12. Let C be the locus of the point in an Argand diagram representing the complex number $z = (\cos\theta - 1) + i\sin\theta$, where $0 \leq \theta < 2\pi$.

(a) Show that $|z + 1| = 1$.
Hence sketch C in an Argand diagram.
(5 marks)

(b) Let P_1 be the point on C representing the complex number $z_1 = (\cos\theta_1 - 1) + i\sin\theta_1$, such that $\arg z_1 = 2\theta_1$, and $0 < \theta_1 < \frac{\pi}{2}$.
Find the value of θ_1 and express z_1 in standard form.
(7 marks)

(c) Let P_2 be the point on C which is farthest away from the point P_1 in (b). Find the complex number represented by P_2 in standard form.
(4 marks)

END OF PAPER

93-CE
A MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1993

ADDITIONAL MATHEMATICS PAPER II

11.15 am-1.15 pm (2 hours)
This paper must be answered in English

Answer **ALL** questions in Section A and any **THREE** questions in Section B.

All working must be clearly shown.

Unless otherwise specified in a question, the **exact values** of numerical answers must be given.

Section A (42 marks)

Answer **ALL** questions in this section.

1. Prove that

$$1^2 \times 2 + 2^2 \times 3 + \ldots + n^2(n+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$$

for any positive integer n.

(5 marks)

2. By expressing $\sqrt{3}\cos x - \sin x$ in the form $r\cos(x+\alpha)$, find the general solution of the equation

$$\sqrt{3}\cos x - \sin x = 1 ,$$

giving your answer in terms of π.

(5 marks)

3. Given $(1 + 4x + x^2)^n = 1 + ax + bx^2 +$ other terms involving higher powers of x, where n is a positive integer.

(a) Express a and b in terms of n.

(b) If $a = 20$, find n and b.

(6 marks)

4. Two lines pass through (4, 3) and each line makes an angle $\frac{\pi}{4}$ with the line $y = \frac{1}{3}x$. Find the equations of the two lines.

(6 marks)

5.

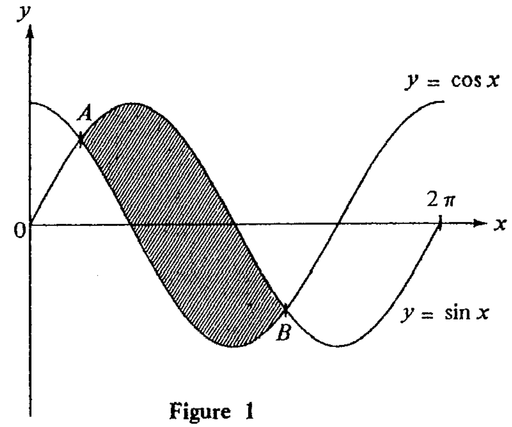

Figure 1

Figure 1 shows the curves of $y = \sin x$ and $y = \cos x$, where $0 \le x \le 2\pi$, intersecting at points A and B.

(a) Find the coordinates of A and B.

(b) Find the area of the shaded region as shown in Figure 1.

(6 marks)

6. The slope of a curve C at any point (x, y) on C is $3x^2 - 6x - 1$. C passes through the point $(1, 0)$.

(a) Find the equation of C.

(b) Find the equation of the tangent to C at the point where C cuts the y-axis.

(7 marks)

7.

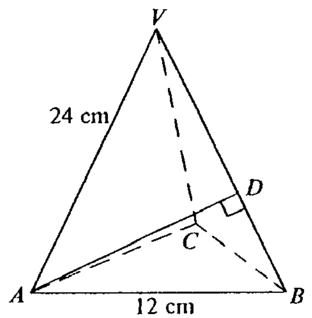

Figure 2

In Figure 2, $VABC$ is a right pyramid whose base ABC is an equilateral triangle. $AB = 12$ cm and $VA = 24$ cm. D is a point on VB such that AD is perpendicular to VB. Find, correct to 3 significant figures,

(a) $\angle VBA$ and AD,

(b) the angle between the faces VAB and VBC.

(7 marks)

Section B (48 marks)

Answer any **THREE** questions in this section.
Each question carries 16 marks.

8. Given $-\pi < x < \pi$ and $t = \tan\frac{x}{2}$.

(a) By expressing $\sin x$ and $\cos x$ in terms of t, show that the equation in x

$$E: a\cos x + b\sin x = c$$

can be expressed as $(a + c)t^2 - 2bt + (c - a) = 0$.

Hence show that if E has solutions, then

$$a^2 + b^2 \geq c^2.$$

(6 marks)

(b) Let x_1, x_2 be the two values of x satisfying the equation

$$5\cos x + 6\sin x = 7.$$

Without evaluating x_1 and x_2, find

(i) $\tan(\frac{x_1 + x_2}{2})$.

(ii) $\tan x_1 \tan x_2$.

(10 marks)

9. Let m, n be integers such that $m > 1$ and $n \geq 0$.

(a) Find $\frac{d}{dx}(\sin^{m-1} x \cos^{n+1} x)$.

(2 marks)

(b) Using the result of (a), show that

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x \, dx .$$

(5 marks)

(c) Using the result of (b) and the substitution $x = \frac{\pi}{2} - y$, show that

$$\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x \, dx = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^n x \cos^{m-2} x \, dx .$$

(4 marks)

(d) Using the results of (b) and (c), evaluate

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx .$$

(5 marks)

10. $S(2s, s^2)$, $T(2t, t^2)$ are two distinct points on the parabola $y = \frac{1}{4}x^2$.

(a) Find the equation of the chord ST.

(2 marks)

(b) Find the equation of the tangent to the parabola at S.

(2 marks)

(c) Suppose ST passes through the point $F(0, 1)$ and the tangents to the parabola at S and T meet at a point P.

(i) Show that $st = -1$.

(ii) Find the angle between PS and PT.

(iii) Show that P lies on the line $y + 1 = 0$.

(iv) Find the equation of the locus of the mid-point of ST as S and T move along the parabola.

(12 marks)

11.

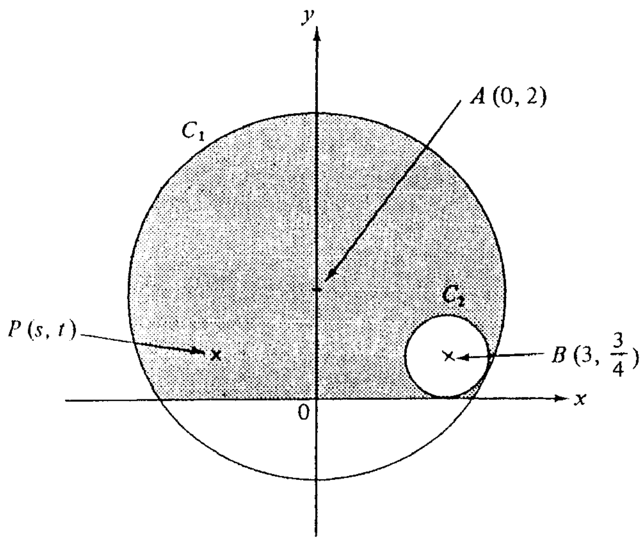

Figure 3

$A\ (0,\ 2)$ is the centre of circle C_1 with radius 4. $B(3,\ \frac{3}{4})$ is the centre of circle C_2 which touches the x-axis. $P(s,\ t)$ is any point in the shaded region as shown in Figure 3.

(a) Find AB and the radius of C_2.

Hence show that C_1 and C_2 touch each other.

(4 marks)

(b) If P is the centre of a circle which touches the x-axis and C_1, show that $4t = 12 - s^2$.

(3 marks)

(c) If P is the centre of a circle which touches the x-axis and C_2, show that $3t = (s-3)^2$.

(3 marks)

(d) Given that there are two circles in the shaded region, each of which touches the x-axis, C_1 and C_2. Using (b) and (c), find the equations of the two circles, giving your answers in the form $(x-h)^2 + (y-k)^2 = r^2$.

(6 marks)

12.

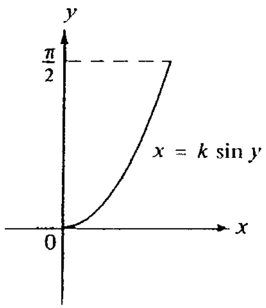

Figure 4 (a)

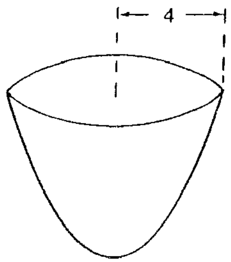

Figure 4 (b)

Figure 4 (a) shows the curve $x = k \sin y$, where $k > 0$ and $0 \leq y \leq \frac{\pi}{2}$. A bowl is formed by revolving the curve about the y-axis.

(a) Show that the capacity of the bowl is $\frac{1}{4}k^2\pi^2$ cubic units.

(6 marks)

(b) Given that the radius of the rim of the bowl is 4 units. (See Figure 4 (b).) The bowl is full of water.

(i) Find the volume of water.

(ii) The water is now pumped out of the bowl at a rate of $(\pi + 2t)$ cubic units per minute, where t is the time in minutes after pumping starts.

Find the time taken to pump out half of the water and the time taken to pump out the remaining water in the bowl. Give both answers in terms of π.

(10 marks)

END OF PAPER