11. O, A, B are the points (0, 0), (10, 4), (5, 10) respectively.
C is a point on OB such that $OC : CB = 1 : r$ and
D is a point on OA such that $OD : DA = r : 1$, where $r > 0$.

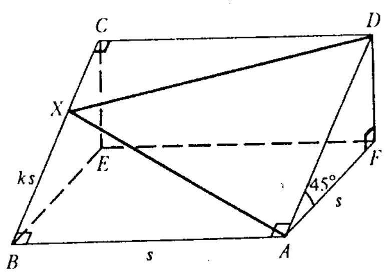
(a) Express the coordinates of C and D in terms of r.

(b) Express the area of ΔODC in terms of r.

(c) If the area of ΔODC is k times the area of ΔOAB, express r in terms of k.
Hence, or otherwise, show that $k \leqslant \frac{1}{4}$.

(d) Using the result in (c), or otherwise, find the maximum area of ΔODC.

12. The figure shows a path AXD on the inclined plane $ABCD$. AX and XD are straight lines. The inclined plane is at $45°$ to the horizontal plane $ABEF$. Let $AB = AF = s$, $BX = ks$, and α be the angle between AX and the horizontal.



(a) Express the length of AX in terms of s and k.

(b) Express $\sin\alpha$ in terms of k.

(c) If the inclination of AX to the horizontal is not to exceed $30°$, find the range of values of k.
Hence, or otherwise, determine the range of values of k so that each of the inclinations of AX and XD to the horizontal does not exceed $30°$.

END OF PAPER

香 港 考 試 局

HONG KONG EXAMINATIONS AUTHORITY

一 九 八 〇 年 香 港 中 學 會 考

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1980

附加數學 試卷二

二小時完卷

上午十一時十五分至下午一時十五分

本試卷必須用英文作答

ADDITIONAL MATHEMATICS PAPER II

Two hours

11.15 a.m.—1.15 p.m.

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

SECTION A (40 marks)

Answer ALL questions in this section.

1. Expand $(1 + 2x)^3 (1 + 3x)^4$ in ascending powers of x as far as the term containing x^2.
(5 marks)

2. Using the substitution $u = x - 1$, find the indefinite integral $\int (x + 2)\sqrt{x - 1}\, dx$.
(5 marks)

3. Find the slope of the tangent to the curve

$$2x^2y + x^2 + y^2 - 4 = 0$$

at the point (2, 0).
(6 marks)

4. If $y = \cos(\sin x)$, find $\frac{d^2y}{dx^2}$.
(6 marks)

5. Solve the equation $\log_4 x - \log_x 16 = 1$.

(6 marks)

6. $A(2, \frac{1}{2})$ and $B(-4, 2)$ are two points on the parabola $x^2 = 8y$.

Find the area of the region enclosed by the parabola and its chord AB.

(6 marks)

7. A is the fixed point $(-1, 2)$. P is a variable point moving on the circle

$$x^2 + y^2 - 2x - 4y - 5 = 0.$$

If M is the mid-point of AP, find the equation of the locus of M.

(6 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.

Each question carries 20 marks.

8. (a) Find the equation of the tangent to the parabola

$$y^2 = 4x$$

at the point $(1, 2)$.

(b) Given the curve

$$C: (x - 1)^2 + (y - 2)^2 + k(y - x - 1) = 0,$$

where k is a non-zero constant,

(i) show that C represents a circle passing through the point $(1, 2)$,

(ii) find the coordinates of the centre of C in terms of k,

(iii) find the equation of the tangent to C at the point $(1, 2)$.

(c) A circle and the parabola

$$y^2 = 4x$$

have a common tangent at $(1, 2)$. Also the centre of this circle lies on the line

$$x - y = 3.$$

Using the results of (a) and (b), or otherwise, find the equation of the circle.

9. $A(5, 3)$ and $B(-5, -3)$ are two given points. P is a variable point such that the product of the slopes of the lines PA and PB is equal to a constant k.

(a) Find the equation of the locus of P.

(b) Write down the range of values of k for which the locus of P is

(i) a circle,

(ii) an ellipse but not a circle,

(iii) a hyperbola.

Name the locus of P if $k = 0$; if $k = \frac{9}{25}$.

(c) Let C be the locus of P when $k = 1$. Find the volume of the solid of revolution obtained by revolving the region enclosed by C and the lines $y = \pm 3$ about the y-axis.

(You may leave your answer in terms of π.)

10. Let α, β be the roots of

$$x^2 - 2x - 1 = 0,$$

where $\alpha > \beta$. For any positive integer n, let

$$U_n = \frac{1}{2\sqrt{2}}(\alpha^n - \beta^n),$$

$$V_n = \frac{1}{2\sqrt{2}}(\alpha^n + \beta^n).$$

(a) Show that

$$U_{n+2} = 2U_{n+1} + U_n,$$

$$V_{n+2} = 2V_{n+1} + V_n.$$

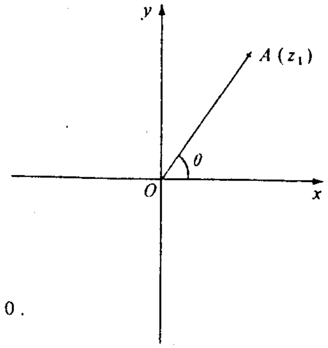
(b) (i) Find U_1 and U_2.

(ii) Suppose U_n and U_{n+1} are integers, deduce that U_{n+2} is also an integer.

(iii) Is U_n an integer for all positive integers n? Give reasons.

(c) Is V_n an integer for all positive integers n? Give reasons.

11. On the Argand plane, A, B and C represent three non-zero complex numbers z_1, z_2 and z_3 respectively, such that $z_2 = \omega z_1$ and $z_3 = \omega z_2$, where $\omega^3 = 1$ and $0 < \text{amp}(\omega) < \pi$.



(a) The position of A is shown in the accompanying figure. If $|z_1| = 3$ and $\text{amp}(z_1) = \theta$, draw an Argand diagram to include the points A, B and C.

(b) Find $|z_2|$ and $|z_3|$.

(c) Show that ABC is an equilateral triangle.

(d) Show that $z_1{}^2 + z_2{}^2 + z_3{}^2 + z_1z_2 + z_2z_3 + z_3z_1 = 0$.

(e) Find the amplitude of $\dfrac{z_3 - z_1}{z_2 - z_1}$.

12. (a) Given that $f(x) = f(a - x)$ for all real values of x, by using the substitution $u = a - x$,

show that $\displaystyle\int_0^a xf(x)dx = a\int_0^a f(u)du - \int_0^a uf(u)du$.

Hence deduce that

$$\int_0^a xf(x)dx = \frac{a}{2}\int_0^a f(x)dx.$$

(b) By using the substitution $u = x - \frac{\pi}{2}$, show that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 u}{\sin^4 u + \cos^4 u}\,du.$$

By using this result and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

evaluate

$$\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx.$$

(c) Using (a) and (b), evaluate

$$\int_0^{\pi} \frac{x\sin^4 x}{\sin^4 x + \cos^4 x}\,dx.$$

END OF PAPER