Solution 1993 (paper 2)	Marks	Remarks 1993
1. For $n = 1$, L.H.S. = 2		
R.H.S. = $\frac{1}{12} \times 2 \times 3 \times 4 = 2$		
$\therefore$ The statement is true for $n = 1$	1	
Assume $1^2 \times 2 + 2^2 \times 3 + \ldots + k^2(k+1) = \frac{k(k+1)(k+2)(3k+1)}{12}$	1	
(for some positive integer k)		
Then $1^2 \times 2 + 2^2 \times 3 + \ldots + k^2(k+1) + (k+1)^2(k+2)$		
$= \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2)$	1	
$= \frac{(k+1)(k+2)}{12}[k(3k+1) + 12(k+1)]$		
$= \frac{(k+1)(k+2)(3k^2+13k+12)}{12}$		
$= \frac{(k+1)(k+2)(k+3)[3(k+1)+1]}{12}$	1	
$\therefore$ The statement is also true for $n = k+1$		
(if it is true for $n = k$)		
(By the principle of mathematical induction)		
$\therefore$ the statement is true for all +ve integers n.	1	
	5	
2. (a) $\sqrt{3}\cos x - \sin x$		OR
$= 2(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x)$		$r\cos\alpha = \sqrt{3}$ $r\sin\alpha = 1$
$= 2\cos(x + \frac{\pi}{6})$	1A+1A	$r = 2,\ \alpha = \frac{\pi}{6}$ or 30°
$2\cos(x + \frac{\pi}{6}) = 1$		
$x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$	1M+1A	1M for $(2n\pi \pm \alpha)$ 1A for $\frac{\pi}{3}$
$x = 2n\pi + \frac{\pi}{6}$ or $2n\pi - \frac{\pi}{2}$	1A	no mark if degree is used
	5	

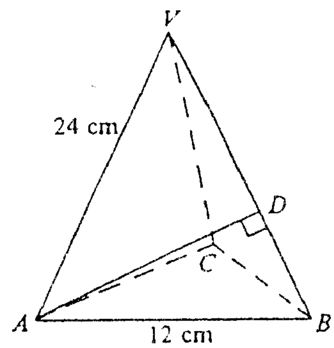
Solution	Marks	Remarks
3. (a) $(1+4x+x^2)^n$		
$= [1 + x(4+x)]^n$	1M	For separating into 2 terms
$= 1 + nx(4+x) + \frac{n(n-1)}{2}x^2(4+x)^2 + \ldots$	1A	Accept ${}_nC_r$ notation (pp − 1) for omitting dots
$\therefore a = 4n$	1A	
$b = n + 8n(n-1)$		
$= 8n^2 - 7n$	1A	
(b) $n = 5$	1A	
$b = 165$	1A	
	6	
4. Let slope of the line be m		
$\dfrac{m - \frac{1}{3}}{1 + \frac{m}{3}} = \pm 1$	1A+1A	$\left\lvert\dfrac{m - \frac{1}{3}}{1 + \frac{m}{3}}\right\rvert = 1$
$m = 2$ or $-\frac{1}{2}$	1A+1A	
Equation of lines are		
$\frac{y-3}{x-4} = 2$ i.e. $y = 2x - 5$		$2x - y - 5 = 0$
$\frac{y-3}{x-4} = -\frac{1}{2}$ $\quad y = -\frac{x}{2} + 5$	1A+1A	$x + 2y - 10 = 0$
	6	

Alternative solution

Solution	Marks
Let the angle of inclination of $y = \frac{1}{3}x$ be θ	
$\tan\theta = \frac{1}{3}$	
Angles of inclination of the two lines $= \theta \pm \frac{\pi}{4}$	1M+1M
Slope $m = \tan(\theta + \frac{\pi}{4}) = \dfrac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}} = 2$	1A+1A
or $m = \tan(\theta - \frac{\pi}{4}) = \dfrac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta\tan\frac{\pi}{4}} = -\frac{1}{2}$	
Equation of the lines are $y = 2x - 5$ and $y = \frac{-x}{2} + 5$	1A+1A

Solution	Marks	Remarks
$\left\lvert 1 + \frac{m}{3} \right\rvert = \left\lvert m - \frac{1}{3} \right\rvert$	2A	
$1 + \frac{2}{3}m + \frac{m^2}{9} = m^2 - \frac{2}{3}m + \frac{1}{9}$		
$2m^2 - 3m - 2 = 0$		
$m = 2$ or $-\frac{1}{2}$	2A	
Equation of lines are		
$y = 2x - 5$ and $y = \frac{-x}{2} + 5$	1A+1A	
5. (a) $\sin x = \cos x$		
$\tan x = 1$		
$x = \frac{\pi}{4}, \frac{5\pi}{4}$	1A	
∴ The coordinates of A and B are		
$(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ and $(\frac{5\pi}{4}, \frac{-\sqrt{2}}{2})$ respectively	1A	Do not accept degrees
(b) Area $= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx$	1M+1A	1A for $\int (\sin x - \cos x)dx$ 1M for limits
$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$	1A	
$= 2\sqrt{2}$	1A	
	6	

Solution	Marks	Remarks 1983
6. (a) $\frac{dy}{dx} = 3x^2 - 6x - 1$	1A	$y = \int (3x^2 - 6x - 1)dx$ omit (PP-1)
$y = x^3 - 3x^2 - x + k$	1A	
Put $x = 1,\ y = 0.\quad k = 3$	1M+1A	withhold 1A for giving $x^3 - 3x^2 - x + 3 = 0$
$\therefore\ y = x^3 - 3x^2 - x + 3$		
(b) At $x = 0$, $y = 3$	1A	
$\frac{dy}{dx} = -1$	1A	marked independently
$\therefore$ Equation of tangent is $y = -x + 3$	1A	
	7	
7. (a) $\cos \angle VBA = \frac{6}{24} = \frac{1}{4}$	1M	
$\angle VBA = 75.5°$ (75.52° No mark)	1A	or 1.32 radian
$AD = 12 \sin \angle VBA$		
$= 11.6$ cm (11.619 No mark)	1A	
(b) The angle between the two planes is $\angle ADC$	1A	
By symmetry, $CD = AD$	1A	
$\sin \frac{\angle ADC}{2} = \frac{\frac{1}{2}AC}{AD}$	1M	
$= \frac{6}{11.619}$		
$\angle ADC = 62.2°$ (62.3° No mark)	1A	or 1.09 radian
Alternative solution		
(b) The angle between the two planes is $\angle ADC$	1A	
By symmetry, $CD = AD$	1A	
$\cos \angle ADC = \frac{AD^2 + CD^2 - AC^2}{2(AD)(CD)}$	1M	
$= \frac{(11.619)^2 + (11.619)^2 - 12^2}{2(11.619)(11.619)}$		
$\angle ADC = 62.2°$	1A	
	7	



Solution	Marks	Remarks 1993
8. (a) $\cos x = \dfrac{1-t^2}{1+t^2}$	1A	
$\sin x = \dfrac{2t}{1+t^2}$	1A	
$a\cos x + b\sin x = c$		
$a\left(\dfrac{1-t^2}{1+t^2}\right) + b\left(\dfrac{2t}{1+t^2}\right) = c$	1M	
$a(1-t^2) + 2bt = c(1+t^2)$		
$(a+c)t^2 - 2bt + (c-a) = 0 \;\ldots\ldots(*)$	1	
If E has solutions in x, $(*)$ has solutions in t		
$(2b)^2 - 4(a+c)(c-a) \geq 0$	1M	
$b^2 - (c^2 - a^2) \geq 0$		
$a^2 + b^2 \geq c^2$	1	
	6	
(b) (i) Put $a = 5$, $b = 6$, $c = 7$ into $(*)$	1M	
$12t^2 - 12t + 2 = 0$	1A	
The roots are $\tan\frac{x_1}{2}$, $\tan\frac{x_2}{2}$		
$\therefore \tan\frac{x_1}{2} + \tan\frac{x_2}{2} = 1$	1M	or $\tan\frac{x_1}{2} = \dfrac{3+\sqrt{3}}{6}$
$\tan\frac{x_1}{2}\tan\frac{x_2}{2} = \frac{1}{6}$	1M	$\tan\frac{x_2}{2} = \dfrac{3-\sqrt{3}}{6}$
$\tan\left(\dfrac{x_1+x_2}{2}\right)$		
$= \dfrac{\tan\frac{x_1}{2} + \tan\frac{x_2}{2}}{1 - \tan\frac{x_1}{2}\tan\frac{x_2}{2}}$	1A	
$= \dfrac{1}{1-\frac{1}{6}} = \dfrac{6}{5}$	1A	
(ii) $\tan x_1 \tan x_2$		
$= \dfrac{2\tan\frac{x_1}{2}}{1-\tan^2\frac{x_1}{2}} \cdot \dfrac{2\tan\frac{x_2}{2}}{1-\tan^2\frac{x_2}{2}}$	1A	
$= \dfrac{4\tan\frac{x_1}{2}\tan\frac{x_2}{2}}{1 - \left(\tan\frac{x_1}{2} + \tan\frac{x_2}{2}\right)^2 + 2\tan\frac{x_1}{2}\tan\frac{x_2}{2} + \left(\tan\frac{x_1}{2}\tan\frac{x_2}{2}\right)^2}$	1M	For expressing the denominator in terms of sum and product.
$= \dfrac{4\left(\frac{1}{6}\right)}{1 - 1 + 2\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2}$		
$= \dfrac{24}{13}$	2A	
	10	

Solution	Marks	Remarks 1993
9. (a) $\frac{d}{dx}(\sin^{m-1}x\cos^{n+1}x)$		
$= (m-1)\sin^{m-2}x\cos^{n+2}x - (n+1)\sin^m x\cos^n x$	1A+1A	
	2	
(b) Integrating with respect to x,		
$[\sin^{m-1}x\cos^{n+1}x]_0^{\pi/2} = (m-1)\int_0^{\pi/2}\sin^{m-2}x\cos^{n+2}x\,dx - (n+1)\int_0^{\pi/2}\sin^m x\cos^n x\,dx$	1M+1A	(pp − 1) for omitting limits
$0 = (m-1)\int_0^{\pi/2}\sin^{m-2}x\cos^{n+2}x\,dx - (n+1)\int_0^{\pi/2}\sin^m x\cos^n x\,dx$	1A	For L.H.S. = 0
$(n+1)\int_0^{\pi/2}\sin^m x\cos^n x\,dx = (m-1)\int_0^{\pi/2}\sin^{m-2}x\cos^n x(1-\sin^2 x)\,dx$	1M	For rewriting $\cos^{n+2}x = \cos^n x(1-\sin^2 x)$
$(n+1+m-1)\int_0^{\pi/2}\sin^m x\cos^n x\,dx = (m-1)\int_0^{\pi/2}\sin^{m-2}x\cos^n x\,dx$		
$\int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{m-1}{m+n}\int_0^{\pi/2}\sin^{m-2}x\cos^n x\,dx$	1	
	5	
(c) Put $x = \frac{\pi}{2} - y$, $dx = -dy$		
$\int_0^{\pi/2}\sin^n x\cos^m x\,dx = \int_{\pi/2}^0 \sin^n(\frac{\pi}{2}-y)\cos^m(\frac{\pi}{2}-y)(-dy)$	1A	
$= \int_0^{\pi/2}\cos^n y\sin^m y\,dy$		
$= \int_0^{\pi/2}\sin^m x\cos^n x\,dx$	1A	
$= \frac{m-1}{m+n}\int_{\pi/2}^0 \sin^{m-2}(\frac{\pi}{2}-y)\cos^n(\frac{\pi}{2}-y)(-dy)$	1A	
$= \frac{m-1}{m+n}\int_0^{\pi/2}\cos^{m-2}y\sin^n y\,dy$		
$= \frac{m-1}{m+n}\int_0^{\pi/2}\sin^n x\cos^{m-2}x\,dx$	1	

Solution	Marks	Remarks
Alternative solution		
Put $x = \frac{\pi}{2} - y$, $dx = -dy$		
The identity in (b) becomes		
$\int_{\pi/2}^{0} \sin^m(\frac{\pi}{2} - y)\cos^n(\frac{\pi}{2} - y)(-dy)$		
$= \frac{m-1}{m+n}\int_{\pi/2}^{0} \sin^{m-2}(\frac{\pi}{2} - y)\cos^n(\frac{\pi}{2} - y)(-dy)$	1A+1A	
$\int_0^{\pi/2} \cos^m y \sin^n y\,dy = \frac{m-1}{m+n}\int_0^{\pi/2}\cos^{m-2} y \sin^n y\,dy$	1A	
$\int_0^{\pi/2} \sin^n x \cos^m x\,dx = \frac{m-1}{m+n}\int_0^{\pi/2}\sin^n x \cos^{m-2} x\,dx$	1	
	4	
(d) $\int_0^{\pi/2} \sin^4 x \cos^6 x\,dx$		
$= \frac{4-1}{4+6}\int_0^{\frac{\pi}{2}} \sin^2 x \cos^6 x\,dx$ (using (b))	1M	For using (b)
$= \frac{3}{10}\cdot\frac{2-1}{2+6}\int_0^{\frac{\pi}{2}} \sin^0 x \cos^6 x\,dx$ (using (b))		
$= \frac{3}{10}\cdot\frac{1}{8}\cdot\frac{6-1}{6}\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$ (using (c))	1M	For using (c)
$= \frac{3}{10}\cdot\frac{1}{8}\cdot\frac{5}{6}\cdot\frac{4-1}{4}\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ (using (c))		
$= \frac{3}{10}\cdot\frac{1}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{2-1}{2}\int_0^{\frac{\pi}{2}} \cos^0 x\,dx$ (using (c))	1M	For evaluating the last integral, accept stopping at $\int\cos^2 x\,dx$, $\int\sin^2 x\,dx$, $\int\sin^0 x\,dx$ or $\int\cos^2 x\sin^2 x\,dx$
$= \frac{3}{10}\cdot\frac{1}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$		
$= \frac{3\pi}{512}$	2A	
	5	

Solution	Marks	Remarks
10. (a) $\frac{y-s^2}{x-2s} = \frac{t^2-s^2}{2t-2s}$	1A	
$2y - 2s^2 = (t+s)x - 2s(t+s)$		
$y = \frac{s+t}{2}x - st$	1A	$(s+t)x - 2y - 2st = 0$
	2	
(b) Put $t = s$	1A	
Equation of tangent is $y = sx - s^2$	1A	
Alternative solutions		
Using the formula $\frac{1}{2}(y + y_1) = \frac{1}{4}xx_1$		
Equation of tangent is $\frac{1}{2}(y + s^2) = \frac{1}{4}x(2s)$	1A	
$y = sx - s^2$	1A	
$\frac{dy}{dx} = \frac{1}{2}x$		
At $(2s, s^2)$, $\frac{dy}{dx} = s$	1A	
Equation of tangent is $\frac{y - s^2}{x - 2s} = s$		
$y = sx - s^2$	1A	
	2	
(c) (i) Substitute (0, 1) into $y = \frac{s+t}{2}x - st$	1M	
$1 = \frac{s+t}{2}(0) - st$		
$st = -1$	1	
(ii) slope of $PS = s$ slope of $PT = t$	1A	
From (i), $st = -1$	1A	
$\therefore PS$ and PT are $\perp$, angle btn them $= \frac{\pi}{2}$	1A	Accept $PS \perp PT$
Alternative solution		
Let θ be the angle between PS and PT		
$\tan\theta = \frac{m_{PS} - m_{PT}}{1 + m_{PS}m_{PT}} = \frac{s-t}{1+st}$	1A	
$\because st = -1$	1A	
$\therefore \theta = \frac{\pi}{2}$	1A	Accept $PS \perp PT$

Solution	Marks	Remarks
(iii) $\begin{cases} y = sx - s^2 \\ y = tx - t^2 \end{cases}$	1A	Equation of PT
Eliminating x,		
$\frac{y + s^2}{s} = \frac{y + t^2}{t}$	1M	For solving y
$ty + s^2t = sy + st^2$		
$(t - s)y = st(t - s)$		
$y = st = -1 \quad (\because s \neq t)$		
$\therefore P$ lies on the line $y + 1 = 0$	1	
Alternative solution		
$\begin{cases} y = sx - s^2 \ldots\ldots (1) \\ y = tx - t^2 \ldots\ldots (2) \end{cases}$	1A	Equation of PT
Since $st = -1$, (2) becomes $y = \frac{-1}{s}x - \frac{1}{s^2}$		
$s^2y = -sx - 1 \ldots\ldots (3)$		
(1) + (3) : $(1 + s^2)y = -(1 + s^2)$	1M	For solving y
$y = -1$		
$\therefore P$ lies on the line $y + 1 = 0$	1	
(iv) Let (x, y) be a point on the locus.		
$\begin{cases} x = s + t \\ y = \frac{1}{2}(s^2 + t^2) \end{cases}$	1A	
$2y = s^2 + t^2$		
$= (s + t)^2 - 2st$	1M	1M for completing square
$2y = x^2 + 2$	1M+1A	1M for using $st = -1$
$\therefore$ The equation of the locus is $2y = x^2 + 2$		
Alternative solution		
$\begin{cases} x = s + t \\ y = \frac{1}{2}(s^2 + t^2) \end{cases}$	1A	
Since $st = -1$		
$\begin{cases} x = s - \frac{1}{s} \\ y = \frac{1}{2}(s^2 + \frac{1}{s^2}) \end{cases}$	1M	For using $st = -1$
$x^2 = (s - \frac{1}{s})^2$	1M	For completing square
$= (s^2 + \frac{1}{s^2}) - 2$		
$x^2 = 2y - 2$	1A	
$\therefore$ The equation of the locus is $x^2 = 2y - 2$		
	12	

Solution	Marks	Remarks
11. (a) $AB = \sqrt{(0-3)^2 + (2-\frac{3}{4})^2}$		
$= \frac{13}{4}$	1A	
Radius of $C_2 = \frac{3}{4}$	1A	
Radius of C_1 − radius of C_2	1M	
$= 4 - \frac{3}{4} = \frac{13}{4} = AB$		
$\therefore C_1$ and C_2 touch each other.	1 / 4	
(b) $PA = \sqrt{s^2 + (t-2)^2}$	1A	
If the circle touches the x-axis and C_1,		
$\sqrt{s^2 + (t-2)^2} = 4 - t$	1M	no mark for $t - 4$
$s^2 + (t-2)^2 = (4-t)^2$		
$4t = 12 - s^2$	1 / 3	
(c) $PB = \sqrt{(s-3)^2 + (t-\frac{3}{4})^2}$	1A	
If the circle touches the x-axis and C_2,		
$\sqrt{(s-3)^2 + (t-\frac{3}{4})^2} = t + \frac{3}{4}$	1M	
$(s-3)^2 + (t-\frac{3}{4})^2 = (t+\frac{3}{4})^2$		
$3t = (s-3)^2$	1 / 3	
(d) $\begin{cases} 4t = 12 - s^2 \\ 3t = (s-3)^2 \end{cases}$		
Eliminating t,		
$\frac{12 - s^2}{4} = \frac{(s-3)^2}{3}$	1M	or eliminating s
$36 - 3s^2 = 4s^2 - 24s + 36$		
$7s^2 - 24s = 0$	1A	
$s = 0$, $t = 3$	1A	
or $s = \frac{24}{7}$, $t = \frac{3}{49}$	1A	
$\therefore$ The equations of the 2 circles are		
$x^2 + (y-3)^2 = 3^2$	1A	
and $(x - \frac{24}{7})^2 + (y - \frac{3}{49})^2 = (\frac{3}{49})^2$	1A / 6	

Solution	Marks	Remarks
12. (a) Capacity $= \int_0^{\frac{\pi}{2}} \pi x^2 dy$	1A+1A	1A for $\int \pi x^2 dy$ 1A if others correct
$= \int_0^{\frac{\pi}{2}} \pi k^2 \sin^2 y dy$	1M	Substituting $x = k\sin y$
$= \pi k^2 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2y) dy$	1M	For $\sin^2 y = \frac{1}{2}(1 - \cos 2y)$
$= \pi k^2 [\frac{1}{2} y - \frac{1}{4}\sin 2y]_0^{\frac{\pi}{2}}$	1A	
$= \frac{1}{4} k^2 \pi^2$	1	
	6	
(b) (i) Put $x = 4$, $y = \frac{\pi}{2}$ in $x = k\sin y$		
$k = 4$	1A	
$\therefore$ Volume of water $= \frac{1}{4}(4)^2\pi^2 = 4\pi^2$	1A	
(ii) Let V be the volume of water remaining after t minutes		
$\frac{dV}{dt} = -(\pi + 2t)$	1A	
$V = -(\pi t + t^2) + c$	1A	
After $t = 0$, $V = 4\pi^2$, $\therefore c = 4\pi^2$	1M+1A	
$\therefore V = 4\pi^2 - (\pi t + t^2)$		
Alternative solution		
Volume remaining, $V = 4\pi^2 - \int_0^t (\pi + 2t) dt$	1M+1A	
$= 4\pi^2 - [\pi t + t^2]_0^t$	1A	
$= 4\pi^2 - (\pi t + t^2)$	1A	
Let V be the volume of water pumped away		
$\frac{dV}{dt} = \pi + 2t$	1A	
$V = \pi t + t^2 + c$	1A	
At $t = 0$, $V = 0$ $\therefore c = 0$	1M+1A	
$\therefore V = \pi t + t^2$		
Volume pumped away $= \int_0^t (\pi + 2t) dt$	1M+1A	
$= [\pi t + t^2]_0^t$	1A	
$= \pi t + t^2$	1A	

Solution	Marks	Remarks
Put $V = 2\pi^2$	1M	
$t^2 + \pi t - 2\pi^2 = 0$		
$t = \pi$ [or -2π (rejected)]	1A	
∴ Time required to pump out half of the water $= \pi$ (minutes)		
Put $V = 0$,		
$t^2 + \pi t - 4\pi^2 = 0$		
$t = \frac{-\pi + \sqrt{17}\pi}{2}$ [or $\frac{-\pi - \sqrt{17}\pi}{2}$ (rejected)]	1A	
∴ Time required to pump out the remaining water		
$= (\frac{\sqrt{17} - 1}{2})\pi - \pi$		
$= (\frac{\sqrt{17} - 3}{2})\pi$ (minutes)	1A	
	10	