2000-CE
A MATH
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 2000

# ADDITIONAL MATHEMATICS PAPER 2

11.15 am – 1.15 pm (2 hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **THREE** questions in Section B.
2. All working must be clearly shown.
3. Unless otherwise specified, numerical answers must be **exact**.
4. The diagrams in the paper are not necessarily drawn to scale.

**FORMULAS FOR REFERENCE**

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

**Section A** (42 marks)
Answer **ALL** questions in this section.

1. Find $\int \sqrt{2x+1}\,\mathrm{d}x$.

(4 marks)

2. Expand $(1+2x)^7(2-x)^2$ in ascending powers of $x$ up to the term $x^2$.

(5 marks)

3.

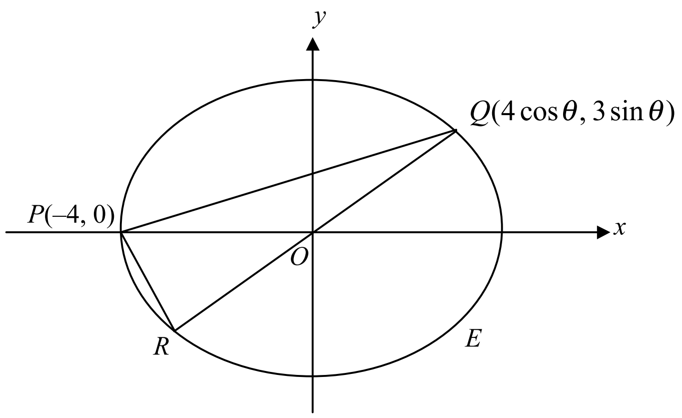


**Figure 1**

Figure 1 shows the ellipse $E : \frac{x^2}{16} + \frac{y^2}{9} = 1$. $P(-4, 0)$ and $Q(4\cos\theta, 3\sin\theta)$ are points on $E$, where $0 < \theta < \frac{\pi}{2}$. $R$ is a point such that the mid-point of $QR$ is the origin $O$.

(a) Write down the coordinates of $R$ in terms of $\theta$.

(b) If the area of $\Delta PQR$ is 6 square units, find the coordinates of $Q$.

(6 marks)



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4. Prove, by mathematical induction, that

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

for all positive integers $n$.

(6 marks)

5.

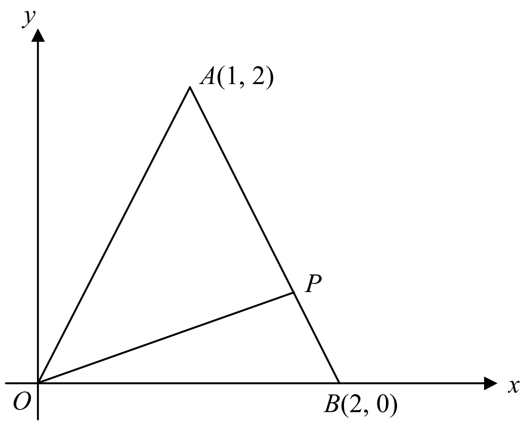


**Figure 2**

In Figure 2, the coordinates of points $A$ and $B$ are (1, 2) and (2, 0) respectively. Point $P$ divides $AB$ internally in the ratio $1 : r$.

(a) Find the coordinates of $P$ in terms of $r$.

(b) Show that the slope of $OP$ is $\frac{2r}{2+r}$.

(c) If $\angle AOP = 45°$, find the value of $r$.

(6 marks)

6.

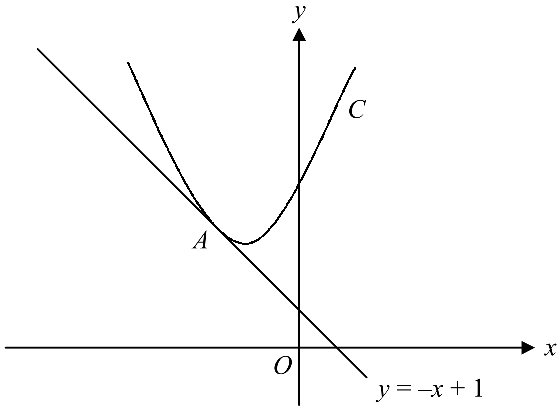


**Figure 3**

The slope at any point $(x, y)$ of a curve $C$ is given by $\frac{dy}{dx} = 2x + 3$. The line $y = -x + 1$ is a tangent to the curve at point $A$. (See Figure 3.) Find

(a) the coordinates of $A$,

(b) the equation of $C$.

(7 marks)

7. (a) By expressing $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x + \theta)$, or otherwise, find the general solution of the equation

$$\cos x - \sqrt{3}\sin x = 2 .$$

(b) Find the number of points of intersection of the curves $y = \cos x$ and $y = 2 + \sqrt{3}\sin x$ for $0 < x < 9\pi$ .

(8 marks)



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**Section B** (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

8. (a) Find $\int \cos 3x \cos x \, \mathrm{d}x$.

(3 marks)

(b) Show that $\dfrac{\sin 5x - \sin x}{\sin x} = 4\cos 3x \cos x$.

Hence, or otherwise, find $\int \dfrac{\sin 5x}{\sin x} \, \mathrm{d}x$.

(4 marks)

(c) Using a suitable substitution, show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x} \, \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x} \, \mathrm{d}x .$$

(4 marks)

(d)

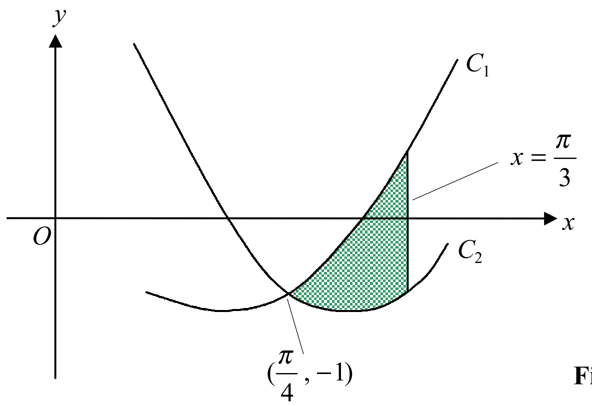


**Figure 4**

In Figure 4, the curves $C_1 : y = \dfrac{\cos 5x}{\cos x}$ and $C_2 : y = \dfrac{\sin 5x}{\sin x}$ intersect at the point $(\dfrac{\pi}{4}, -1)$. Find the area of the shaded region bounded by $C_1, C_2$ and the line $x = \dfrac{\pi}{3}$.

(5 marks)

9. Given a family of circles

$$F : x^2 + y^2 + (4k+4)x + (3k+1)y - (8k+8) = 0,$$

where $k$ is real. $C_1$ is the circle $x^2 + y^2 - 2y = 0$.

(a) Show that

(i) $C_1$ is a circle in $F$,

(ii) $C_1$ touches the $x$-axis.

(4 marks)

(b) Besides $C_1$, there is another circle $C_2$ in $F$ which also touches the $x$-axis.

(i) Find the equation of $C_2$.

(ii) Show that $C_1$ and $C_2$ touch externally.

(7 marks)

(c)

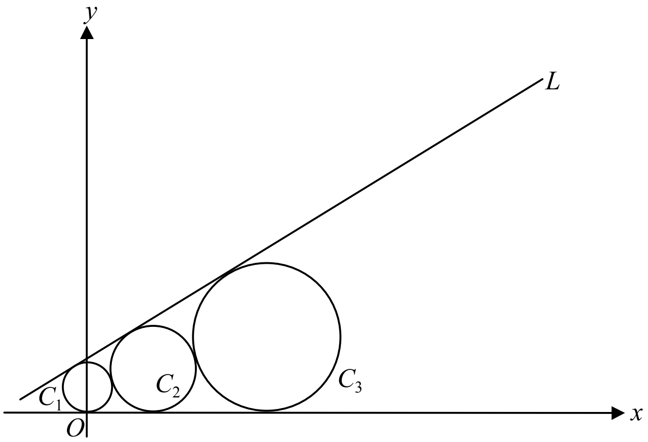


**Figure 5**

Figure 5 shows the circles $C_1$ and $C_2$ in (b). $L$ is a common tangent to $C_1$ and $C_2$. $C_3$ is a circle touching $C_2$, $L$ and the $x$-axis but it is not in $F$. (See Figure 5.) Find the equation of $C_3$.

(Hint : The centres of the three circles are collinear.)

(5 marks)



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10.

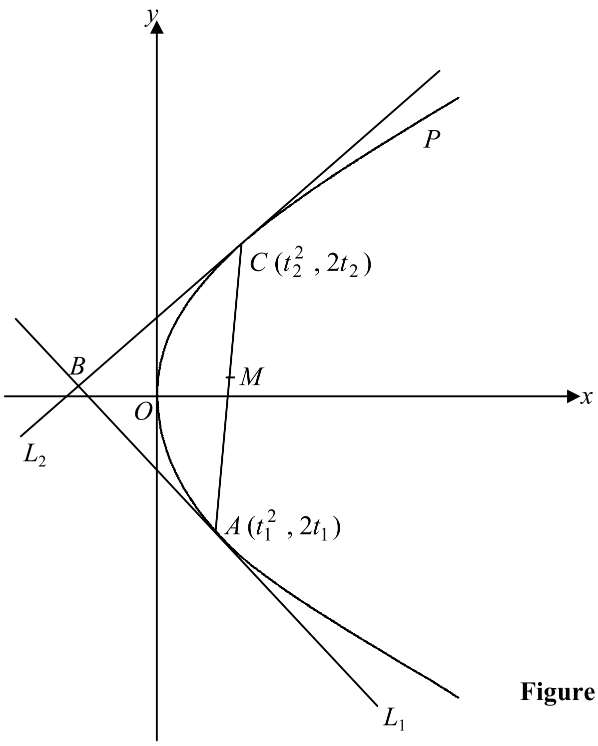


**Figure 6(a)**

Figure 6(a) shows a parabola $P: y^2 = 4x$. $A(t_1^2, 2t_1)$ and $C(t_2^2, 2t_2)$ are two distinct points on $P$, where $t_1 < 0 < t_2$. $L_1$ and $L_2$ are tangents to $P$ at $A$ and $C$ respectively and they intersect at point $B$. Let $M$ be the mid-point of $AC$.

(a) Show that

(i) the equation of $L_1$ is $x - t_1 y + t_1^2 = 0$,

(ii) the coordinates of $B$ are $(t_1 t_2, t_1 + t_2)$,

(iii) $BM$ is parallel to the $x$-axis.

(7 marks)

(b)

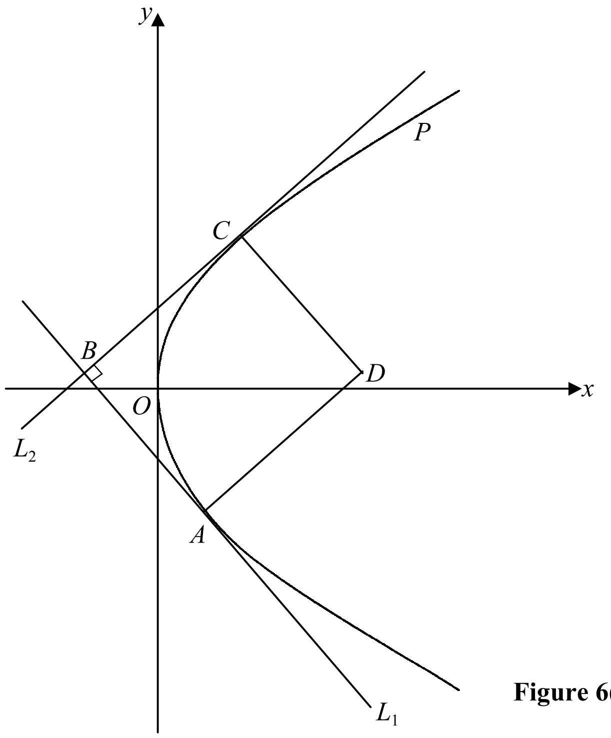


**Figure 6(b)**

Suppose $L_1$ and $L_2$ are perpendicular to each other and $D$ is a point such that $ABCD$ is a rectangle. (See Figure 6(b).)

(i) Find the value of $t_1t_2$.

(ii) Show that the coordinates of $D$ are $(t_1^2 + t_2^2 + 1,\ t_1 + t_2)$.

(iii) Find the equation of the locus of $D$ as $A$ and $C$ move along the parabola $P$.

(9 marks)

11. (a)

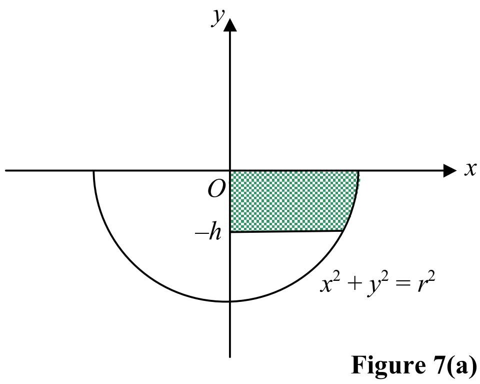


**Figure 7(a)**

In Figure 7 (a), the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the $x$-axis, the $y$-axis and the line $y = -h$, where $h > 0$. If the shaded region is revolved about the $y$-axis, show that the volume of the solid generated is $(r^2 h - \frac{1}{3}h^3)\pi$ cubic units.

(4 marks)

(b)

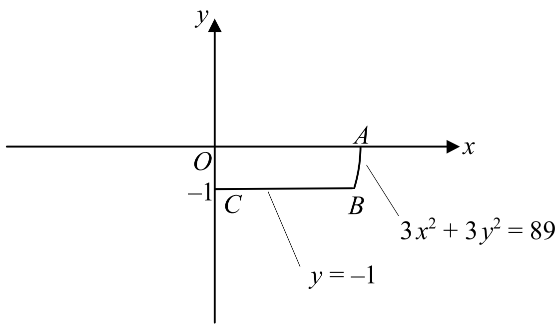


**Figure 7(b)**

In Figure 7 (b), $A$ and $C$ are points on the $x$-axis and $y$-axis respectively, $AB$ is an arc of the circle $3x^2 + 3y^2 = 89$ and $BC$ is a segment of the line $y = -1$. A mould is formed by revolving $AB$ and $BC$ about the $y$-axis. Using (a), or otherwise, show that the capacity of the mould is $\frac{88\pi}{3}$ cubic units.

(2 marks)

(c)

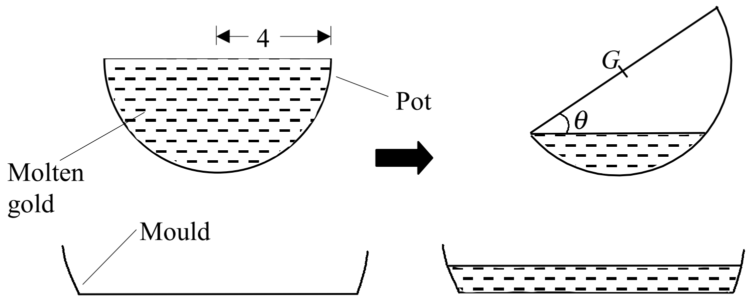


**Figure 7(c)** **Figure 7(d)**

A hemispherical pot of inner radius 4 units is completely filled with molten gold. (See Figure 7 (c).) The molten gold is then poured into the mould mentioned in (b) by steadily tilting the pot. Suppose the pot is tilted through an angle $\theta$ and $G$ is the centre of the rim of the pot. (See Figure 7 (d).)

(i) Find, in terms of $\theta$,

(1) the distance between $G$ and the surface of the molten gold remaining in the pot,

(2) the volume of gold poured into the mould.

(ii) When the mould is completely filled with molten gold, show that

$$8\sin^3\theta - 24\sin\theta + 11 = 0 .$$

Hence find the value of $\theta$.

(10 marks)


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12. (a)

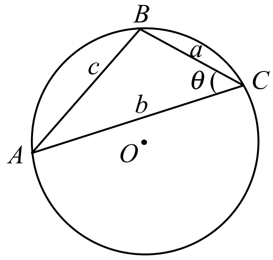



**Figure 8(a)**

In Figure 8 (a), a triangle $ABC$ is inscribed in a circle with centre $O$ and radius $r$. $AB = c$, $BC = a$ and $CA = b$. Let $\angle BCA = \theta$.

(i) Express $\cos\theta$ in terms of $a$, $b$ and $c$.

(ii) Show that $r = \dfrac{c}{2\sin\theta}$.

(iii) Using (i) and (ii), or otherwise, show that

$$r = \frac{abc}{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}.$$

(7 marks)

(b) **In this part, numerical answers should be given correct to two significant figures.**

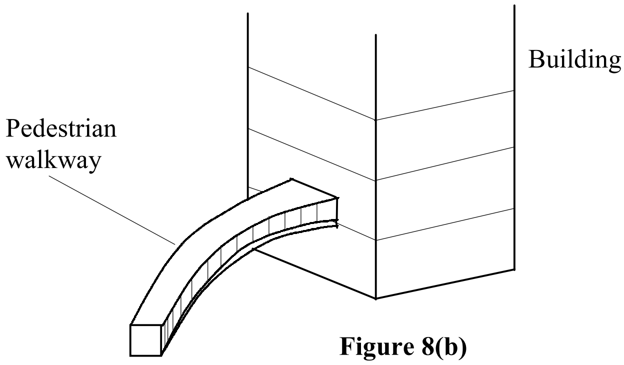



**Figure 8(b)**

12. (b) **(continued)**

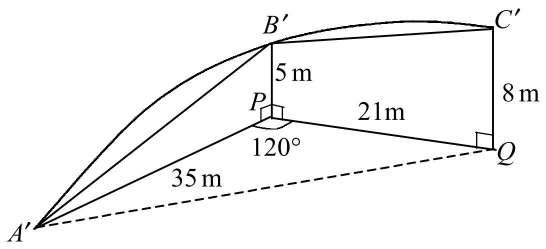



**Figure 8(c)**

Figure 8 (b) shows a pedestrian walkway joining the horizontal ground and the first floor of a building. To estimate its length, the walkway is modelled by a circular arc $A'B'C'$ as shown in Figure 8 (c), where $A'$ denotes the entrance to the walkway on the ground and $C'$ the exit leading to the first floor of the building. $P$ and $Q$ are the feet of perpendiculars from $B'$ and $C'$ to the ground respectively. It is given that $A'P = 35$ m, $PQ = 21$ m, $B'P = 5$ m, $C'Q = 8$ m and $\angle A'PQ = 120°$.

(i) Find the radius of the circular arc $A'B'C'$.

(ii) Estimate the length of the walkway.

(9 marks)

**END OF PAPER**

2000

*Additional Mathematics*

**Paper 2**

**Section A**

1. $\frac{1}{3}(2x+1)^{\frac{3}{2}}+c$, where $c$ is a constant

2. $4+52x+281x^2+\cdots$

3. (a) $(-4\cos\theta\ ,-3\sin\theta)$

   (b) $(2\sqrt{3}\ ,\frac{3}{2})$

5. (a) $(\frac{2+r}{1+r}\ ,\frac{2r}{1+r})$

   (c) $\frac{2}{5}$

6. (a) $(-2\ ,3)$

   (b) $y=x^2+3x+5$

7. (a) $x=2n\pi-\frac{\pi}{3}$, where $n$ is an integer

   (b) 4

**Section B**

Q.8 (a) $\int \cos 3x \cos x \, dx = \int \frac{1}{2}(\cos 4x + \cos 2x) \, dx$

$= \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + c$, where $c$ is a constant

(b) $\frac{\sin 5x - \sin x}{\sin x} = \frac{2\sin 2x \cos 3x}{\sin x}$

$= \frac{2(2\sin x \cos x)\cos 3x}{\sin x}$

$= 4\cos x \cos 3x$

$\int \frac{\sin 5x}{\sin x} dx = \int (1 + 4\cos 3x \cos x) dx$

$= x + 4(\frac{\sin 4x}{8} + \frac{\sin 2x}{4}) + c$, where $c$ is a constant

$= x + \frac{1}{2}\sin 4x + \sin 2x + c$

(c) Put $x = \frac{\pi}{2} - \theta$ :

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{\sin 5(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta)} (-d\theta)$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5\theta}{\cos \theta} d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x} dx$

(d) Area of shaded region

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\frac{\cos 5x}{\cos x} - \frac{\sin 5x}{\sin x}) dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 5x}{\cos x} dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x} dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 5x}{\sin x} dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x} dx$ (using (c))

$= [x + \frac{1}{2}\sin 4x + \sin 2x]_{\frac{\pi}{6}}^{\frac{\pi}{4}} - [x + \frac{1}{2}\sin 4x + \sin 2x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= 2 - \sqrt{3}$

Q.9 (a) (i) Put $k=-1$ into $F$, the equation becomes

$$x^2+y^2+(-4+4)x+(-3+1)y-(-8+8)=0$$

i.e. $x^2+y^2-2y=0$.

$\therefore$ $C_1$ is a circle in $F$.

(ii) Co-ordinates of centre of $C_1=(0,1)$.

radius of $C_1=1$

Since the $y$-coordinate of centre is equal to the radius, $C_1$ touches the $x$-axis.

(b) (i) Put $y=0$ in $F$:

$$x^2+(4k+4)x-(8k+8)=0$$

Since the circle touches the $x$-axis,

$$(4k+4)^2+4(8k+8)=0$$

$$16k^2+64k+48=0$$

$$16(k+1)(k+3)=0$$

$k=-1$ (rejected) or $k=-3$,

$\therefore$ the equation of $C_2$ is

$$x^2+y^2+[4(-3)+4]x+[3(-3)+1]y-[(-3)\times 8+8]=0$$

$$x^2+y^2-8x-8y+16=0$$

(ii) Co-ordinates of centre of $C_1=(0,1)$, radius $=1$.

Co-ordinates of centre of $C_2=(4,4)$, radius $=4$.

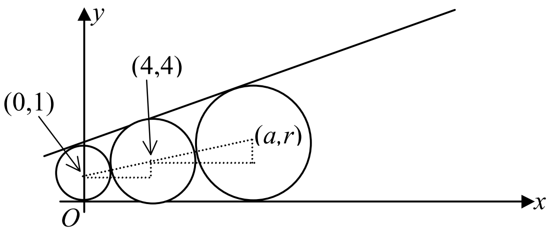
Distance between centres $=\sqrt{(4-0)^2+(4-1)^2}$

$=5$

$=$ sum of radii of $C_1$ and $C_2$

$\therefore$ $C_1$ and $C_2$ touch externally.

(c) Let radius of $C_3$ be $r$ and coordinates of its centre be $(a, r)$.




Considering the similar triangles,

$$\frac{r+4}{1+4} = \frac{r-4}{4-1}$$

$$3r + 12 = 5r - 20$$

$$r = 16$$

$$\frac{a-4}{4-0} = \frac{r+4}{4+1} = \frac{16+4}{4+1}$$

$$a = 20$$

$\therefore$ the equation of $C_3$ is $(x-20)^2 + (y-16)^2 = 256$.

Q.10 (a) (i) $y^2 = 4x$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}$

At point $A$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2t_1} = \frac{1}{t_1}$

Equation of $L_1$ is

$\frac{y-2t_1}{x-t_1^2} = \frac{1}{t_1}$

$t_1 y - 2t_1^2 = x - t_1^2$

$x - t_1 y + t_1^2 = 0$

(ii) Equation of $L_2$ is $x - t_2 y + t_2^2 = 0$.

$$\begin{cases} x - t_1 y + t_1^2 = 0 & \text{------(1)} \\ x - t_2 y + t_2^2 = 0 & \text{------(2)} \end{cases}$$

(1) – (2) : $(t_2 - t_1)y + (t_1^2 - t_2^2) = 0$

$y = t_1 + t_2$

$x = t_1(t_1 + t_2) - t_1^2 = t_1 t_2$

$\therefore$ the coordinates of $B$ are $(t_1 t_2, t_1 + t_2)$.

(iii)The coordinates of $M$ are $(\frac{t_1^2 + t_2^2}{2}, \frac{2t_1 + 2t_2}{2})$,

i.e. $(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2)$.

As the $y$-coordinates of $B$ and $M$ are equal,
$BM$ is parallel to the $x$-axis.

(b) (i) (Slope of $L_1$) (Slope of $L_2$) = –1

$(\frac{1}{t_1})(\frac{1}{t_2}) = -1$

$t_1 t_2 = -1$

(ii) Since $ABCD$ is a rectangle, mid-point of $BD$ coincides with mid-point of $AC$, i.e. point $M$.

$$\frac{x+t_1t_2}{2}=\frac{t_1^2+t_2^2}{2}$$

$$x=t_1^2+t_2^2-t_1t_2$$

$$=t_1^2+t_2^2+1 \quad (\because t_1t_2=-1)$$

Since $BD$ is parallel to the $x$-axis, the $y$-coordinate of $D$ = $y$-coordinate of $B$ = $t_1+t_2$.

$\therefore$ the coordinates of $D$ are $(t_1^2+t_2^2+1, t_1+t_2)$.

(iii) Let $(x, y)$ be the coordinates of $D$.

$$\begin{cases} x=t_1^2+t_2^2+1 \\ y=t_1+t_2 \end{cases}$$

$$x=(t_1+t_2)^2-2t_1t_2+1$$

$$=y^2-2(-1)+1$$

$$x=y^2+3$$

$\therefore$ the equation of the locus is $x-y^2-3=0$.

Q.11 (a) Volume $= \int_{-h}^{0} \pi x^2 \, \mathrm{d}y$

$= \int_{-h}^{0} \pi (r^2 - y^2) \, \mathrm{d}y$

$= \pi [r^2 y - \frac{1}{3} y^3]_{-h}^{0}$

$= \pi (r^2 h - \frac{1}{3} h^3)$ cubic units

(b) Put $h = 1, r = \sqrt{\frac{89}{3}}$:

Using (a),

capacity of the mould $= \pi [\frac{89}{3}(1) - \frac{1}{3}(1)^3]$

$= \frac{88\pi}{3}$ cubic units

(c) (i) (1) Distance $= 4\sin\theta$ .

(2) Put $r = 4, h = 4\sin\theta$ .

Using (a), amount of gold poured into the pot

$= \pi [4^2 (4\sin\theta) - \frac{1}{3}(4\sin\theta)^3]$

$= \pi (64\sin\theta - \frac{64}{3}\sin^3\theta)$

(ii) When the mould is completely filled,

$\pi (64\sin\theta - \frac{64}{3}\sin^3\theta) = \frac{88\pi}{3}$

$64\sin^3\theta - 192\sin\theta + 88 = 0$

$8\sin^3\theta - 24\sin\theta + 11 = 0$ ---- (*)

Put $\sin\theta = \frac{1}{2}$:

$8\sin^3\theta - 24\sin\theta + 11 = 0.$

$\therefore \sin\theta = \frac{1}{2}$ is a root of (*)

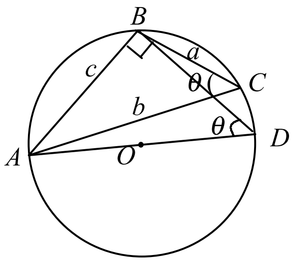
$(2\sin\theta - 1)(4\sin^2\theta + 2\sin\theta - 11) = 0$

$\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{-2 \pm \sqrt{180}}{8}$ (rejected)

$\therefore \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

Q.12 (a)




(i) $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2ab}$

(ii) Consider $\Delta ABD$ :

$AD = 2r$

$\angle BDA = \angle BCA = \theta$

$\angle ABD = 90°$

$\therefore \sin\theta = \dfrac{c}{2r}$

$r = \dfrac{c}{2\sin\theta}$

(iii) $\sin^2\theta + \cos^2\theta = 1$

$(\dfrac{c}{2r})^2 + (\dfrac{a^2 + b^2 - c^2}{2ab})^2 = 1$

$\dfrac{c^2}{4r^2} = 1 - \dfrac{(a^2 + b^2 - c^2)^2}{4a^2b^2}$

$r^2 = \dfrac{a^2b^2c^2}{4a^2b^2 - (a^2 + b^2 - c^2)^2}$

$r = \dfrac{abc}{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}$

(b) (i) Consider $\Delta A'B'C'$ : $A'B' = \sqrt{(A'P)^2 + (PB')^2}$

$= \sqrt{35^2 + 5^2}$

$= \sqrt{1250}$

$B'C' = \sqrt{(PQ)^2 + (QC' - PB')^2}$

$= \sqrt{21^2 + (8-5)^2}$

$= \sqrt{450}$

$$(A'Q)^2 = (A'P)^2 + (PQ)^2 - 2(A'P)(PQ)^2 \cos\angle A'PQ$$
$$= 35^2 + 21^2 - 2(35)(21)\cos 120°$$
$$= 2401$$
$$A'C' = \sqrt{(A'Q)^2 + (QC')^2}$$
$$= \sqrt{2401 + 8^2}$$
$$= \sqrt{2465}$$

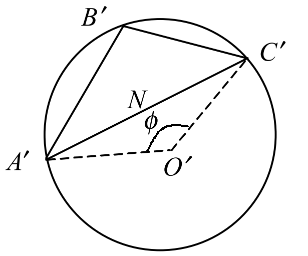
Using (a) (iii), put $a = \sqrt{450}$, $b = \sqrt{2465}$, $c = \sqrt{1250}$ :

$$r = \frac{\sqrt{1250}\sqrt{450}\sqrt{2465}}{\sqrt{4(450)(2465) - (450 + 2465 - 1250)^2}}$$

= 29 m (correct to 2 sig. figures)

$\therefore$ the radius of arc $A'B'C'$ is 29 m.

(ii)




Let $O'$ be the centre of the circle passing through $A'$, $B'$ and $C'$,

$\phi$ be the angle subtended by arc $A'B'C'$ at $O'$.

Consider $\Delta O'A'N$ ($N$ is the mid-point of $A'C'$)

$$\sin\frac{\phi}{2} = \frac{\frac{1}{2}A'C'}{r}$$

$$\sin\frac{\phi}{2} = \frac{\frac{1}{2}\sqrt{2465}}{28.86}$$
$$= 0.8602$$

$\phi = 2.07$

Length of walkway

$= \text{length of } \overset{\frown}{A'B'C'} = r\phi$

$= 28.86\ (2.07)$

= 60 m (correct to 2 sig. figures)

$\therefore$ the length of the walkway is 60 m.