3. ALGEBRA AREA

1. Matrices and Determinants

(1991-AL-P MATH 1 #01) (4 marks)

1. Factorize the determinant $\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

(1992-AL-P MATH 1 #03) (7 marks)

3. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 1 & \lambda \end{pmatrix}$.

   If B^{-1} exists and $B^{-1}AB = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, find λ , a and b .

   Hence find A^{100} .

(1993-AL-P MATH 1 #06) (7 marks)

6. (a) Show that if A is a 3×3 matrix such that $A^T = -A$, then $\det A = 0$.

   (b) Given that

$$B = \begin{pmatrix} 1 & -2 & 74 \\ 2 & 1 & -67 \\ -74 & 67 & 1 \end{pmatrix} ,$$

   use (a), or otherwise, to show $\det(I - B) = 0$.

   Hence deduce that $\det(I - B^4) = 0$.

(1994-AL-P MATH 1 #01) (6 marks)

1. Let $A = \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$.

   Find $P^{-1}AP$.

   Find A^n , where n is a positive integer.

(1995-AL-P MATH 1 #01) (6 marks)

1. (a) Let $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$ where a , $b \in \mathbf{R}$ and $a \neq b$.

   Prove that $A^n = \begin{pmatrix} a^n & \dfrac{a^n - b^n}{a - b} \\ 0 & b^n \end{pmatrix}$ for all positive integers n .

   (b) Hence, or otherwise, evaluate $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95}$.

(1996-AL-P MATH 1 #01) (6 marks)

1. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

Evaluate $A^3 - 5A^2 + 8A - 4I$.

Hence, or otherwise, find A^{-1} .

(1997-AL-P MATH 1 #07) (7 marks)

7. (a) Let A be a 3×3 non-singular matrix. Show that

$$\det\left(A^{-1} - xI\right) = -\frac{x^3}{\det A}\det\left(A - x^{-1}I\right) .$$

(b) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$.

Show that 4 is a root of $\det(A - xI) = 0$ and hence find the other roots in surd form.

Solve $\det\left(A^{-1} - xI\right) = 0$.

(1998-AL-P MATH 1 #09) (15 marks)

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a , b , c , $d \in \mathbf{R}$, $a \neq 0$ and $\det A = 0$.

(a) Show that $A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$ for some $k \in \mathbf{R}$.

(b) Find P in the form of $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ such that $PA = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ for some α , $\beta \in \mathbf{R}$.

If $a + d \neq 0$, find Q in the form of $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ such that $PQP^{-1}Q = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ for some $\gamma \in \mathbf{R}$.

(c) Find S such that $S\begin{pmatrix} 3 & 7 \\ 6 & 14 \end{pmatrix}S^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ for some $\lambda \in \mathbf{R}$.

Hence, or otherwise, evaluate $\begin{pmatrix} 3 & 7 \\ 6 & 14 \end{pmatrix}^n$ where n is a positive integer.

(1999-AL-P MATH 1 #09) (15 marks)

9. (a) Let A and B be two square matrices of the same order. If $AB = BA = 0$, show that

$(A + B)^n = A^n + B^n$ for any positive integer n .

(b) Let $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where a , b are not both zero. If $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, show that $AB = BA = 0$ if and only if $p = r = 0$ and $aq + bs = 0$.

(c) Let $C = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where x , z are non-zero and distinct. Find non-zero matrices D and E such that $C = D + E$ and $DE = ED = 0$.

(d) Evaluate $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{99}$.

(2000-AL-P MATH 1 #01) (5 marks)

1. Let $M = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & b & a \\ \mu & c & -b \end{pmatrix}$ where $b^2 + ac = 1$. Show by induction that

$$M^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ n\left[\lambda(1+b) + \mu a\right] & 1 & 0 \\ n\left[\lambda c + \mu(1-b)\right] & 0 & 1 \end{pmatrix}$$ for all positive integers n .

Hence or otherwise, evaluate $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 2 \\ 1 & -4 & -3 \end{pmatrix}^{2000}$.

(2002-AL-P MATH 1 #12) (15 marks)

12. (a) Let A be a 3×3 matrix such that

$$A^3 + A^2 + A + I = 0 ,$$

where I is the 3×3 identity matrix.

(i) Prove that A has an inverse, and find A^{-1} in terms of A .

(ii) Prove that $A^4 = I$.

(iii) Prove that $\left(A^{-1}\right)^3 + \left(A^{-1}\right)^2 + A^{-1} + I = 0$.

(iv) Find a 3×3 invertible matrix B such that $B^3 + B^2 + B + I \neq 0$.

(b) Let $X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.

(i) Using (a)(i) or otherwise, find X^{-1} .

(ii) Let n be a positive integer. Find X^n .

(iii) Find two 3×3 matrices Y and Z , other than X , such that $Y^3 + Y^2 + Y + I = 0$, $Z^3 + Z^2 + Z + I = 0$.

(2003-AL-P MATH 1 #08) (15 marks)

8. (a) If $\det\begin{pmatrix} -2-\alpha & \sqrt{3} \\ \sqrt{3} & -\alpha \end{pmatrix} = 0$, find the two values of α .

(b) Let α_1 and α_2 be the values obtained in (a), where $\alpha_1 < \alpha_2$. Find θ_1 and θ_2 such that

$$\begin{pmatrix} -2-\alpha_1 & \sqrt{3} \\ \sqrt{3} & -\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \ 0 \le \theta_1 < \pi ,$$

$$\begin{pmatrix} -2-\alpha_2 & \sqrt{3} \\ \sqrt{3} & -\alpha_2 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \ 0 \le \theta_2 < \pi .$$

Let $P = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}$. Evaluate P^n , where n is a positive integer.

Prove that $P^{-1}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P$ is a matrix of the form $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

(c) Evaluate $\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}^n$, where n is a positive integer.

(2004-AL-P MATH 1 #08) (15 marks)

8. Let $A = \begin{pmatrix} \alpha - k & \alpha - \beta - k \\ k & \beta + k \end{pmatrix}$, where α , β , $k \in \mathbf{R}$ with $\alpha \ne \beta$.

Define $X = \frac{1}{\alpha - \beta}(A - \beta I)$ and $Y = \frac{1}{\beta - \alpha}(A - \alpha I)$, where I is the 2×2 identity matrix.

(a) Evaluate XY , YX , $X + Y$, X^2 and Y^2 .

(b) Prove that $A^n = \alpha^n X + \beta^n Y$ for all positive integers n .

(c) Evaluate $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}^{2004}$.

(d) If α and β are non-zero real numbers, guess an expression for A^{-1} in terms of α , β , X and Y , and verify it.

(2007-AL-P MATH 1 #05) (6 marks)

5. Let P be a non-singular 2×2 real matrix and $Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where α and β are two distinct real numbers.

Define $M = P^{-1}QP$ and denote the 2×2 identity matrix by I .

(a) Find real numbers λ and μ , in terms of α and β , such that $M^2 = \lambda M + \mu I$.

(b) Prove that $\det\left(M^2 + \alpha\beta I\right) = \alpha\beta(\alpha + \beta)^2$.

(2011-AL-P MATH 1 #08) (15 marks)

8. (a) Let $A = \begin{pmatrix} 4-b & a \\ b & 4-a \end{pmatrix}$ be a real matrix and $P = \begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}$, where $ab > 0$.

(i) Prove that P is a non-singular matrix.

(ii) Evaluate $P^{-1}AP$.

(iii) For any positive integer n , find d_1 and d_2 such that $A^n = P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}$.

(b) Let $B = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$. For any positive integer n , find $B + B^3 + B^5 + \ldots + B^{2n-1}$.

(SP-DSE-MATH-EP(M2) #10) (8 marks)

10. Let $0^\circ < \theta < 180^\circ$ and define $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Prove, by mathematical induction, that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

for all positive integers n .

(b) Solve $\sin 3\theta + \sin 2\theta + \sin\theta = 0$.

(c) It is given that $A^3 + A^2 + A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Find the value(s) of a .

(SP-DSE-MATH-EP(M2) #11) (12 marks)

11. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Let I and O be the 3×3 identity matrix and zero matrix respectively.

(i) Prove that $P^3 - 2P^2 - P + I = \mathbf{0}$.

(ii) Using the result of (i), or otherwise, find P^{-1} .

(b) (i) Prove that $D = P^{-1}AP$.

(ii) Prove that D and A are non-singular.

(iii) Find $\left(D^{-1}\right)^{100}$.

Hence, or otherwise, find $\left(A^{-1}\right)^{100}$.

(PP-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) It is given that $\cos(x+1) + \cos(x-1) = k\cos x$ for any real x . Find the value of k .

(b) Without using a calculator, find the value of $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$.

(PP-DSE-MATH-EP(M2) #11) (14 marks)

11. Let $A = \begin{pmatrix} \alpha + \beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}$ where α and β are distinct real numbers. Let I be the 2×2 identity matrix.

(a) Show that $A^2 = (\alpha + \beta)A - \alpha\beta I$.

(b) Using (a), or otherwise, show that $(A - \alpha I)^2 = (\beta - \alpha)(A - \alpha I)$ and $(A - \beta I)^2 = (\alpha - \beta)(A - \beta I)$.

(c) Let $X = s(A - \alpha I)$ and $Y = t(A - \beta I)$ where s and t are real numbers.

Suppose $A = X + Y$.

(i) Find s and t in terms of α and β .

(ii) For any positive integer n , prove that

$$X^n = \frac{\beta^n}{\beta - \alpha}(A - \alpha I) \text{ and } Y^n = \frac{\alpha^n}{\alpha - \beta}(A - \beta I) .$$

(iii) For any positive integer n , express A^n in the form of $pA + qI$, where p and q are real numbers.

(Note: It is known that for any 2×2 matrices H and K ,

if $HK = KH = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $(H + K)^n = H^n + K^n$ for any positive integer n.)

(2012-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) Solve the equation

$$\begin{vmatrix} 1 - x & 4 \\ 2 & 3 - x \end{vmatrix} = 0 \text{ .... (*)}$$

(b) Let x_1 , x_2 ($x_1 < x_2$) be the roots of (*). Let $P = \begin{pmatrix} a & c \\ b & 1 \end{pmatrix}$. It is given that

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = x_1\begin{pmatrix} a \\ b \end{pmatrix} , \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = x_2\begin{pmatrix} c \\ 1 \end{pmatrix} \text{ and } |P| = 1 .$$

where a , b and c are constants.

(i) Find P .

(ii) Evaluate $P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P$.

(iii) Using (b)(ii), evaluate $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}^{12}$.

(2013-AL-P MATH 1 #11) (15 marks)

11. (a) Define $A = \begin{pmatrix} a & -2 \\ -2 & a+3 \end{pmatrix}$, where $a \in \mathbf{R}$.

Let λ , $\mu \in \mathbf{R}$ and $b > 0$ such that $A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $A\begin{pmatrix} b \\ 1 \end{pmatrix} = \mu\begin{pmatrix} b \\ 1 \end{pmatrix}$.

(i) Express λ in terms of a .

(ii) Prove that $b = 2$ and express μ in terms of a .

(iii) Define $M = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Denote the transpose of M by M^T .

(1) Evaluate $M^T M$.

(2) Using mathematical induction, prove that $A^n = \frac{1}{5}MD^nM^T$ for any $n \in \mathbf{N}$,

where $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

(b) Let x , $y \in \mathbf{R}$.

(i) Prove that if $(x \quad y)\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}^{2013}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, then $x = y = 0$.

(ii) Someone claims that if $(x \quad y)\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}^{2013}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, then $x = y = 0$. Do you agree? Explain your answer.

(2013-DSE-MATH-EP(M2) #08) (5 marks)

8. Let M be the matrix $\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}$, where $k \neq 0$.

(a) Find M^{-1} .

(b) If $M\begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, find the value of k .

(2013-DSE-MATH-EP(M2) #13) (13 marks)

13. For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define $\operatorname{tr}(M) = a + d$.

Let A and B be 2×2 matrices such that $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) (i) For any matrix $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $\operatorname{tr}(MN) = \operatorname{tr}(NM)$.

(ii) Show that $\operatorname{tr}(A) = 4$.

(iii) Find the value of $|A|$.

(b) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$ and $C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and distinct scalars λ_1 and λ_2 .

(i) Prove that $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$.

(ii) Prove that λ_1 and λ_2 are the roots of the equation $\lambda^2 - \operatorname{tr}(C)\cdot\lambda + |C| = 0$.

(c) Find the two values of λ such that $A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$.

(2014-DSE-MATH-EP(M2) #07) (7 marks)

7. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Prove, by mathematical induction, that for all positive integer n , $A^{n+1} = 2^n A$.

(b) Using the result of (a), Willy proceeds in the following way:

$A^2 = 2A$
$A^2A^{-1} = 2AA^{-1}$
$A = 2I$

Explain why Willy arrives at a wrong conclusion.

(2014-DSE-MATH-EP(M2) #12) (11 marks)

12. Let $M = \begin{pmatrix} k-1 & k \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$, where k and p are real numbers and $p \neq -1$.

(a) (i) Find A^{-1} in terms of p .

(ii) Show that $A^{-1}MA = \begin{pmatrix} -1 & k-p \\ 0 & k \end{pmatrix}$.

(iii) Suppose $p = k$. Using (ii), find M^n in terms of k and n , where n is a positive integer.

(b) A sequence is defined by

$x_1 = 0$, $x_2 = 1$ and $x_n = x_{n-1} + 2x_{n-2}$ for $n = 3$, 4 , 5 ...

It is known that this sequence can be expressed in the matrix form $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$.

Using the result of (a)(iii), express x_n in terms of n .

(2015-DSE-MATH-EP(M2) #06) (6 marks)

6. (a) Let M be a 3×3 real matrix such that $M^T = -M$, where M^T is the transpose of M . Prove that $|M| = 0$.

(b) Let $A = \begin{pmatrix} -1 & a & b \\ -a & -1 & -8 \\ -b & 8 & -1 \end{pmatrix}$, where a and b are real numbers. Denote that 3×3 identity matrix by I .

(i) Using (a), or otherwise, prove that $|A + I| = 0$.

(ii) Someone claims that $A^3 + I$ is a singular matrix. Do you agree: Explain your answer.

(2015-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) Let λ and μ be real numbers such that $\mu - \lambda \neq 2$. Denote the 2×2 identity matrix by I .

Define $A = \dfrac{1}{\lambda - \mu + 2}(I - \mu I + M)$ and $B = \dfrac{1}{\lambda - \mu + 2}(I + \lambda I - M)$, where $M = \begin{pmatrix} \lambda & 1 \\ \lambda - \mu + 1 & \mu \end{pmatrix}$.

(i) Evaluate AB , BA and $A + B$.

(ii) Prove that $A^2 = A$ and $B^2 = B$.

(iii) Prove that $M^n = (\lambda + 1)^n A + (\mu - 1)^n B$ for all positive integers n .

(b) Using (a), or otherwise, evaluate $\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315}$.

(2016-DSE-MATH-EP(M2) #08) (8 marks)

8. Let n be a positive integer.

(a) Define $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Evaluate

(i) A^2 ,

(ii) A^n ,

(iii) $\left(A^{-1}\right)^n$.

(b) Evaluate

(i) $\sum_{k=0}^{n-1} 2^k$,

(ii) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n$.

(2017-DSE-MATH-EP(M2) #12) (12 marks)

12. Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Denote the 2×2 identity matrix by I .

(a) Using mathematical induction, prove that $A^n = 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all positive integers n .

(b) Let $B = \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}$.

(i) Define $P = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n = 3^n I + 3^{n-1} n \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ for any positive integer n .

(iii) Does there exist a positive integer m such that $\left|A^m - B^m\right| = 4m^2$? Explain your answer.

(2018-DSE-MATH-EP(M2) #7) (8 marks)

7. Define $M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$. Let $X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$ be a non-zero real matrix such that $MX = XM$.

(a) Express b and c in terms of a .

(b) Prove that X is a non-singular matrix.

(c) Denote the transpose of X be X^T. Express $\left(X^T\right)^{-1}$ in terms of a .

(2019-DSE-MATH-EP(M2) #02) (5 marks)

2. Let $P(x)=\begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbf{R}$. It is given that the coefficient of x^3 in the expansion of $P(x)$ is 160 . Find

(a) λ ,

(b) $P'(0)$.

(2019-DSE-MATH-EP(M2) #11) (12 marks)

11. Let $M=\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$. Denote the 2×2 identity matrix by I .

(a) Find a pair of real numbers a and b such that $M^2=aM+bI$.

(b) Prove that $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$ for all positive integers n .

(c) Does there exist a pair of 2×2 real matrices A and B such that $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$ for all positive integers n ? Explain your answer.

(2020-DSE-MATH-EP(M2) #08) (8 marks)

8. Define $P=\begin{pmatrix} -5 & -2 \\ 15 & 6 \end{pmatrix}$ and $Q=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $M=\begin{pmatrix} 1 & a \\ b & c \end{pmatrix}$ such that $|M|=1$ and $PM=MQ$, where a , b and c are real numbers.

(a) Find a , b and c .

(b) Define $R=\begin{pmatrix} 6 & 2 \\ -15 & -5 \end{pmatrix}$.

(i) Evaluate $M^{-1}RM$.

(ii) Using the result of (b)(i), prove that $(\alpha P+\beta R)^{99}=\alpha^{99}P+\beta^{99}R$ for any real numbers α and β .

(2021-DSE-MATH-EP(M2) #11) (12 marks)

11. Define $P = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$, where $\frac{\pi}{2} < \theta < \pi$.

(a) Let $A = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbf{R}$.

Prove that $PAP^{-1} = \begin{pmatrix} -\alpha\cos 2\theta + \beta\sin 2\theta & -\beta\cos 2\theta - \alpha\sin 2\theta \\ -\beta\cos 2\theta - \alpha\sin 2\theta & \alpha\cos 2\theta - \beta\sin 2\theta \end{pmatrix}$.

(b) Let $B = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$.

(i) Find θ such that $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda, \mu \in \mathbf{R}$.

(ii) Using the result of (b)(i), prove that $B^n = 2^{n-2}\begin{pmatrix} (-1)^n + 3 & \sqrt{3}(-1)^{n+1} + \sqrt{3} \\ \sqrt{3}(-1)^{n+1} + \sqrt{3} & 3(-1)^n + 1 \end{pmatrix}$ for any positive integer n .

(iii) Evaluate $\left(B^{-1}\right)^{555}$.

ANSWERS

(1991-AL-P MATH 1 #01) (4 marks)

1. $(a-c)(b-c)(a-b)(a+b+c)$

(1992-AL-P MATH 1 #03) (7 marks)

3. λ can be any non-zero number.

$a=1$, $b=3$

$$A^{100}=\begin{pmatrix}1 & 0\\ \dfrac{3^{100}-1}{2} & 3^{100}\end{pmatrix}$$

(1993-AL-P MATH 1 #06) (7 marks)

(1994-AL-P MATH 1 #01) (6 marks)

1. $P^{-1}AP=\begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}$

$$A^n=\frac{1}{6}\begin{pmatrix}2\cdot 7^n+4 & 8\cdot 7^n-8\\ 7^n-1 & 4\cdot 7^n+2\end{pmatrix}$$

(1995-AL-P MATH 1 #01) (6 marks)

1. (b) $\begin{pmatrix}1 & 3^{95}-1\\ 0 & 3^{95}\end{pmatrix}$

(1996-AL-P MATH 1 #01) (6 marks)

1. $A^3-5A^2+8A-4I=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

$$A^{-1}=\frac{1}{4}\begin{pmatrix}6 & 0 & 4\\ -2 & 2 & -2\\ -2 & 0 & 0\end{pmatrix}$$

(1997-AL-P MATH 1 #07) (7 marks)

7. (b) The other roots $=2\pm\sqrt{3}$

$x=\dfrac{1}{4}$ or $2\pm\sqrt{3}$

(1998-AL-P MATH 1 #09) (15 marks)

9. (b) $P=\begin{pmatrix}1 & 0\\ -k & 1\end{pmatrix}$, $Q=\begin{pmatrix}1 & \dfrac{-b}{a+kb}\\ 0 & 1\end{pmatrix}$

(c) $S=\begin{pmatrix}\dfrac{3}{17} & \dfrac{7}{17}\\ -2 & 1\end{pmatrix}$

$$A^n=\begin{pmatrix}3\cdot 17^{n-1} & 7\cdot 17^{n-1}\\ 6\cdot 17^{n-1} & 14\cdot 17^{n-1}\end{pmatrix}$$

(1999-AL-P MATH 1 #09) (15 marks)

9. (c) $D=\begin{pmatrix}x & \dfrac{xy}{x-z}\\ 0 & 0\end{pmatrix}$, $E=\begin{pmatrix}0 & \dfrac{-yz}{x-z}\\ 0 & z\end{pmatrix}$

(d) $\begin{pmatrix}2^{99} & 5\left(2^{99}-1\right)\\ 0 & 1\end{pmatrix}$

(2000-AL-P MATH 1 #01) (5 marks)

1. $\begin{pmatrix}1 & 0 & 0\\ -6000 & 1 & 0\\ 6000 & 0 & 1\end{pmatrix}$

(2002-AL-P MATH 1 #12) (15 marks)

12. (a) (i) $A^{-1}=-\left(A^2+A+I\right)$

(iv) I

(b) (i) $\begin{pmatrix}-1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$

(ii) $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 1 & 1\\ -1 & -1 & 0\\ -1 & 0 & -1\end{pmatrix}$,

$\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}$,

$\begin{pmatrix}-1 & -1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$

(iii) $-I$ or X^3

(2003-AL-P MATH 1 #08) (15 marks)

8. (a) $\alpha = 1$ or $\alpha = -3$

(b) $P^n = \begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ if n is odd.

$P^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if n is even.

$P^{-1}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$

(c)

$$\begin{pmatrix} \frac{1+(-1)^n 3^{n+1}}{4} & \frac{3^{\frac{1}{2}}\left(1+(-1)^{n+1}3^n\right)}{4} \\ \frac{3^{\frac{1}{2}}\left(1+(-1)^{n+1}3^n\right)}{4} & \frac{3\left(1+(-1)^n 3^{n-1}\right)}{4} \end{pmatrix}$$

(2004-AL-P MATH 1 #08) (15 marks)

8. (a) $XY = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $YX = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$X + Y = I$, $X^2 = X$, $Y^2 = Y$

(c) $\begin{pmatrix} \frac{2\left(7^{2004}\right)+1}{3} & \frac{2\left(7^{2004}\right)-2}{3} \\ \frac{7^{2004}-1}{3} & \frac{7^{2004}+2}{3} \end{pmatrix}$

(d) $A^{-1} = \frac{1}{\alpha}X + \frac{1}{\beta}Y$

(2007-AL-P MATH 1 #05) (6 marks)

5. (a) $\lambda = \alpha + \beta$, $\mu = -\alpha\beta$

(2011-AL-P MATH 1 #08) (15 marks)

8. (a) (ii) $\begin{pmatrix} 4 & 0 \\ 0 & 4-a-b \end{pmatrix}$

(iii) $d_1 = 4^n$, $d_2 = (4-a-b)^n$

(b)

$$\frac{1}{75}\begin{pmatrix} 4^{2n+2} - 15n - 16 & 4^{2n+2} + 60n - 16 \\ 4^{2n+1} + 15n - 4 & 4^{2n+1} - 60n - 4 \end{pmatrix}$$

(SP-DSE-MATH-EP(M2) #10) (8 marks)

10. (b) $\frac{\pi}{2}$, $\frac{2\pi}{3}$

(c) $a = -1$ or 0

(SP-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) (ii) $P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) (iii) $\left(D^{-1}\right)^{100} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2^{100}} \end{pmatrix}$

$$\left(A^{-1}\right)^{100} = \begin{pmatrix} \frac{1}{2^{100}} & 0 & 0 \\ \frac{1}{2^{100}}-1 & 1 & 0 \\ \frac{1}{2^{100}}-1 & 0 & 1 \end{pmatrix}$$

(PP-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) $k = 2\cos 1$

(b) 0

(PP-DSE-MATH-EP(M2) #11) (14 marks)

11. (c) (i) $s = \frac{\beta}{\beta - \alpha}$, $t = \frac{\alpha}{\alpha - \beta}$

(iii) $A^n = \frac{\alpha^n - \beta^n}{\alpha - \beta}A + \frac{\alpha\beta^n - \alpha^n\beta}{\alpha - \beta}I$

(2012-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) $x = -1$ or 5

(b) (i) $\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

(iii) $\begin{pmatrix} \frac{5^{12}+2}{3} & \frac{2\cdot 5^{12}-2}{3} \\ \frac{5^{12}-1}{3} & \frac{2\cdot 5^{12}+1}{3} \end{pmatrix}$

(2013-AL-P MATH 1 #11) (15 marks)

11. (a) (i) $\lambda = a + 4$

(iii) (1) $5I$

(2013-DSE-MATH-EP(M2) #08) (5 marks)

8. (a) $M^{-1} = \frac{1}{k^2}\begin{pmatrix} 0 & 0 & k \\ k & 0 & -1 \\ -k & k^2 & 1 \end{pmatrix}$

(b) 1

(2013-DSE-MATH-EP(M2) #13) (13 marks)

13. (a) (iii) 3

(c) 1 or 3

(2014-DSE-MATH-EP(M2) #07) (7 marks)

7. (b) $|A| = 0$

(2014-DSE-MATH-EP(M2) #12) (11 marks)

12. (a) (i) $A^{-1} = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}$

(iii)

$$M^n = \frac{1}{1+k}\begin{pmatrix} k^{n+1} + (-1)^n & k^{n+1} + (-1)^{n+1}k \\ k^n + (-1)^{n+1} & k^n + (-1)^n k \end{pmatrix}$$

(b) $x_n = \frac{2^{n-1} + (-1)^{n-2}}{3}$

(2015-DSE-MATH-EP(M2) #06) (6 marks)

6. (b) (ii) $|A^3 + I| = 0$, agreed.

(2015-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) (i) $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2^{630} & 6^{315} - 2^{630} \\ 0 & 6^{315} \end{pmatrix}$

(2016-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) (i) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$

(b) (i) $2^n - 1$

(ii) $\begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$

(2017-DSE-MATH-EP(M2) #12) (12 marks)

12. (b) (i) A

(2018-DSE-MATH-EP(M2) #7) (8 marks)

7. (a) $b = -2a$, $c = -3a$

(c) $\frac{1}{9a}\begin{pmatrix} -3 & 2 \\ -6 & 1 \end{pmatrix}$

(2019-DSE-MATH-EP(M2) #02) (5 marks)

2. (a) 2

(b) 145

(2019-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) $a = -4$, $b = 5$

(c) Yes

$A = \frac{1}{6}\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}$, $B = \frac{1}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix}$

(2020-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $a = 2$, $b = -3$, $c = -5$

(b) (i) $M^{-1}RM = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(2021-DSE-MATH-EP(M2) #11) (12 marks)

11. (b) (i) $\frac{5\pi}{6}$

(iii) $\frac{1}{2^{556}}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$