2002-CE
A MATH

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 2002

ADDITIONAL MATHEMATICS

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **FOUR** questions in Section B.
2. Write your answers in the answer book provided. **For Section A, there is no need to start each question on a fresh page.**
3. All working must be clearly shown.
4. Unless otherwise specified, numerical answers must be **exact**.
5. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.
6. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$
$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$
$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$	

* *

Section A (62 marks)
Answer **ALL** questions in this section.

1. If n is a positive integer and the coefficient of x^2 in the expansion of

$$(1+x)^n + (1+2x)^n$$

is 75, find the value(s) of n.

(4 marks)

2. Find the equation of the tangent to the curve $C: y = (x-1)^4 + 4$ which is parallel to the line $y = 4x + 8$.

(4 marks)

3. Let $x \sin y = 2002$.

Find $\frac{dy}{dx}$. (4 marks)

4. Find $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \, dx$.

[Hint : Let $x = \sin\theta$.]

(4 marks)

5. $P(x, y)$ is a variable point such that the distance from P to the line $x - 4 = 0$ is always equal to twice the distance between P and the point (1, 0).

(a) Show that the equation of the locus of P is $3x^2 + 4y^2 - 12 = 0$.

(b) Sketch the locus of P.

(5 marks)

6.

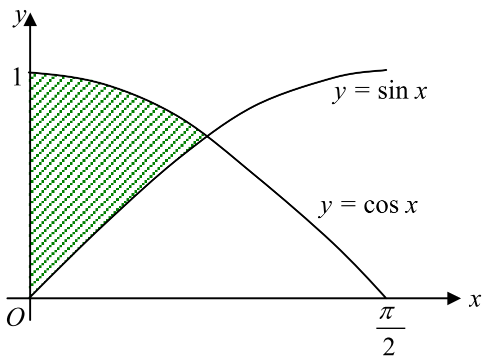

Figure 1

Figure 1 shows the curves $y = \sin x$ and $y = \cos x$. Find the area of the shaded region.

(5 marks)

7. Solve the following inequalities :

(a) $|x - 1| > 2$;

(b) $\left| |y| - 1 \right| > 2$.

(5 marks)

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8. Given $0 < x < \frac{\pi}{2}$.

Show that $\frac{\tan x - \sin^2 x}{\tan x + \sin^2 x} = \frac{4}{2 + \sin 2x} - 1$.

Hence, or otherwise, find the least value of $\frac{\tan x - \sin^2 x}{\tan x + \sin^2 x}$.

(5 marks)

9. Let α, β be the roots of the equation $z^2 + z + 1 = 0$.

Find α and β in polar form.

Hence, or otherwise, find $\alpha^6 + \beta^6$.

(5 marks)

10.

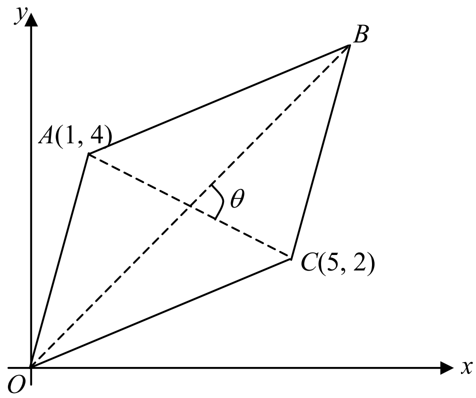

Figure 2

Figure 2 shows a parallelogram $OABC$. The position vectors of the points A and C are $\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i} + 2\mathbf{j}$ respectively.

(a) Find $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

(b) Let θ be the acute angle between OB and AC. Find θ correct to the nearest degree.

(6 marks)

11.

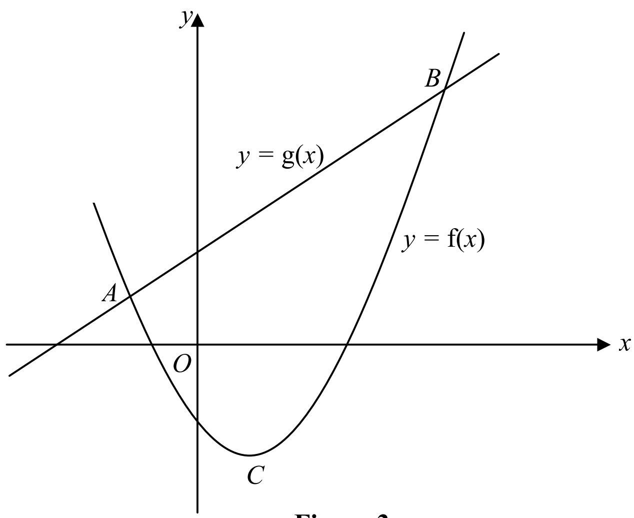

Figure 3

Let $f(x) = x^2 - 2x - 6$ and $g(x) = 2x + 6$. The graphs of $y = f(x)$ and $y = g(x)$ intersect at points A and B (see Figure 3). C is the vertex of the graph of $y = f(x)$.

(a) Find the coordinates of points A, B and C.

(b) Write down the range of values of x such that $f(x) \le g(x)$.

Hence write down the value(s) of k such that the equation $f(x) = k$ has only one real root in this range.

(7 marks)

12. (a) Prove, by mathematical induction, that

$$2(2) + 3(2^2) + 4(2^3) + \cdots + (n+1)(2^n) = n(2^{n+1})$$

for all positive integers n.

(b) Show that

$$1(2) + 2(2^2) + 3(2^3) + \cdots + 98(2^{98}) = 97(2^{99}) + 2$$

.(8 marks)

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Section B (48 marks)
Answer any **FOUR** questions in this section.
Each question carries 12 marks.

13.

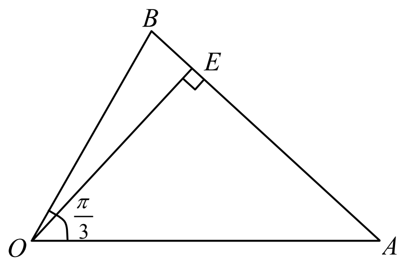

Figure 4

In Figure 4, OAB is a triangle. Point E is the foot of perpendicular from O to AB. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. It is given that $OA = 3$, $OB = 2$ and $\angle AOB = \frac{\pi}{3}$.

(a) Find $\mathbf{a} \cdot \mathbf{b}$.

(2 marks)

(b) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
[Hint : Let $BE : EA = t : (1 - t)$.]

(5 marks)

(c) F is a variable point on OE. A student says that $\overrightarrow{BA} \cdot \overrightarrow{BF}$ is always a constant. Explain whether the student is correct or not.

If you agree with the student, find the value of that constant.

If you do not agree with the student, find two possible values of $\overrightarrow{BA} \cdot \overrightarrow{BF}$.

(5 marks)

14.

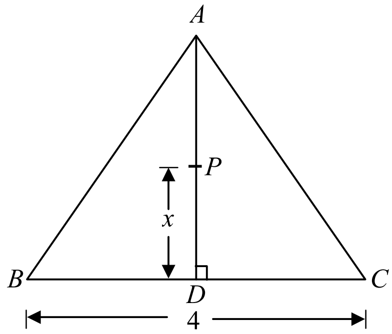

Figure 5

Figure 5 shows an isosceles triangle ABC with $AB = AC$ and $BC = 4$. D is the foot of perpendicular from A to BC and P is a point on AD. Let $PD = x$ and $r = PA + PB + PC$, where $0 \le x \le AD$.

(a) Suppose that $AD = 3$.

(i) Show that $\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{2x}{\sqrt{x^2+4}} - 1$.

(ii) Find the range of values of x for which

(1) r is increasing,

(2) r is decreasing.

Hence, or otherwise, find the least value of r.

(iii) Find the greatest value of r.

(9 marks)

(b) Suppose that $AD = 1$. Find the least value of r.

(3 marks)

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15. (a)

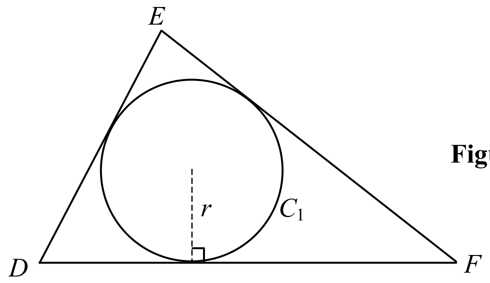

Figure 6

DEF is a triangle with perimeter p and area A. A circle C_1 of radius r is inscribed in the triangle (see Figure 6). Show that $A = \frac{1}{2}pr$.

(4 marks)

(b)

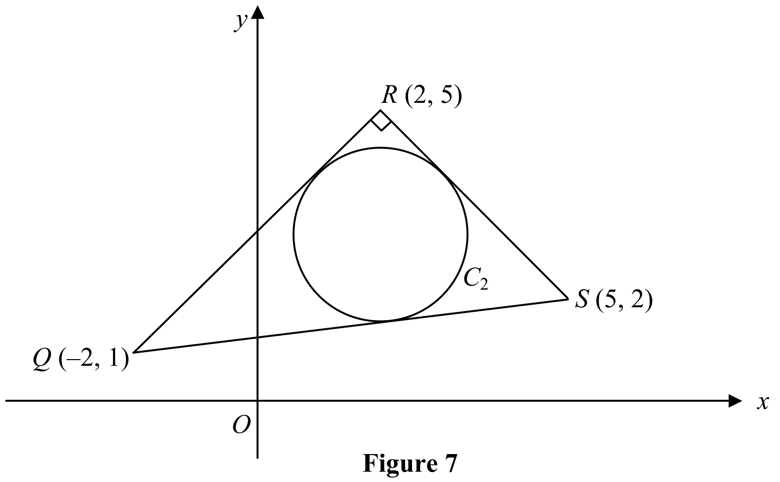

Figure 7

In Figure 7, a circle C_2 is inscribed in a right-angled triangle QRS. The coordinates of Q, R and S are (–2, 1), (2, 5) and (5, 2) respectively.

(i) Using (a), or otherwise, find the radius of C_2.

(ii) Find the equation of C_2.

(8 marks)

16. (a)

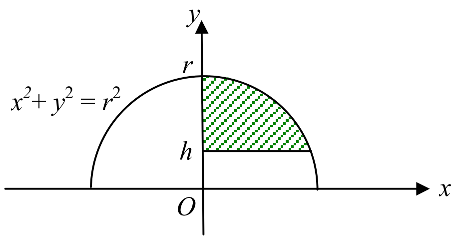

Figure 8

In Figure 8, the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the y-axis and the line $y = h$, where $0 \le h < r$. The shaded region is revolved about the y-axis. Show that the volume of the solid generated is $\frac{\pi}{3}(2r^3 - 3r^2h + h^3)$.

(4 marks)

(b)

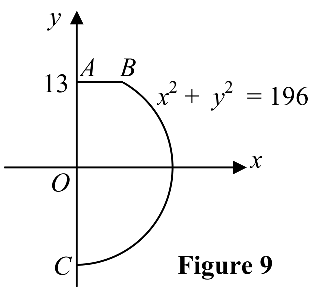

Figure 9

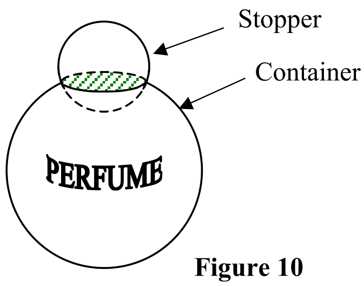

Figure 10

In Figure 9, A and C are points on the y-axis, BC is an arc of the circle $x^2 + y^2 = 196$ and AB is a segment of the line $y = 13$. A pot is formed by revolving BC about the y-axis.

(i) Find the capacity of the pot.

(ii) Figure 10 shows a perfume bottle. The container is in the shape of the pot described above and the stopper is a solid sphere of radius 6. Find the capacity of the perfume bottle.

(8 marks)

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17.

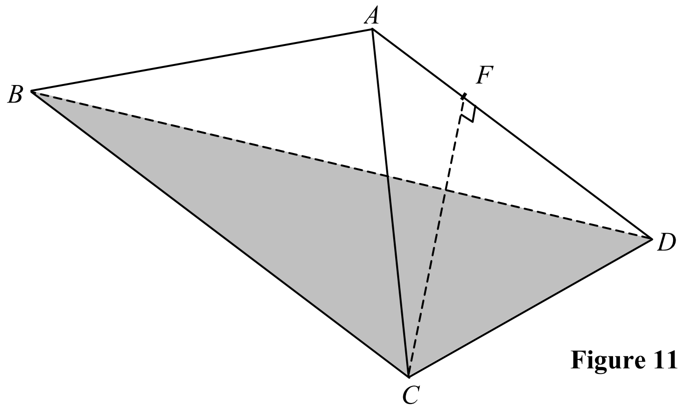


Figure 11

Figure 11 shows a tetrahedron $ABCD$ such that $AB = 28$, $CD = 30$, $AC = AD = 25$ and $BC = BD = 40$. F is the foot of perpendicular from C to AD.

(a) Find $\angle BFC$, giving your answer correct to the nearest degree.

(8 marks)

(b) A student says that $\angle BFC$ represents the angle between the planes ACD and ABD.

Explain whether the student is correct or not.

(4 marks)

18. (a) Let $z = \cos\theta + i\sin\theta$, where $-\pi < \theta \le \pi$.

Show that $|z^2 + 1|^2 = 2(1 + \cos 2\theta)$.

Hence, or otherwise, find the greatest value of $|z^2 + 1|$.

(5 marks)

(b) w is a complex number such that $|w| = 3$.

(i) Show that the greatest value of $|w^2 + 9|$ is 18.

(ii) Explain why the equation

$$w^4 - 81 = 100\,i\,(w^2 - 9)$$

has only two roots.

(7 marks)

END OF PAPER

2002

Additional Mathematics

Section A

1. 6

2. $y = 4x - 3$

3. $\dfrac{-\tan y}{x}$

4. $\dfrac{\pi}{6}$

6. $\sqrt{2} - 1$

7. (a) $x > 3$ or $x < -1$

   (b) $y > 3$ or $y < -3$

8. $\dfrac{1}{3}$

9. $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$, $\cos(-\dfrac{2\pi}{3}) + i\sin(-\dfrac{2\pi}{3})$

   2

10. (a) $6\mathbf{i} + 6\mathbf{j}$, $4\mathbf{i} - 2\mathbf{j}$

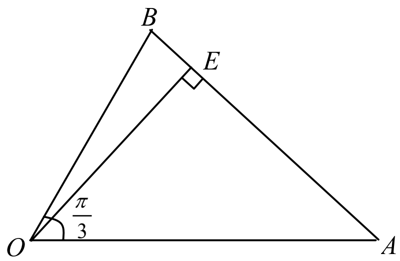
    (b) 72°

11. (a) $A(-2, 2)$, $B(6, 18)$, $C(1, -7)$

    (b) $-2 \le x \le 6$

    $2 < k \le 18$ or $k = -7$

Section B



Q.13 (a) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\angle AOB$

$= (3)(2)\cos\frac{\pi}{3}$

(b) $\overrightarrow{OE} = \frac{t\overrightarrow{OA} + (1-t)\overrightarrow{OB}}{t + (1-t)}$

$= t\mathbf{a} + (1-t)\mathbf{b}$

Since $OE \perp AB$,

$\overrightarrow{OE}\cdot\overrightarrow{AB} = 0$

$[\,t\mathbf{a} + (1-t)\mathbf{b}\,]\cdot(\mathbf{b}-\mathbf{a}) = 0$

$t\mathbf{a}\cdot\mathbf{b} - t\mathbf{a}\cdot\mathbf{a} + (1-t)\,\mathbf{b}\cdot\mathbf{b} - (1-t)\mathbf{a}\cdot\mathbf{b} = 0$

$3t - 9t + 4(1-t) - 3(1-t) = 0$

$1 - 7t = 0$

$t = \frac{1}{7}$

$\therefore \overrightarrow{OE} = \frac{1}{7}\mathbf{a} + \frac{6}{7}\mathbf{b}$

(c) $\overrightarrow{BA}\cdot\overrightarrow{BF}$

$= |\overrightarrow{BA}||\overrightarrow{BF}|\cos\angle ABF$

$= |\overrightarrow{BA}||\overrightarrow{BE}|$

= a constant (since $|\overrightarrow{BA}|$ and $|\overrightarrow{BE}|$ are constants)

$\therefore$ the student is correct.

By Cosine Law,

$|\overrightarrow{BA}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos\angle AOB$

$= 3^2 + 2^2 - 2(3)(2)\cos\frac{\pi}{3}$

$= 7$

$|\overrightarrow{BA}| = \sqrt{7}$

From (b), $\overrightarrow{BE} = \frac{1}{7}\overrightarrow{BA}$

$$|\overrightarrow{BE}| = \frac{1}{7}|\overrightarrow{BA}|$$

$$= \frac{\sqrt{7}}{7}$$

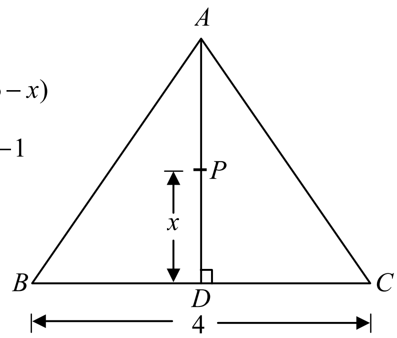
$$\therefore\ \overrightarrow{BA}\cdot\overrightarrow{BF} = |\overrightarrow{BA}||\overrightarrow{BE}|$$

$$= \sqrt{7}\,(\frac{\sqrt{7}}{7}) = 1$$

Q.14 (a) (i) $PB = PC = \sqrt{x^2+4}$, $PA = (3-x)$

$r = PA + PB + PC = 2\sqrt{x^2+4} + (3-x)$

$\frac{dr}{dx} = 2(\frac{1}{2})\frac{2x}{\sqrt{x^2+4}} - 1 = \frac{2x}{\sqrt{x^2+4}} - 1$



(ii) (1) $\frac{dr}{dx} \geq 0$

$\frac{2x}{\sqrt{x^2+4}} - 1 \geq 0$

$2x \geq \sqrt{x^2+4}$

$x \geq \frac{2}{\sqrt{3}}$

$\therefore$ r is increasing on $3 \geq x \geq \frac{2}{\sqrt{3}}$.

(2) $\frac{dr}{dx} \leq 0$

$x \leq \frac{2}{\sqrt{3}}$

$\therefore$ r is decreasing on $0 \leq x \leq \frac{2}{\sqrt{3}}$

r is the least at $x = \frac{2}{\sqrt{3}}$.

Least value of $r = 2\sqrt{(\frac{2}{\sqrt{3}})^2 + 4} + (3 - \frac{2}{\sqrt{3}})$

$= 2\sqrt{3} + 3$

(iii) The greatest value of r occurs at the end-points.

At $x = 0$, $r = 2\sqrt{0+4} + (3-0) = 7$

At $x = 3$, $r = 2\sqrt{3^2+4} + (3-3) = 2\sqrt{13}$

$\therefore$ the greatest value of r is $2\sqrt{13}$.

(b) $r = 2\sqrt{x^2+4} + (1-x)$

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{2x}{\sqrt{x^2+4}} - 1$$

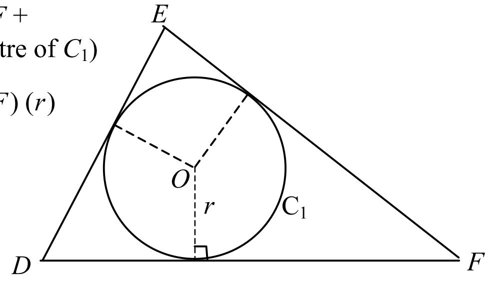
From (a), r is decreasing on $0 \le x \le 1$.

r is the least at $x = 1$.

Least value $= 2\sqrt{1+4} + (1-1)$

$= 2\sqrt{5}$

Q.15 (a) A = Area of ΔODE + area of ΔOEF + area of ΔODF (where O is centre of C_1)

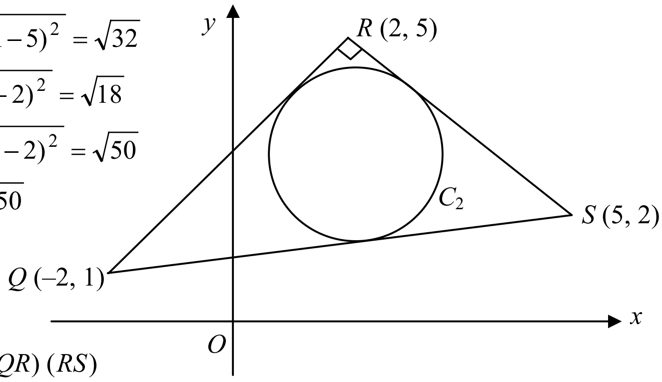


$$= \frac{1}{2}(DE)(r) + \frac{1}{2}(EF)(r) + \frac{1}{2}(DF)(r)$$

$$= \frac{1}{2}(DE + EF + FD)(r)$$

$$A = \frac{1}{2}pr$$

(b) (i) $QR = \sqrt{(-2-2)^2 + (1-5)^2} = \sqrt{32}$



$RS = \sqrt{(2-5)^2 + (5-2)^2} = \sqrt{18}$

$SQ = \sqrt{(-2-5)^2 + (1-2)^2} = \sqrt{50}$

$p = \sqrt{32} + \sqrt{18} + \sqrt{50}$

$= 12\sqrt{2}$

Area of $\Delta QRS = \frac{1}{2}(QR)(RS)$

$= \frac{1}{2}(\sqrt{32})(\sqrt{18})$

$= 12$

Using (a), $A = \frac{1}{2}pr$

$12 = \frac{1}{2}(12\sqrt{2})\,r$

$r = \sqrt{2}$

(ii) Equation of QR

$$\frac{y-5}{x-2}=\frac{5-1}{2-(-2)}$$

$$x-y+3=0$$

Let (h, k) be the centre of C_2.

$$(\frac{h-k+3}{\sqrt{2}})=\sqrt{2}$$

$$h-k+1=0 \text{ - - - - - } (1)$$

Equation of RS

$$\frac{y-5}{x-2}=\frac{5-2}{2-5}$$

$$x+y-7=0$$

$$-(\frac{h+k-7}{\sqrt{2}})=\sqrt{2}$$

$$h+k-5=0 \text{ - - - - - } (2)$$

Solve (1) and (2), $h=2$, $k=3$.

$\therefore$ the equation of C_2 is $(x-2)^2+(y-3)^2=2$.

Q.16 (a) Volume $= \int_h^r \pi x^2 \, \mathrm{d}y$

$$= \pi \int_h^r (r^2 - y^2) \, \mathrm{d}y$$

$$= \pi \left[r^2 y - \frac{1}{3} y^3 \right]_h^r$$

$$= \pi \left[r^3 - \frac{1}{3} r^3 - r^2 h + \frac{1}{3} h^3 \right]$$

$$= \frac{\pi}{3}(2r^3 - 3r^2 h + h^3)$$

(b) (i) Using the result in (a), substitute $r = 14$, $h = 13$:

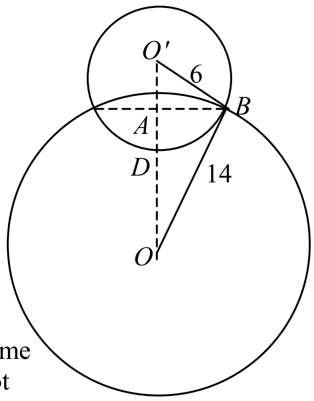
Capacity of the pot

$$= \frac{4}{3}\pi(14)^3 - \frac{\pi}{3}[2(14)^3 - 3(14)^2 \ (13) + (13)^3]$$

$$= 3645\,\pi$$

(ii) Let O' be the centre of the stopper.

$OB = 14$, $BO' = 6$, $OA = 13$



$$AB = \sqrt{OB^2 - OA^2}$$

$$= \sqrt{14^2 - 13^2} = \sqrt{27}$$

$$AO' = \sqrt{BO'^2 - AB^2}$$

$$= \sqrt{6^2 - 27}$$

$$= 3$$

Capacity of the perfume bottle

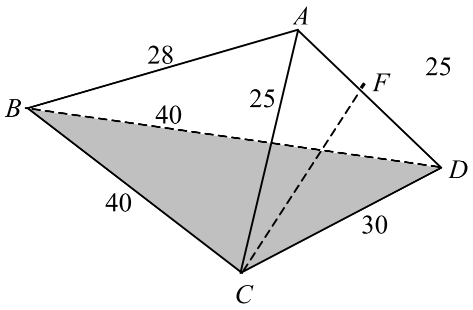
= Capacity of pot found in (i) – Volume of the stopper lying inside the pot

$$= 3645\pi - \frac{\pi}{3}[2(6)^3 - 3(6)^2(3) + (3)^3]$$

$$= 3645\pi - 45\,\pi$$

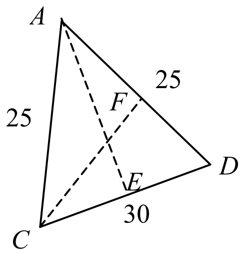
$$= 3600\,\pi$$

Q.17 (a)



Consider ΔACD :

Let E be the foot of perpendicular from A to CD.



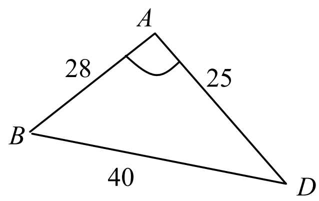
$$\cos\angle ADE = \frac{DE}{AD} = \frac{15}{25} = \frac{3}{5}$$

$$CF = CD\sin\angle ADE$$

$$= 30\left(\frac{\sqrt{5^2 - 3^2}}{5}\right)$$

$$= 24$$

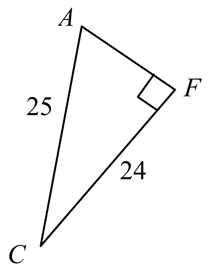
Consider ΔABD :



$$\cos\angle BAD = \frac{28^2 + 25^2 - 40^2}{2(28)(25)}$$

$$= -\frac{191}{1400}$$

Consider ΔACF :

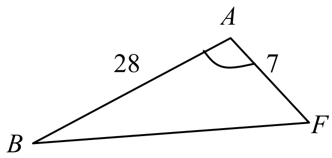


$$AF^2 = AC^2 - CF^2$$

$$= 25^2 - 24^2 = 49$$

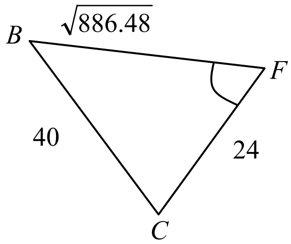
$$AF = 7$$

Consider ΔABF :



$$BF^2 = AB^2 + AF^2 - 2(AB)(AF)\cos\angle BAF$$
$$= 28^2 + 7^2 - 2(28)(7)(-\frac{191}{1400})$$
$$BF = \sqrt{886.48}$$

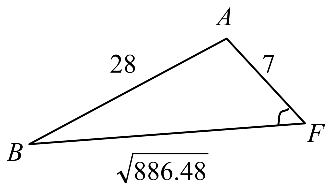
Consider ΔBCF :



$$\cos\angle BFC = \frac{BF^2 + CF^2 - BC^2}{2(BF)(CF)}$$
$$= \frac{886.48 + 24^2 - 40^2}{2(\sqrt{886.48})(24)}$$
$$= -0.096$$

$\angle BFC = 96°$ (correct to the nearest degree)

(b) Consider ΔABF :



$$BF^2 + FA^2 = 886.48 + 49$$
$$= 935.48$$
$$\neq AB^2$$
$$\therefore \angle AFB \neq 90°$$

Since $CF \perp AD$ but BF is not perpendicular to AD, $\angle BFC$ does not represent the angle between the two planes. The student is incorrect.

Q.18 (a) $z^2 = \cos 2\theta + i\sin 2\theta$

$z^2 + 1 = (\cos 2\theta + 1) + i\sin 2\theta$

$|z^2 + 1|^2 = (\cos 2\theta + 1)^2 + \sin^2 2\theta$

$= \cos^2 2\theta + 2\cos 2\theta + 1 + \sin^2 2\theta$

$= 2(1 + \cos 2\theta)$

Since $-\pi < \theta \le \pi$, $-2\pi < 2\theta \le 2\pi$.

$\cos 2\theta \le 1$

$\therefore$ the greatest value of $|z^2 + 1| = \sqrt{2(1+1)}$

$= 2$

(b) (i) $w = 3z$

$|w^2 + 9| = |(3z)^2 + 9|$

$= 9|z^2 + 1|$

From (a), $|z^2 + 1| \le 2$.

$\therefore$ greatest value of $|w^2 + 9| = 9(2)$

$= 18$

(ii) $w^4 - 81 = 100i(w^2 - 9)$

$(w^2 + 9)(w^2 - 9) - 100i(w^2 - 9) = 0$

$(w^2 - 9)(w^2 + 9 - 100i) = 0$

$w^2 - 9 = 0 ---(1)$ or $w^2 + 9 - 100i = 0 ---(2)$

Consider (1) : $w = \pm 3$

which satisfies the condition $|w| = 3$

Consider (2) : $w^2 + 9 = 100i$

From (i), $|w^2 + 9| \le 18$ but $|100i| = 100$.

So equation (2) has no solutions

$\therefore$ the equation has only two roots.