2013-ASL
M & S

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2013

MATHEMATICS AND STATISTICS AS-LEVEL

8.30 am – 11.30 am (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer **ALL** questions in Section A, **using the AL(E) answer book.**

3. Answer any **FOUR** questions in Section B, **using the AL(C) answer book.**

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers should be either exact or given to 4 decimal places.

Not to be taken away before the end of the examination session

Section A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Expand $e^{\frac{x}{16}}$ in ascending powers of x as far as the term in x^2.
Hence prove that $\sqrt{e} \approx 1.625$.

(b) Expand $(1+x)^{\frac{-1}{2}}$ in ascending powers of x as far as the term in x^2.

(c) (i) Using (a) and (b), prove that $\dfrac{e^{\frac{x}{16}}}{\sqrt{1+x}} \approx 1 - \dfrac{7}{16}x + \dfrac{177}{512}x^2$ -------------------- (*).

(ii) Mary estimates the value of $\sqrt{e}$ by putting $x = 8$ into (*) and obtains $\sqrt{e} \approx 58.875$.
What is wrong with Mary's estimation? Explain your answer.

(7 marks)

2. Let $x = \ln \dfrac{1+t}{1-t}$, where $-1 < t < 1$.

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$.

(b) Let $y = 1 + e^{-x} - e^{-2x}$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ when $t = \dfrac{1}{2}$.

(6 marks)

3. The value $\mathrm{R}(t)$, in thousand dollars, of a machine can be modelled by

$$\mathrm{R}(t) = Ae^{-0.5t} + B,$$

where t (≥ 0) is the time, in years, since the machine has been purchased. At $t = 0$, its value is 500 thousand dollars and in the long run, its value is 10 thousand dollars.

(a) Find the values of A and B.

(b) The machine can generate revenue at a rate of $\mathrm{P}'(t) = 600e^{-0.3t}$ thousand dollars per year, where t is the number of years since the machine has been purchased. Richard purchased the machine for his factory and used it for 5 years before he sold it. How much did he gain in this process? Correct your answer to the nearest thousand dollars.

(6 marks)

4.

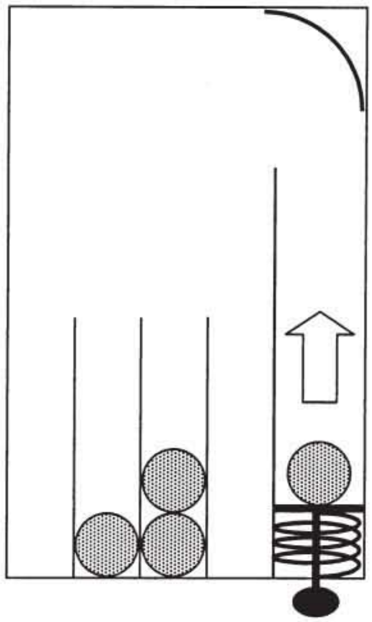


Figure 1

In a game, a player will ping 4 balls one by one and each ball will randomly fall into 4 different slots as shown in Figure 1. A prize will be given if all the 4 balls are aligned in a horizontal or a vertical row.

(a) What is the probability that a player wins the prize?

(b) What is the probability that a player wins the prize given the first two balls are in two different slots?

(6 marks)

5. Let A and B be two events. It is given that $P(A)=a$, $P(B' \mid A)=\frac{27}{32}$ and $P(A \mid B')=\frac{27}{31}$.

(a) Find $P(A \cap B')$ in terms of a .

(b) Find $P(B)$ in terms of a .

(c) It is given that $P(A \cap B)=0.1$.

(i) Find the value of a .

(ii) Determine whether A and B are independent or not.

(7 marks)

6. Let $\$X$ be the amount of money won in playing a certain game. It is known that $X \sim B(10, p)$. Two plans are proposed for calculating the game fee ($\$F$).

Plan 1: $F = (1+\theta)E(X)$,
Plan 2: $F = E(X) + 0.1\,Var(X)$,

where θ is a constant, $E(X)$ is the expected value of X and $Var(X)$ is the variance of X. It is known that the game fees are the same for both plans if $p = \frac{1}{4}$.

(a) Find θ .

(b) Show that the variance of X is the greatest when $p = \frac{1}{2}$.

(c) Determine which plan will give a lower game fee when $p = \frac{1}{2}$.

(8 marks)

Section B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Let $k \neq 0$ be a constant.

Define $f(x) = \dfrac{x}{kx-1}$ for all $x \neq \dfrac{1}{k}$, and $g(x) = f\left(\dfrac{1}{x}\right)$ for all $x \neq 0$ and $x \neq k$.

Let C_1 be the curve $y = f(x)$ and C_2 be the curve $y = g(x)$. It is given that C_2 has a vertical asymptote $x = 2$.

(a) Find the value of k .

(2 marks)

(b) Find the points of intersection of the curves C_1 and C_2 .

(2 marks)

(c) Sketch the curves C_1 and C_2 on the same diagram, indicating their asymptotes, intercepts and points of intersection.

(6 marks)

(d) Find the exact value of the area enclosed by C_1 , C_2 and the y-axis.

(5 marks)

8. In a certain country, the daily rate of change of the amount of oil production P, in million barrels per day, can be modelled by

$$\frac{dP}{dt} = \frac{k-3t}{1+ae^{-bt}}$$

where $t\ (\geq 0)$ is the time measured in days. When $\ln\left(\dfrac{k-3t}{\dfrac{dP}{dt}} - 1\right)$ is plotted against t, the graph is a straight line with slope -0.3 and the intercept on the horizontal axis 0.32. Moreover, P attains its maximum when $t = 3$.

(a) Find the values of a, b and k.

(5 marks)

(b) (i) Using trapezoidal rule with 6 subintervals, estimate the total amount of oil production from $t = 0$ to $t = 3$.

(ii)

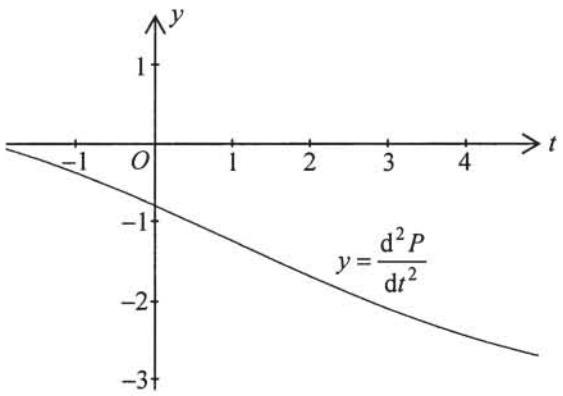


Figure 2

Figure 2 shows the graph of $y = \dfrac{d^2P}{dt^2}$. Using the graph, determine whether the estimation in (i) is an under-estimate or an over-estimate.

(4 marks)

(c) The daily rate of change of the demand for oil D, in million barrels per day, can be modelled by

$$\frac{dD}{dt} = 1.63^{2-0.1t}$$

where $t\ (\geq 0)$ is the time measured in days.

(i) Let $y = \alpha^{\beta x}$, where α, β ($\alpha > 0$, $\alpha \neq 1$ and $\beta \neq 0$) are constants. Find $\dfrac{dy}{dx}$ in terms of x.

(ii) Find the demand of oil from $t = 0$ to $t = 3$.

(iii) Does the overall oil production meet the overall demand of oil from $t = 0$ to $t = 3$? Explain your answer.

(6 marks)

9. The population size N (in trillion) of a culture of bacteria increases at the rate of $\frac{dN}{dt} = t\ln(2t+1)$, where t (≥ 0) is the time measured in days.
It is given that $N = 21$ when $t = 0$.

(a) (i) Find $\int \frac{t^2}{2t+1}\,dt$.

(ii) Find $\frac{d}{dt}[t^2 \ln(2t+1)]$.

(iii) Find the population of the culture of bacteria at $t = 5$. Correct your answer to the nearest trillion.

(8 marks)

(b) A certain kind of drug is then added to the culture of bacteria at $t = 5$. A researcher estimates that the population size M (in trillion) of the bacteria can then be modelled by

$$M = 40e^{-2\lambda(t-5)} - 20e^{-\lambda(t-5)} + K \qquad (5 \leq t \leq 18) ,$$

where t is the time measured in days. K and λ are constants. It is given that $M = 27$ when $t = 11$.

(i) Using (a), find the value of K correct to the nearest integer.
Hence, find the value of λ correct to 1 decimal place.

(ii) By using the value of K correct to the nearest integer and the value of λ correct to 1 decimal place, determine whether M is always decreasing in this model.
Hence, explain whether the population of the bacteria will drop to 23 trillion.

(7 marks)

10. The speeds of the vehicles (X km/h) on a highway follow a normal distribution with mean μ km/h and standard deviation σ km/h . It is known that 12.3% of vehicles have speeds more than 82.64 km/h and 24.2% of vehicles have speeds less than 75.2 km/h . A machine is used to detect the speeds of the vehicles at a spot on the highway. A notice will be issued to the driver if the speed of his/her vehicle is detected to be over 80 km/h .

(a) Find μ and σ .

(3 marks)

(b) (i) A vehicle passes the spot. What is the probability that a notice will be issued?

(ii) Suppose that 10 vehicles pass the spot on the highway. What is the probability that at most 2 notices will be issued?

(4 marks)

(c) On a certain day, the machine does not work properly and there is an error in detecting the speeds of the vehicles. The error (Y km/h) is defined as follows:

Y = speed detected − actual speed ,

and it can be modelled by the following probability distribution:

Error (Y)	2	$2+\theta$
Probability	0.5	0.5

where θ is a non-zero constant. A vehicle passes the spot.

(i) Find the probability that a notice will be issued but the speed of the vehicle is not over 80 km/h for the following two cases:

(1) $\theta = 1$,

(2) $\theta = -3$.

(ii) Find the range of values of θ such that the probability that a notice will not be issued but the speed of the vehicle is over 80 km/h is at most 7.125% .

(8 marks)

11. According to the school regulation, air-conditioners can only be switched on if the temperature at 8 am exceeds 26 °C. From past experience, the probability that the temperature at 8 am does NOT exceed 26 °C is q ($q > 0$). Assume that there are five school days in a week. For two consecutive school days, the probability that the air-conditioners are switched on for not more than one day is $\frac{7}{16}$.

(a) (i) Show that the probability that the air-conditioners are switched on for not more than one day on two consecutive school days is $2q - q^2$.

(ii) Find the value of q.

(2 marks)

(b) The air-conditioners are said to be *fully engaged* in a week if the air-conditioners are switched on for all five school days in a week.

(i) Find the probability that the fifth week is the second week that the air-conditioners are *fully engaged.*

(ii) What is the expected number of consecutive weeks when the air-conditioners are not *fully engaged*?

(5 marks)

(c) On a certain day, the temperature at 8 am exceeds 26 °C and all the 5 classrooms on the first floor are reserved for class activities after school. There are 2 air-conditioners in each classroom. The number of air-conditioners being switched off in the classroom after school depends on the number of students staying in the classroom. Assume that the number of students in each classroom is independent.

Case	I	II	III
Number of air-conditioners being switched off	2	1	0
Probability	0.25	0.3	0.45

(i) What is the probability that all air-conditioners are switched off on the first floor after school?

(ii) Find the probability that there are exactly 2 classrooms with no air-conditioners being switched off and at most 1 classroom with exactly 1 air-conditioner being switched off on the first floor after school.

(iii) Given that there are 6 air-conditioners being switched off on the first floor after school, find the probability that at least 1 classroom has no air-conditioners being switched off.

(8 marks)

12. A group of 5 members is waiting for a mini-bus to Mong Kok at a mini-bus station. It is known that there is one mini-bus every fifteen minutes and the number of empty seats on a mini-bus can be modelled by a Poisson distribution with mean λ . The probability that each of three consecutive mini-buses has at least one empty seat is 0.6465 . Assume the number of empty seats for each mini-bus is independent and the 5 members want to travel together.

(a) Find λ . Correct your answer to the nearest integer.

(2 marks)

(b) By using the λ corrected to the nearest integer, find the probability that

(i) the 5 members cannot get on the first arriving mini-bus together,

(ii) the 5 members will have to wait for more than two mini-buses.

(4 marks)

(c) After waiting for a long time, the 5 members decided to break up into a group of 2 members and a group of 3 members.

- All the 5 members will wait for the coming mini-buses if the mini-bus has less than two empty seats.
- The group of 2 members will get on a mini-bus if the mini-bus has exactly two empty seats and the group of 3 members will wait for the coming mini-buses.
- The group of 3 members will get on a mini-bus if the mini-bus has three or four empty seats and the group of 2 members will wait for the coming mini-buses.
- All the 5 members will get on a mini-bus if the mini-bus has at least five empty seats.

By using the λ corrected to the nearest integer, find the probability that

(i) the group of 2 members gets on the first arriving mini-bus and the group of 3 members gets on the next mini-bus,

(ii) none of the members have to wait for more than two mini-buses,

(iii) the group of 2 members will go first given that some members have to wait for more than two mini-buses.

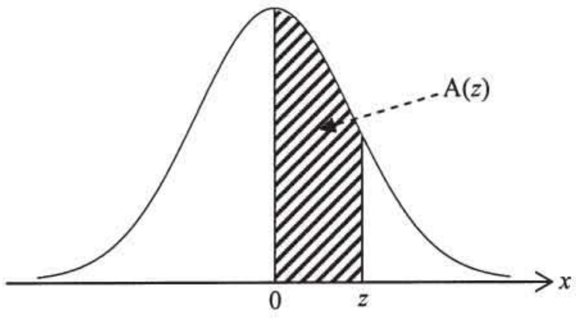
(9 marks)

END OF PAPER

Standard Normal Distribution Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

Note : An entry in the table is the area under the standard normal curve between $x = 0$ and $x = z$ $(z \geq 0)$. Areas for negative values of z can be obtained by symmetry.



$$A(z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$