OB=1$ and $\angle AOB=\dfrac{2\pi}{3}$. C is a point on AB such that $AC:CB=3:2$. D is a point on OC produced such that $\overrightarrow{OD}=k\overrightarrow{OC}$ and AB is perpendicular to AD . Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$.

(a) Find $\mathbf{a}\cdot\mathbf{b}$.

(b) Show that $\overrightarrow{AD}=\left(\dfrac{2k}{5}-1\right)\mathbf{a}+\dfrac{3k}{5}\mathbf{b}$.
Hence find the value of k .

(c) Determine whether the triangles OCB and ACD are similar.

(2004-CE-A MATH #06) (5 marks)

6.

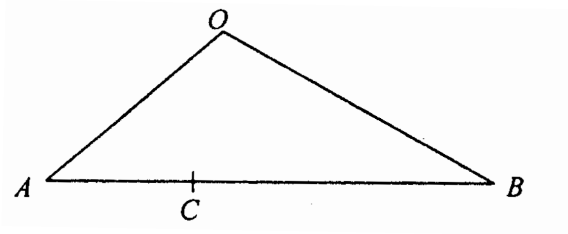


Figure 2

In Figure 2, OAB is a triangle. C is a point on AB such that $AC : CB = 1 : 2$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$ and $\angle AOB = \dfrac{2\pi}{3}$, find $\left|\overrightarrow{OC}\right|$.

(2004-CE-A MATH #13) (12 marks)

13.

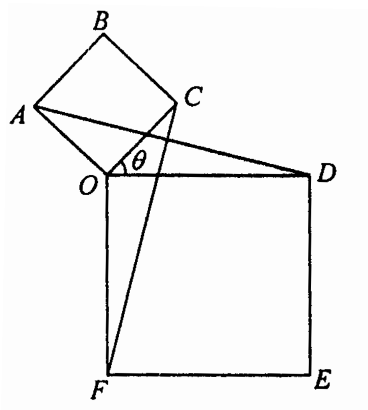


Figure 6

In Figure 6, $OABC$ and $ODEF$ are two squares such that $OA = 1$, $OF = 2$ and $\angle COD = \theta$, where $0 < \theta < \dfrac{\pi}{2}$.

Let $\overrightarrow{OD} = 2\mathbf{i}$ and $\overrightarrow{OF} = -2\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are two perpendicular unit vectors.

(a) (i) Express $\overrightarrow{OC}$ and $\overrightarrow{OA}$ in terms of θ , $\mathbf{i}$ and $\mathbf{j}$.

(ii) Show that $\overrightarrow{AD} = (2 + \sin\theta)\mathbf{i} - \cos\theta\,\mathbf{j}$.

(b) Show that $\overrightarrow{AD}$ is always perpendicular to $\overrightarrow{FC}$.

(c) Find the value(s) of θ such that points B , C and E are collinear. Give your answer(s) correct to the nearest degree.

(2005-AL-P MATH 1 #12) (15 marks)

12. (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbf{R}^3$.

(i) Prove that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

Hence deduce that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.

(ii) Suppose $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \neq 0$. Prove that

$$\mathbf{x} = \left(\frac{\mathbf{x}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\right)\mathbf{a} + \left(\frac{\mathbf{x}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}\right)\mathbf{b} + \left(\frac{\mathbf{x}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}\right)\mathbf{c}$$

for any vector $\mathbf{x}$ in $\mathbf{R}^3$.

(iii) Suppose $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = 1$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$.

(1) Prove that $\left|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\right| = 1$.

(2) Using (a) (ii), prove that

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{a})\mathbf{a} + (\mathbf{x}\cdot\mathbf{b})\mathbf{b} + (\mathbf{x}\cdot\mathbf{c})\mathbf{c}$$

for any vector $\mathbf{x}$ in $\mathbf{R}^3$.

(b) Let $\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})$, $\mathbf{w} = \frac{1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = 6\mathbf{i}-\mathbf{j}+10\mathbf{k}$.

Find real numbers α , β and γ such that $\mathbf{r} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$.

(2005-CE-A MATH #11) (6 marks)

11.

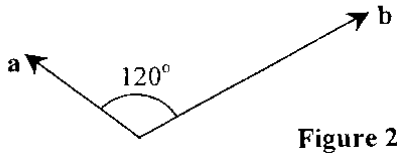


Figure 2

Figure 2 shows two vectors $\mathbf{a}$ and $\mathbf{b}$, where $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, and the angle between the two vectors is $\frac{2\pi}{3}$.

(a) Find $\mathbf{a}\cdot\mathbf{b}$.

(b) Let $\mathbf{c}$ be a vector such that $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$. Find $|\mathbf{c}|$.

(2005-CE-A MATH #14) (12 marks)

14.

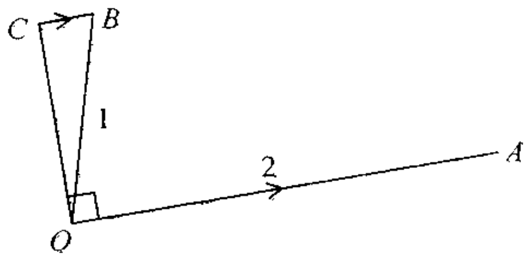


Figure 4

In Figure 4, $OA = 2$, $OB = 1$ and $\cos\angle AOB = \dfrac{1}{4}$. C is a point such that $CB \parallel OA$ and $OC \perp OA$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Find CB in terms of $\mathbf{a}$.

Hence, or otherwise, show that $\mathbf{c} = \mathbf{b} - \dfrac{1}{8}\mathbf{a}$.

(b)

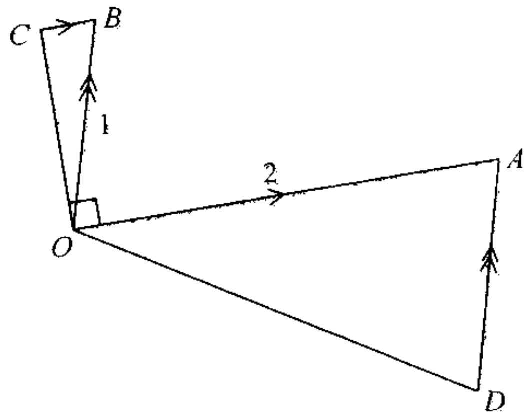


Figure 5

D is a point such that $DA \parallel OB$ and $OD = OA$ (see Figure 5). Let $\overrightarrow{OD} = \mathbf{d}$.

(i) By finding DA , or otherwise, express $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) P is a point on the line segment CD such that $CP : PD = r : 1$. Express $\overrightarrow{OP}$ in terms of r , $\mathbf{a}$ and $\mathbf{b}$.

(iii) If M is the mid-point of AB , find the ratio in which OM divides CD .

(2006-CE-A MATH #07) (5 marks)

7. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors such that $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = 2$ and the angle between them is $\dfrac{5\pi}{6}$.

(a) Find $\mathbf{a} \cdot \mathbf{b}$.

(b) Find $|\mathbf{a} + 2\mathbf{b}|$.

(2006-CE-A MATH #18) (12 marks)

18. Figure 9 shows a triangle OAB . Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OA} = \mathbf{b}$ and M be the mid-point of OA .

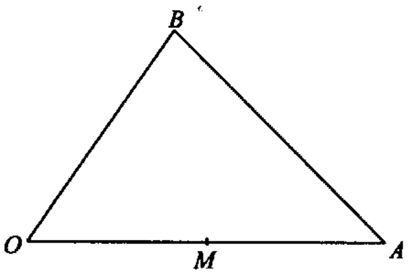


Figure 9

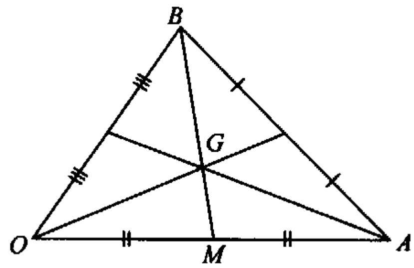


Figure 10

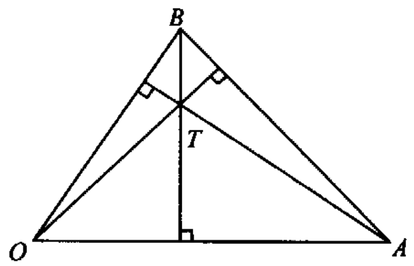


Figure 11

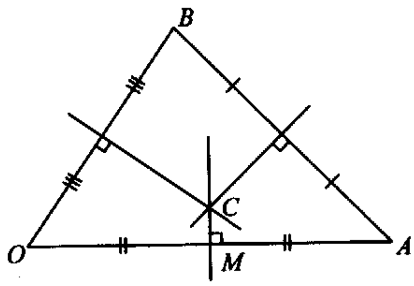


Figure 12

(a) Let G be the centroid of ΔOAB (see Figure 10). It is given that $BG : GM = 2 : 1$. Express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Let T be the orthocentre of ΔOAB (see Figure 11). Show that $\overrightarrow{OT} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} = 0$ and write down the value of $\overrightarrow{OT} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}$.

(c) Let C be the circumcentre of ΔOAB (see Figure 12). Show that $2\overrightarrow{OC} \cdot \mathbf{a} = |\mathbf{a}|^2$ and find $\overrightarrow{OC} \cdot \mathbf{b}$ in terms of $|\mathbf{b}|$.

(d) Consider the points G , T and C described in (a), (b) and (c) respectively.

(i) Using the above results, find the values of $(\overrightarrow{GT} - 2\overrightarrow{CG}) \cdot \mathbf{a}$ and $(\overrightarrow{GT} - 2\overrightarrow{CG}) \cdot \mathbf{b}$.

(ii) Show that G , T and C are collinear.

Note: You may use the following property for vectors in the two-dimensional space:
If $\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v} = 0$, where $\mathbf{u}$ and $\mathbf{v}$ are non-parallel, then $\mathbf{w} = 0$.

(2007-CE-A MATH #08) (5 marks)

8.

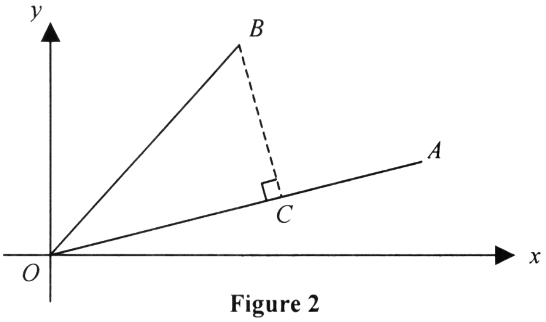


Figure 2

In Figure 2, OCA is a straight line and $BC \perp OA$. It is given that $\overrightarrow{OA} = 6\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + 6\mathbf{j}$. Let $\overrightarrow{OC} = k\overrightarrow{OA}$.

(a) Express $\overrightarrow{BC}$ in terms of k, $\mathbf{i}$ and $\mathbf{j}$.

(b) Find the value of k.

(2007-CE-A MATH #17) (12 marks)

17.

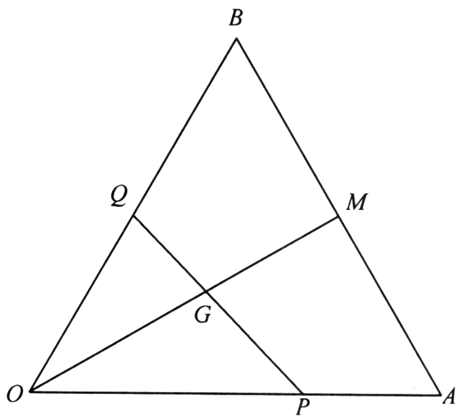


Figure 10

In Figure 10, OAB is an equilateral triangle with $OA = 1$. M is the mid-point of AB and P divides the line segment OA in the ratio $2:1$. Q is a point on OB such that PQ intersects OM at G and $PG : GQ = 4:3$. Let OA and OB be $\mathbf{a}$ and $\mathbf{b}$ respectively.

(a) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Let $OQ : QB = k : (1-k)$.

(i) Find $\overrightarrow{OG}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

(ii) Show that $\overrightarrow{PQ} = \dfrac{1}{2}\mathbf{b} - \dfrac{2}{3}\mathbf{a}$.

(c) (i) Find $\mathbf{a} \cdot \mathbf{b}$ and hence find $\left|\overrightarrow{PQ}\right|$.

(ii) Find $\angle QGM$ correct to the nearest degree.

(2008-CE-A MATH #07) (5 marks)

7. It is given that $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OB} = 5\mathbf{i} + 6\mathbf{j}$. If P is a point on AB such that $\overrightarrow{PB} = 2\overrightarrow{AP}$, find the unit vector in the direction of $\overrightarrow{OP}$.

(2008-CE-A MATH #15) (12 marks)

15.

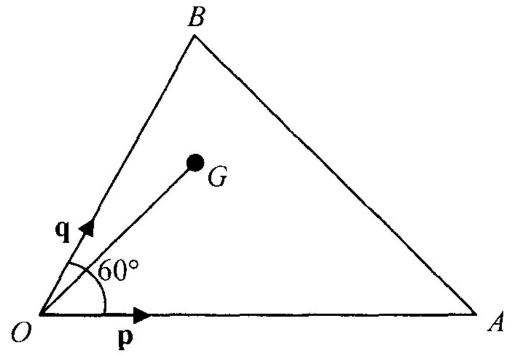


Figure 3

In Figure 3, $\mathbf{p}$ and $\mathbf{q}$ are unit vectors with angle between them $60°$. Let $\overrightarrow{OA} = 4\mathbf{p}$, $\overrightarrow{OB} = 3\mathbf{q}$ and $\overrightarrow{OG} = \frac{2}{3}\mathbf{p} + \frac{5}{3}\mathbf{q}$.

(a) Find $\mathbf{p} \cdot \mathbf{q}$.

(b) Show that $OG \perp AB$.
Hence show that G is the orthocentre of ΔOAB .

(c)

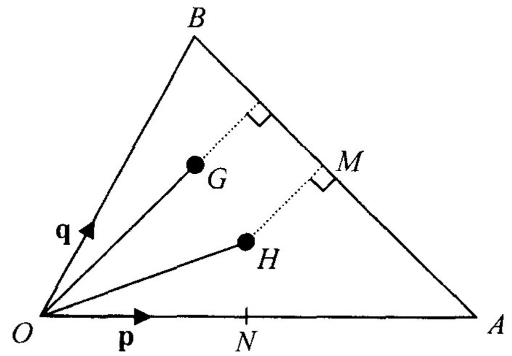


Figure 4

In Figure 4, H is the circumcentre of ΔOAB , M and N are the mid-points of AB and OA respectively. Let $HM : OG = t : 1$.

By expressing $\overrightarrow{HM}$ and $\overrightarrow{HN}$ in terms of t , $\mathbf{p}$ and $\mathbf{q}$, find $\overrightarrow{OH}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(2009-CE-A MATH #07) (4 marks)

7.

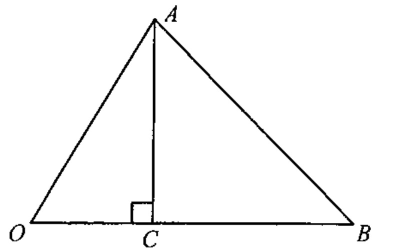


Figure 1

In Figure 1, AC is an altitude of ΔOAB. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively. It is given that $|\mathbf{a}| = 6$, $|\mathbf{b}| = 8$ and $\mathbf{a} \cdot \mathbf{b} = 24$. Find

(a) $\angle AOB$,

(b) $|\mathbf{c}|$.

(2009-CE-A MATH #14) (12 marks)

14.

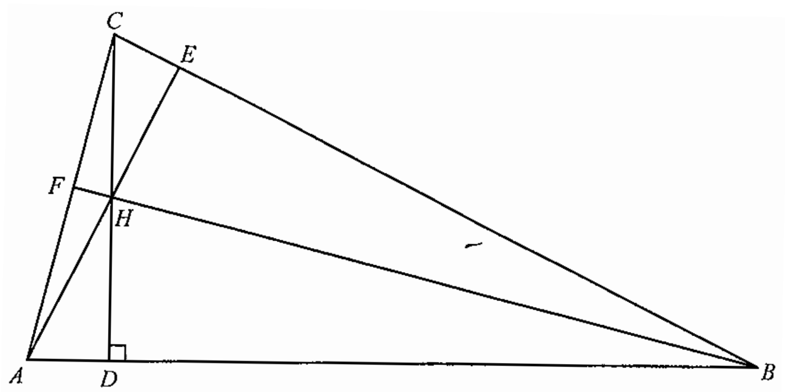


Figure 4

In Figure 4, CD is an altitude of ΔABC and H is the mid-point of CD. AH and BH are produced to meet BC and AC at E and F respectively.

Let $\mathbf{p}$, $\lambda\mathbf{p}$ ($\lambda > 1$) and $\mathbf{q}$ be $\overrightarrow{AD}$, $\overrightarrow{AB}$ and $\overrightarrow{DH}$ respectively. Let $\frac{BE}{EC} = r$.

(a) Find $\overrightarrow{AH}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(b) Express $\overrightarrow{AE}$ in terms of λ, r, $\mathbf{p}$ and $\mathbf{q}$. Hence show that $r = \lambda$.

(c) It is given that $|\mathbf{p}| = 1$, $|\mathbf{q}| = 2$ and H is the orthocentre of ΔABC.

(i) Find $\overrightarrow{AE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(ii) Find $\frac{AF}{FC}$.

(2010-CE-A MATH #12) (7 marks)

12. It is given that $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j}$, $\left|\overrightarrow{OB}\right| = 5\sqrt{2}$ and $\cos\angle AOB = \dfrac{3}{\sqrt{10}}$.

(a) Evaluate $\overrightarrow{OA} \cdot \overrightarrow{OB}$.

(b) Find $\overrightarrow{OB}$.

(2010-CE-A MATH #14) (12 marks)

14.

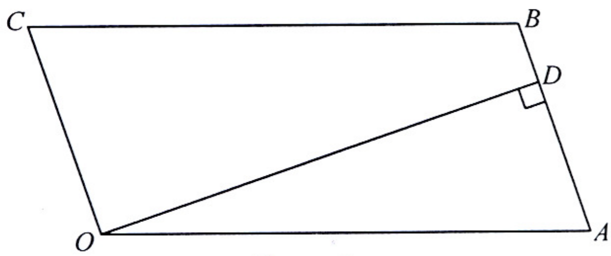


Figure 5

In Figure 5, $OABC$ is a parallelogram with $OA = 7$, $OC = 3$ and $\angle AOC = \theta$ where $\cos\theta = -\dfrac{1}{3}$. D is a point on AB such that $OD \perp AB$ and $AD : DB = 1 : r$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) By expressing $\overrightarrow{OD}$ in terms of $\mathbf{a}$, $\mathbf{c}$ and r , find the value of r .

(b) E is a point on OD produced such that C , B and E are collinear.

(i) Express $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

(ii) Are A , O , C and E concyclic? Explain your answer.

(2011-CE-A MATH #09) (6 marks)

9. It is given that $\overrightarrow{OA} = \mathbf{i} + \mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j}$.

(a) Find the value of $\cos\angle AOB$.

(b) Let $\overrightarrow{OC} = k\mathbf{i} + \mathbf{j}$. If OB is the angle bisector of ΔAOC , find the value of k .

(2011-CE-A MATH #12) (12 marks)

12.

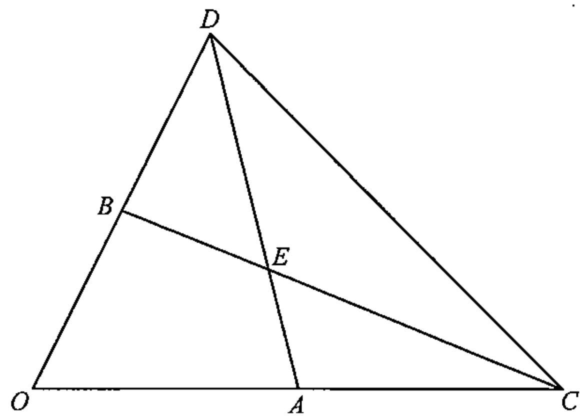


Figure 2

Figure 2 shows a triangle OCD . A and B are points on OC and OD respectively such that $OA : AC = OB : BD = 1 : h$, where $h > 0$. AD and BC intersect at E such that $AE : ED = \mu : (1-\mu)$ and $BE : EC = \lambda : (1-\lambda)$, where $0 < \mu < 1$ and $0 < \lambda < 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) By considering $\overrightarrow{OE}$, show that $\mu = \lambda$.

(b) F is a point on CD such that O , E and F are collinear. Show that OF is a median of ΔOCD .

(c) Using the above results, show that in a triangle, the centroid divides every median in $2 : 1$.

(SP-DSE-MATH-EP(M2) #09) (6 marks)

9.

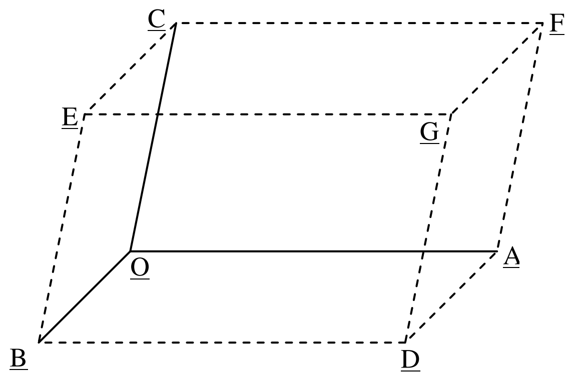


Figure 2

Let $\overrightarrow{OA} = 4\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$. Figure 2 shows the parallelepiped $OADBECFG$ formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

(a) Find the area of the parallelogram $OADB$.

(b) Find the volume of the parallelepiped $OADBECFG$.

(c) If C' is a point different from C such that the volume of the parallelepiped formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC'}$ is the same as that of $OADBECFG$, find a possible vector of $\overrightarrow{OC'}$.

(SP-DSE-MATH-EP(M2) #14) (10 marks)

14.

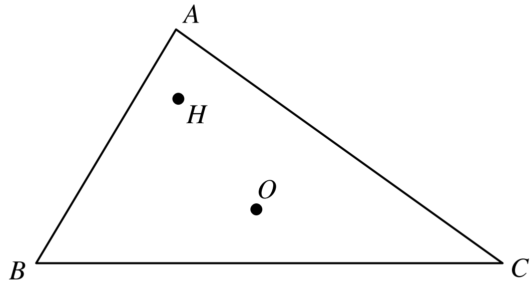


Figure 3

In Figure 3, ΔABC is an acute-angled triangle, where O and H are the circumcentre and orthocentre respectively. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OH} = \mathbf{h}$.

(a) Show that $(\mathbf{h} - \mathbf{a}) \mathbin{//} (\mathbf{b} + \mathbf{c})$.

(b) Let $\mathbf{h} - \mathbf{a} = t(\mathbf{b} + \mathbf{c})$, where t is a non-zero constant.
Show that
(i) $t(\mathbf{b} + \mathbf{c}) + \mathbf{a} - \mathbf{b} = s(\mathbf{c} + \mathbf{a})$ for some scalars,
(ii) $(t - 1)(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = 0$.

(c) Express $\mathbf{h}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(PP-DSE-MATH-EP(M2) #12) (13 marks)

12.

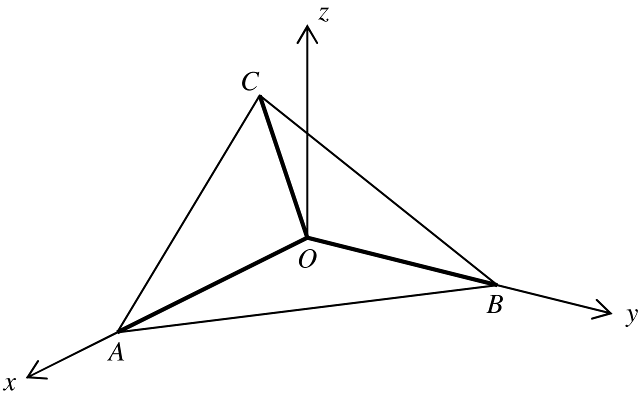


Figure 2

Let $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$ and $\overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (see Figure 2). Let M and N be points on the straight lines AB and OC respectively such that $AM : MB = a : (1-a)$ and $ON : NC = b : (1-b)$, where $0 < a < 1$ and $0 < b < 1$. Suppose that MN is perpendicular to both AB and OC .

(a) (i) Show that $\overrightarrow{MN} = (a+b-1)\mathbf{i} + (b-a)\mathbf{j} + b\mathbf{k}$.

(ii) Find the values of a and b .

(iii) Find the shortest distance between the straight lines AB and OC .

(b) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Let G be the projection of O on the plane ABC , find the coordinates of the intersecting point of the two straight lines OG and MN .

(2012-DSE-MATH-EP(M2) #07) (5 marks)

7.

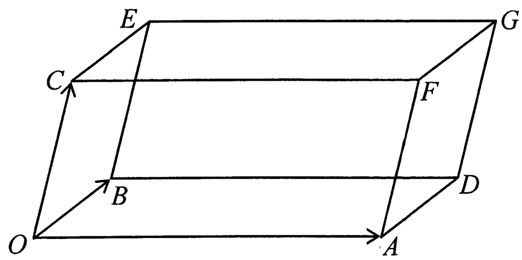


Figure 3

Figure 3 shows a parallelepiped $OADBECFG$. Let $\overrightarrow{OA} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OC} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

(a) Find the area of the parallelogram $OADB$.

(b) Find the distance between point C and the plane $OADB$.

(2012-DSE-MATH-EP(M2) #12) (12 marks)

12.

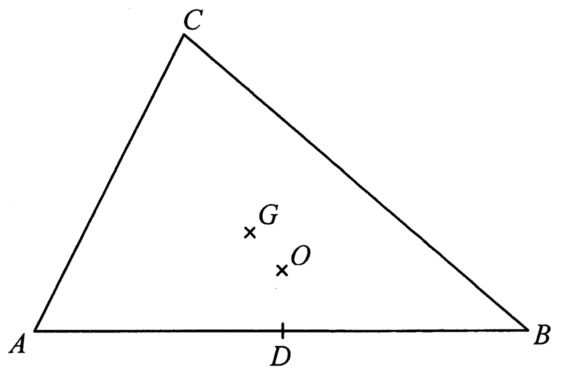


Figure 6

Figure 6 shows an acute angled scalene triangle ABC, where D is the mid-point of AB, G is the centroid and O is the circumcentre. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Express $\overrightarrow{AG}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) It is given that E is a point on AB such that CE is an altitude. Extend OG to meet CE at F.

(i) Prove that $\Delta DOG \sim \Delta CFG$.
Hence find $FG : GO$.

(ii) Show that $\overrightarrow{AF} = \mathbf{b} + \mathbf{c}$.
Hence prove that F is the orthocentre of ΔABC.

(2013-DSE-MATH-EP(M2) #10) (5 marks)

10.

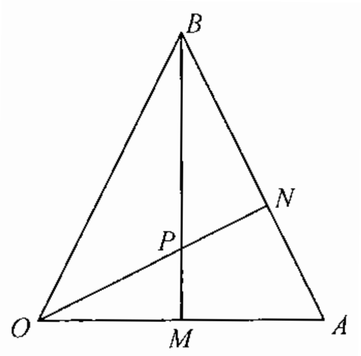


Figure 2

Let $\overrightarrow{OA} = 2\mathbf{i}$ and $\overrightarrow{OB} = \mathbf{i} + 2\mathbf{j}$. M is the mid-point of OA and N lies on AB such that $BN : NA = k : 1$. BM intersects ON at P.

(a) Express $\overrightarrow{ON}$ in terms of k.

(b) If A, N, P and M are concyclic, find the value of k.

(2013-DSE-MATH-EP(M2) #14) (12 marks)

14.

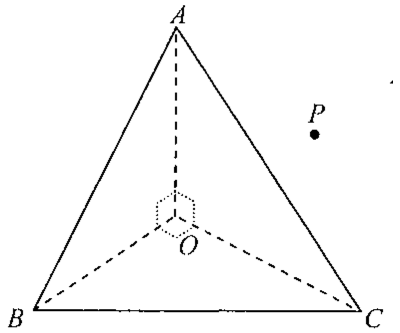


Figure 5

Figure 5 shows a fixed tetrahedron $OABC$ with $\angle AOB = \angle BOC = \angle COA = \frac{\pi}{2}$. P is a variable point such that $\overrightarrow{AP} \cdot \overrightarrow{BP} + \overrightarrow{BP} \cdot \overrightarrow{CP} + \overrightarrow{CP} \cdot \overrightarrow{AP} = 0$. Let D be a fixed point such that $\overrightarrow{OD} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OD} = \mathbf{d}$.

(a) (i) Show that $\overrightarrow{AP} \cdot \overrightarrow{BP} = \mathbf{p} \cdot \mathbf{p} - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p}$.

(ii) Using (a)(i), show that $\mathbf{p} \cdot \mathbf{p} = 2\mathbf{p} \cdot \mathbf{d}$.

(iii) Show that $\left|\mathbf{p} - \mathbf{d}\right| = \left|\mathbf{d}\right|$.

hence show that P lies on the sphere centred at D with fixed radius.

(b) (i) Alice claims that O lies on the sphere mentioned in (a)(iii). Do you agree? Explain your answer.

(ii) Suppose P_1 , P_2 and P_3 are three distinct points on the sphere in (a)(iii) such that $\overrightarrow{DP_1} \times \overrightarrow{DP_2} = \overrightarrow{DP_2} \times \overrightarrow{DP_3}$. Alice claims that the radius of the circle passing through P_1 , P_2 and P_3 is OD . Do you agree? Explain your answer.

(2014-DSE-MATH-EP(M2) #08) (8 marks)

8. Let $\overrightarrow{OP} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OR} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OQ}$.

Hence find the volume of tetrahedron $OPQR$.

(b) Find the acute angle between the plane OPQ and the line OR, correct to the nearest 0.1°.

(2014-DSE-MATH-EP(M2) #11) (13 marks)

11.

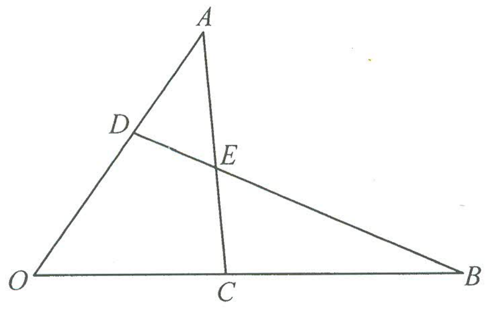


Figure 4

In Figure 4, C and D are points on OB and OA respectively such that $AD : DO = OC : CB = t : (1-t)$, where $0 < t < 1$, BD and AC intersect at E such that $AE : EC = m : 1$ and $BE : ED = n : 1$, where m and n are positive. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) By considering ΔOAC , express $\overrightarrow{OE}$ in terms of m , t , $\mathbf{a}$ and $\mathbf{b}$.

(ii) By considering ΔOBD , express $\overrightarrow{OE}$ in terms of n , t , $\mathbf{a}$ and $\mathbf{b}$.

(iii) Show that $m = \dfrac{t}{(1-t)^2}$ and $n = \dfrac{1-t}{t^2}$.

(iv) Chris claims that

"if $m = n$, then E is the centroid of ΔOAB ".

Do you agree? Explain your answer.

(b) It is given that $OA = 1$ and $OB = 2$. Francis claims that

"if AC is perpendicular to OB , then BD is always perpendicular to OA ".

Do you agree? Explain your answer.

(2015-DSE-MATH-EP(M2) #10) (12 marks)

10. OAB is a triangle. P is the mid-point of OA . Q is a point lying on AB such that $AQ : QB = 1 : 2$ while R is a point lying on OB such that $OR : RB = 3 : 1$. PR and OQ intersect at C .

(a) (i) Let t be a constant such that $PC : CR = t : (1-t)$.

By expressing $\overrightarrow{OQ}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, find the value of t .

(ii) Find $CQ : OQ$.

(b) Suppose that $\overrightarrow{OA} = 20\mathbf{i} - 6\mathbf{j} - 12\mathbf{k}$, $\overrightarrow{OB} = 16\mathbf{i} - 16\mathbf{j}$ and $\overrightarrow{OD} = \mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, where O is the origin.

Find

(i) the area of ΔOAB ,

(ii) the volume of the tetrahedron $ABCD$.

(2016-DSE-MATH-EP(M2) #12) (13 marks)

12. Let $\overrightarrow{OA} = 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OP} = \mathbf{i} + t\mathbf{j}$, where t is a constant and O is the origin. It is given that P is equidistant from A and B .

(a) Find t .

(b) Let $\overrightarrow{OC} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$. Denote the plane which contains A , B and C by Π .

(i) Find a unit vector which is perpendicular to Π .

(ii) Find the angle between CD and Π .

(iii) It is given that E is a point lying on Π such that $\overrightarrow{DE}$ is perpendicular to Π . Let F be a point such that $\overrightarrow{PF} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$. Describe the geometric relationship between D , E and F . Explain your answer.

(2017-DSE-MATH-EP(M2) #03) (5 marks)

3. P is a point lying on AB such that $AP : PB = 3 : 2$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $|\mathbf{a}| = 45$, $|\mathbf{b}| = 20$ and $\cos\angle AOB = \frac{1}{4}$. Find

(i) $\mathbf{a} \cdot \mathbf{b}$,

(ii) $\left|\overrightarrow{OP}\right|$.

(2017-DSE-MATH-EP(M2) #10) (12 marks)

10. ABC is a triangle. D is the mid-point of AC . E is a point lying on BC such that $BE : EC = 1 : r$. AB produced and DE produced meet at the point F . It is given that $DE : EF = 1 : 10$. Let $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OC} = 8\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, where O is a the origin.

(a) By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r , find r .

(b) (i) Find $\overrightarrow{AD} \cdot \overrightarrow{DE}$.

(ii) Are B , D , C and F concyclic? Explain your answer.

(c) Let $\overrightarrow{OP} = 3\mathbf{i} + 10\mathbf{j} - 4\mathbf{k}$. Denote the circumcenter of ΔBCF by Q . Find the volume of the tetrahedron $ABPQ$.

(2018-DSE-MATH-EP(M2) #12) (13 marks)

12. The position vectors of the points A , B , C and D are $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $-\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $7\mathbf{i}-\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-5\mathbf{k}$ respectively. Denote the plane which contains A , B and C by Π . Let E be the projection of D on Π .

(a) Find

(i) $\overrightarrow{AB}\times\overrightarrow{AC}$,

(ii) the volume of the tetrahedron $ABCD$.

(iii) $\overrightarrow{DE}$.

(b) Let F be a point lying on BC such that DF is perpendicular to BC .

(i) Find $\overrightarrow{DF}$.

(ii) Is $\overrightarrow{BC}$ perpendicular to $\overrightarrow{EF}$? Explain your answer.

(c) Find the angle between ΔBCD and Π .

(2019-DSE-MATH-EP(M2) #12) (13 marks)

12. Let $\overrightarrow{OA}=\mathbf{i}-4\mathbf{j}+2\mathbf{k}$, $\overrightarrow{OB}=-5\mathbf{i}-4\mathbf{j}+8\mathbf{k}$ and $\overrightarrow{OC}=-5\mathbf{i}-12\mathbf{j}+t\mathbf{k}$, where O is the origin and t is a constant. It is given that $\left|\overrightarrow{AC}\right|=\left|\overrightarrow{BC}\right|$.

(a) Find t .

(b) Find $\overrightarrow{AB}\times\overrightarrow{AC}$.

(c) Find the volume of the pyramid $OABC$.

(d) Denote the plane which contains A , B and C by Π . It is given that P , Q and R are points lying on Π such that $\overrightarrow{OP}=p\mathbf{i}$, $\overrightarrow{OQ}=q\mathbf{j}$ and $\overrightarrow{OR}=r\mathbf{k}$. Let D be the projection of O on Π .

(i) Prove that $pqr\neq 0$.

(ii) Find $\overrightarrow{OD}$.

(iii) Let E be a point such that $\overrightarrow{OE}=\frac{1}{p}\mathbf{i}+\frac{1}{q}\mathbf{j}+\frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D , E and O . Explain your answer.

(2020-DSE-MATH-EP(M2) #12) (12 marks)

12. Let $\overrightarrow{OP} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OQ} = 5\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$, where O is the origin. R is a point lying on PQ such that $PR : RQ = 1 : 3$.

(a) Find $\overrightarrow{OP} \times \overrightarrow{OR}$.

(b) Define $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OR}$. Find the area of the quadrilateral $OPSR$.

(c) Let N be a point such that $\overrightarrow{ON} = \lambda\left(\overrightarrow{OP} \times \overrightarrow{OR}\right)$, where λ is a real number.

(i) Is $\overrightarrow{NR}$ perpendicular to $\overrightarrow{PQ}$? Explain your answer.

(ii) Let μ be a real number such that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i} + \mu\mathbf{j} - 10\mathbf{k}$.

(1) Find λ and μ .

(2) Denote the angle between ΔOPQ and ΔNPQ by θ . Find $\tan\theta$.

(2021-DSE-MATH-EP(M2) #12) (13 marks)

12. The position vectors of the points A , B , C and D are $t\mathbf{i} + 14\mathbf{j} + s\mathbf{k}$, $12\mathbf{i} - s\mathbf{j} - 2\mathbf{k}$ and $(s+2)\mathbf{i} - 16\mathbf{j} + 10\mathbf{k}$ and $-t\mathbf{i} + (s+2)\mathbf{j} + 14\mathbf{k}$ respectively, where $s, t \in \mathbf{R}$. Suppose that $\overrightarrow{AB}$ is parallel to $5\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$. Denote the plane which contains A , B and C by Π .

(a) Find

(i) s and t .

(ii) the area of ΔABC ,

(iii) the volume of the tetrahedron $ABCD$,

(iv) the shortest distance from D to Π .

(b) Let E be the projection of D on Π . Is E the circumcentre of ΔABC ? Explain your answer.

ANSWERS

(1991-CE-A MATH 1 #08) (16 marks)

8. (a) $\overrightarrow{CA} = (3-x)\mathbf{i} - (y+1)\mathbf{j}$

$\overrightarrow{OB} = (x-7)\mathbf{i} + (y-1)\mathbf{j}$

$\overrightarrow{AB} = (x-1)\mathbf{i} + y\mathbf{j}$

(b) (ii) (1) $x = 4, y = 2$

(1992-CE-A MATH 1 #08) (16 marks)

8. (a) $\mathbf{a} \cdot \mathbf{a} = 4$

$\mathbf{a} \cdot \mathbf{b} = 3$

(b) $OD = 1$

$\overrightarrow{OD} = \dfrac{1}{3}\mathbf{b}$

(c) (i) $\overrightarrow{OH} = \dfrac{k}{k+1}\mathbf{a} + \dfrac{1}{3(k+1)}\mathbf{b}$

$k = 2$

(ii) (1) $\overrightarrow{OC} = \dfrac{m}{m+1}\mathbf{a} + \dfrac{1}{m+1}\mathbf{b}$

(2) $\overrightarrow{OC} = \dfrac{2(n+1)}{3}\mathbf{a} + \dfrac{(n+1)}{9}\mathbf{b}$

(3) $m = 6,\ n = \dfrac{2}{7}$

(1993-CE-A MATH 1 #06) (7 marks)

6. (a) $\overrightarrow{AB} = -2\mathbf{i} + 3\mathbf{j}$

(b) $\overrightarrow{AB} \cdot \overrightarrow{AB} = 13$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 13$

(1993-CE-A MATH 1 #08) (16 marks)

8. (a) $\overrightarrow{OP} = \dfrac{1}{1+r}\mathbf{a} + \dfrac{r}{1+r}\mathbf{b}$

$\overrightarrow{OQ} = \dfrac{1}{(1+r)^2}\mathbf{a} + \dfrac{r(r+2)}{(1+r)^2}\mathbf{b}$

(b) $\overrightarrow{OT} = \dfrac{1}{1+r}\mathbf{b}$

(c) $r = \dfrac{-1+\sqrt{5}}{2}$

(d) (i) $\mathbf{a} \cdot \mathbf{a} = 4$

$\mathbf{a} \cdot \mathbf{b} = 16$

(ii) $r = \dfrac{1}{2}$

(1994-CE-A MATH 1 #03) (6 marks)

3. (a) $\overrightarrow{PQ} = 2\mathbf{i} - \mathbf{j}$

$\left|\overrightarrow{PQ}\right| = \sqrt{5}$

(b) $\cos\angle QPR = \dfrac{-4}{\sqrt{63}}$

(1994-CE-A MATH 1 #10) (16 marks)

10. (a) $\overrightarrow{OC} = \dfrac{1}{3}\mathbf{a} + \dfrac{2}{3}\mathbf{b}$

$\overrightarrow{DA} = \mathbf{a} - \dfrac{1}{2}\mathbf{b}$

(c) $k = \dfrac{1}{5}$

(d) (i) 1

(ii) $k = \dfrac{2}{7}$

Distance $= \dfrac{\sqrt{7}}{7}$

(1995-CE-A MATH 1 #07) (8 marks)

7. (a) $\overrightarrow{OR} = \dfrac{2-6k}{k+1}\mathbf{i} + \dfrac{3+4k}{k+1}\mathbf{j}$

(b) $\overrightarrow{OP} \cdot \overrightarrow{OR} = \dfrac{13}{k+1}$

$\overrightarrow{OQ} \cdot \overrightarrow{OR} = \dfrac{52k}{k+1}$

(c) $k = \dfrac{1}{2}$

(1995-CE-A MATH 1 #08) (16 marks)

8. (a) (i) $\overrightarrow{AE} = h\mathbf{p} + h\mathbf{q}$

(ii) $\overrightarrow{AE} = \dfrac{\lambda k}{1+\lambda}\mathbf{p} + \dfrac{1}{1+\lambda}\mathbf{q}$

(b) (i) 3

(ii) (1) $\overrightarrow{DF} = k\mathbf{p} - \mathbf{q}$

$k = \dfrac{7}{12}$

(2) $\dfrac{7}{\sqrt{19}}$

(1996-CE-A MATH 1 #07) (6 marks)

7. (a) Unit vector $= \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

$\overrightarrow{OC} = \frac{64}{25}\mathbf{i} + \frac{48}{25}\mathbf{j}$

(1996-CE-A MATH 1 #10) (16 marks)

10. (a) (i) $\overrightarrow{AE} = \frac{2}{1+t}\mathbf{a} + \frac{t}{1+t}\mathbf{b}$

(ii) $\overrightarrow{AE} = \frac{9-7t}{1+t}\mathbf{a} + \frac{8t}{1+t}\mathbf{b}$

(b) (i) $\frac{9}{7}$

(ii) (1) 2

(2) $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

$\overrightarrow{AD} \cdot \overrightarrow{DE} = 0$

(1997-CE-A MATH 1 #07) (7 marks)

7. (a) $|\mathbf{a}| = 2\sqrt{5}$

(b) $\mathbf{a} \cdot \mathbf{b} = 8$

(c) $n = \frac{11}{5}, m = \frac{-2}{5}$

(1997-CE-A MATH 1 #09) (16 marks)

9. (a) (i) $\overrightarrow{AF} = \mathbf{a} + 2\mathbf{b}$

(ii) $\overrightarrow{DP} = (m+1)\mathbf{a} - \mathbf{b}$

(b) (ii) (1) $\overrightarrow{AE} = \frac{1}{r+1}\mathbf{a} + \frac{2}{r+1}\mathbf{b}$

(2) $\overrightarrow{AE} = \frac{8}{k+1}\mathbf{a} + \frac{k}{k+1}\mathbf{b}$

$r = \frac{9}{8}, k = 16$

(c) $\theta_1 = 76°$

(1998-CE-A MATH 1 #05) (6 marks)

5. (a) $\overrightarrow{AB} = 3\mathbf{i} + 5\mathbf{j}$

$\overrightarrow{AC} = -3\mathbf{i} + 8\mathbf{j}$

(b) $\overrightarrow{AB} \cdot \overrightarrow{AC} = 31$

$\angle BAC = 52°$

(1998-CE-A MATH 1 #09) (16 marks)

9. (a) (i) $\mathbf{a} \cdot \mathbf{b} = 3$

(ii) $\overrightarrow{OC} = (1-t)\mathbf{a} + t\mathbf{b}$

(iii) $\mathbf{a} \cdot \overrightarrow{OC} = 4 - t$

$\mathbf{b} \cdot \overrightarrow{OC} = 3 + 6t$

(b) (ii) $k = \frac{4-t}{4}, s = \frac{1+2t}{3}$

(c) $t = \frac{8}{11}$

(1999-CE-A MATH 1 #07) (6 marks)

7. (a) 5

(b) 10

(c) −1.6

(1999-CE-A MATH 1 #10) (16 marks)

10. (a) $\overrightarrow{OC} = \frac{7}{15}\mathbf{a} + \frac{8}{15}\mathbf{b}$

$\overrightarrow{AD} = \frac{16}{21}\mathbf{b} - \mathbf{a}$

(b) (i) $\overrightarrow{OE} = \frac{7r}{15}\mathbf{a} + \frac{8r}{15}\mathbf{b}$

(ii) $\overrightarrow{OE} = (1-k)\mathbf{a} + \frac{16k}{21}\mathbf{b}$

(c) (i) 1 : 2

(2000-CE-A MATH 1 #08) (7 marks)

8. (b) (i) $\sqrt{5} - 1$

(ii) $\angle BOD = 48°$

(2000-CE-A MATH 1 #09) (16 marks)

9. (a) (i) $\overrightarrow{OD} = \frac{\mathbf{a} + \mathbf{b}}{2}$

(iii) $\overrightarrow{EF} = \mathbf{a} + (1-k)\mathbf{b}$

(b) (i) $\mathbf{a} \cdot \mathbf{b} = 3$

$\mathbf{b} \cdot \mathbf{b} = 4$

(ii) (1) $\frac{7}{4}$

(2001-AL-P MATH 1 #04) (5 marks)

4. (a) $\overrightarrow{AB} \times \overrightarrow{AC} = bc\,\mathbf{i} + ac\,\mathbf{j} + ab\,\mathbf{k}$.

(2001-CE-A MATH #08) (6 marks)

8. (a) 6

(b) $\frac{1}{3}$

(2001-CE-A MATH #14) (12 marks)

14. (a) $\overrightarrow{OR} = \frac{s\overrightarrow{OP} + r\overrightarrow{OQ}}{r+s}$

(b) (i) $\overrightarrow{OG} = \frac{1}{8}\mathbf{a} + \frac{3}{4}\mathbf{b}$

(ii) $\overrightarrow{OY} = \frac{k}{8}\mathbf{a} + \frac{3k}{8}\mathbf{b}$

(iii) (1) $\frac{6}{7}$

(2) parallel

(2002-AL-P MATH 1 #04) (6 marks)

4. (b) $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k}), \mathbf{u} = -\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$

(2002-CE-A MATH #10) (6 marks)

10. (a) $\overrightarrow{OB} = 6\mathbf{i} + 6\mathbf{j}$

$\overrightarrow{AC} = 4\mathbf{i} - 2\mathbf{j}$

(b) $\theta = 72°$

(2002-CE-A MATH #13) (12 marks)

13. (a) 3

(b) $\overrightarrow{OE} = t\mathbf{a} + (1-t)\mathbf{b}$

(c) $\overrightarrow{BA} \cdot \overrightarrow{BF} = 1$

(2003-AL-P MATH 1 #05) (6 marks)

5. (b) (i) $|\mathbf{m} \times \mathbf{n}| = 6$

(ii) 4

(2003-CE-A MATH #06) (5 marks)

6. (a) $\overrightarrow{OP} = \frac{\mathbf{a}+3\mathbf{b}}{4}$

(b) $\overrightarrow{OC} = \frac{1}{3}\mathbf{a} + \mathbf{b}$

(2003-CE-A MATH #14) (12 marks)

14. (a) $\frac{-3}{2}$

(b) $k = \frac{35}{9}$

(2004-CE-A MATH #06) (5 marks)

6. (a) $\overrightarrow{OC} = \frac{2\mathbf{a}+\mathbf{b}}{3}$

(b) $\left|\overrightarrow{OC}\right| = \frac{2}{3}$

(2004-CE-A MATH #13) (12 marks)

13. (a) (i) $\overrightarrow{OC} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$

$\overrightarrow{OA} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

(c) $\theta = 24°$

(2005-AL-P MATH 1 #12) (15 marks)

12. (b) $\alpha = 5\sqrt{3}, \beta = -2\sqrt{2}, \gamma = 3\sqrt{6}$

(2005-CE-A MATH #11) (6 marks)

11. (a) $\frac{-15}{2}$

(b) $\sqrt{19}$

(2005-CE-A MATH #14) (12 marks)

14. (a) $\overrightarrow{CB} = \frac{1}{8}\mathbf{a}$

(b) (i) $DA = 1$

$\mathbf{d} = \mathbf{a} - \mathbf{b}$

(ii) $\overrightarrow{OP} = \frac{8r-1}{8(r+1)}\mathbf{a} + \frac{1-r}{1+r}\mathbf{b}$

(iii) 9 : 16

(2006-CE-A MATH #07) (5 marks)

7. (a) -3

(b) $\sqrt{7}$

(2006-CE-A MATH #18) (12 marks)

18. (a) $\overrightarrow{OG} = \dfrac{\mathbf{a}+\mathbf{b}}{3}$

(b) $\overrightarrow{OT}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} = 0$

(c) $\overrightarrow{OC}\cdot\mathbf{b} = \dfrac{|\mathbf{b}|^2}{2}$

(d) (i) $(\overrightarrow{GT} - 2\overrightarrow{CG})\cdot\mathbf{a} = 0$

$(\overrightarrow{GT} - 2\overrightarrow{CG})\cdot\mathbf{b} = 0$

(2007-CE-A MATH #08) (5 marks)

8. (a) $\overrightarrow{BC} = (6k-2)\mathbf{i} + (3k-6)\mathbf{j}$

(b) $\dfrac{2}{3}$

(2007-CE-A MATH #17) (12 marks)

17. (a) $\overrightarrow{OM} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$

(b) (i) $\overrightarrow{OG} = \dfrac{2\mathbf{a}+4k\mathbf{b}}{7}$

(c) (i) $\mathbf{a}\cdot\mathbf{b} = \dfrac{1}{2}$

$\left|\overrightarrow{PQ}\right| = \dfrac{\sqrt{13}}{6}$

(ii) $\angle QGM = 104°$

(2008-CE-A MATH #07) (5 marks)

7. $\dfrac{3\mathbf{i}+4\mathbf{j}}{5}$

(2008-CE-A MATH #15) (12 marks)

15. (a) $\dfrac{1}{2}$

(c) $\overrightarrow{HM} = \dfrac{2t}{3}\mathbf{p} + \dfrac{5t}{3}\mathbf{q}$

$\overrightarrow{HN} = \dfrac{2t}{3}\mathbf{p} + \left(\dfrac{5t}{3} - \dfrac{3}{2}\right)\mathbf{q}$

$\overrightarrow{OH} = \dfrac{5}{3}\mathbf{p} + \dfrac{2}{3}\mathbf{q}$

(2009-CE-A MATH #07) (4 marks)

7. (a) $\angle AOB = 60°$

(b) $|\mathbf{c}| = 3$

(2009-CE-A MATH #14) (12 marks)

14. (a) $\overrightarrow{AH} = \mathbf{p} + \mathbf{q}$

(b) $\overrightarrow{AE} = \dfrac{(r+\lambda)\mathbf{p} + 2r\mathbf{q}}{r+1}$

(c) (i) $\dfrac{9}{5}(\mathbf{p}+\mathbf{q})$

(ii) $\dfrac{AF}{FC} = \dfrac{9}{8}$

(2010-CE-A MATH #12) (7 marks)

12. (a) 15

(b) $\overrightarrow{OB} = 5\mathbf{i} + 5\mathbf{j}$ or $\mathbf{i} + 7\mathbf{j}$

(2010-CE-A MATH #14) (12 marks)

14. (a) $\overrightarrow{OD} = \mathbf{a} + \dfrac{1}{1+r}\mathbf{c}$

$r = \dfrac{2}{7}$

(b) (i) $\overrightarrow{OE} = \dfrac{9}{7}\mathbf{a} + \mathbf{c}$

(2011-CE-A MATH #09) (6 marks)

9. (a) $\cos\angle AOB = \dfrac{3}{\sqrt{10}}$

(b) 7

(2011-CE-A MATH #12) (12 marks)

12. (a) $\overrightarrow{OE} = (1+\mu)\mathbf{a} + \mu(1+h)\mathbf{b}$

$\overrightarrow{OE} = (1-\lambda)\mathbf{b} + \lambda(1+h)\mathbf{a}$

(SP-DSE-MATH-EP(M2) #09) (6 marks)

9. (a) 13

(b) 65

(c) $\overrightarrow{OC'} = (3+4s)\mathbf{i} + (1+3s+3t)\mathbf{j} + (5+t)\mathbf{k}$

where s and t are not both zero.

(SP-DSE-MATH-EP(M2) #14) (10 marks)

14. (c) $\mathbf{h} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

(PP-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) (ii) $a = \frac{1}{2}, b = \frac{1}{3}$

(iii) $\frac{\sqrt{6}}{6}$

(b) (i) $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

(ii) $P = (1, 1, -1)$

(2012-DSE-MATH-EP(M2) #07) (5 marks)

7. (a) 3

(b) $\frac{11}{3}$

(2012-DSE-MATH-EP(M2) #12) (12 marks)

12. (a) $\overrightarrow{AG} = \frac{-2\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$

(b) (i) $FG : GO = 2 : 1$

(2013-DSE-MATH-EP(M2) #10) (5 marks)

10. (a) $\overrightarrow{ON} = \frac{2(k+1)\mathbf{i} + 2\mathbf{j}}{k+1}$

(b) $k = \frac{3}{2}$

(2013-DSE-MATH-EP(M2) #14) (12 marks)

(2014-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $\overrightarrow{OP} \times \overrightarrow{OQ} = 6\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Volume = 1

(b) 6.8°

(2014-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) (i) $\overrightarrow{OE} = \frac{\mathbf{a} + mt\mathbf{b}}{1+n}$

(ii) $\overrightarrow{OE} = \frac{n(1-t)\mathbf{a} + \mathbf{b}}{1+n}$

(iv) $t = \frac{1}{2}$

(b) $\overrightarrow{BD} \cdot \overrightarrow{OA} \neq 0$

(2015-DSE-MATH-EP(M2) #10) (12 marks)

10. (a) (i) $\overrightarrow{OQ} = \frac{1-t}{2}\overrightarrow{OA} + \frac{3t}{4}\overrightarrow{OB}$

(ii) 7 : 16

(b) (i) 176

(ii) 42

(2016-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) -1

(b) (i) $\frac{-1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

(ii) $\sin^{-1}\left(\frac{3\sqrt{11}}{11}\right)$

(iii) D is the mid-point of the line segment joining E and F.

(2017-DSE-MATH-EP(M2) #03) (5 marks)

3. (a) $\overrightarrow{OP} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$

(b) (i) 225

(ii) 24

(2017-DSE-MATH-EP(M2) #10) (12 marks)

10. (a) $\overrightarrow{AE} = \frac{2r+6}{r+1}\mathbf{i} + \frac{r-6}{r+1}\mathbf{j} + \frac{r}{r+1}\mathbf{k}$

$\overrightarrow{AF} = \frac{-8r+36}{r+1}\mathbf{i} + \frac{41r-36}{r+1}\mathbf{j} + \frac{11r}{r+1}\mathbf{k}$

$r = \frac{6}{5}$

(b) (i) 0

(c) 7

(2018-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) (i) $32\mathbf{i} + 8\mathbf{j} - 28\mathbf{k}$

(ii) 24

(iii) $-\frac{32}{13}\mathbf{i} - \frac{8}{13}\mathbf{j} + \frac{28}{13}\mathbf{k}$

(b) (i) $-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

(ii) $\overrightarrow{BC} \cdot \overrightarrow{EF} = 0$

(c) $\cos^{-1}\left(\frac{3\sqrt{13}}{13}\right)$

(2019-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) 2

(b) $48\mathbf{i} - 36\mathbf{j} + 48\mathbf{k}$

(c) 48 cubic units

(d) (i) $p = 6$, $q = -8$, $r = 6$

(ii) $\frac{24}{41}(4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$

(iii) D , E and O are collinear

(2020-DSE-MATH-EP(M2) #12) (12 marks)

12. (a) $6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

(b) 9

(c) (i) ... Yes

(ii) (1) $\lambda = \frac{2}{9}$, $\mu = -25$

(2) $\frac{2}{3}$

(2021-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) (i) $s = 10$, $t = 18$

(ii) 270

(iii) 2 160,

(iv) 24

(b) E is not the circumcentre of ΔABC