1. Binomial Expansion

Learning Unit	Learning Objective
Foundation Knowledge Area	
1. Binomial expansion	1.1 recognise the expansion of $(a+b)^n$, where n is a positive integer

Section A

1. (a) Expand e^{-18x} in ascending powers of x as far as the term in x^2.

   (b) Let n be a positive integer. If the coefficient of x^2 in the expansion of $e^{-18x}(1+4x)^n$ is -38. Find n.

   (6 marks) (2019 DSE-MATH-M1 Q6)

2. Let k be a constant.

   (a) Expand $e^{kx}+e^{2x}$ in ascending powers of x as far as the term in x^2.

   (b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx}+e^{2x}-1)$ are equal, find k.

   (6 marks) (2018 DSE-MATH-M1 Q6)

3. (a) Expand $(1+e^{3x})^2$ in ascending powers of x as far as the term in x^2.

   (b) Find the coefficient of x^2 in the expansion of $(5-x)^4(1+e^{3x})^2$.

   (6 marks) (2017 DSE-MATH-M1 Q5)

4. Let k be a constant.

   (a) Expand e^{kx} in ascending powers of x as far as the term in x^2.

   (b) If the coefficient of x in the expansion of $(1+2x)^7e^{kx}$ is 8, find the coefficient of x^2.

   (5 marks) (2016 DSE-MATH-M1 Q5)

5. (a) Expand e^{-4x} in ascending powers of x as far as the term in x^2.

   (b) Find the coefficient of x^2 in the expansion of $\frac{(2+x)^5}{e^{4x}}$.

   (5 marks) (2015 DSE-MATH-M1 Q5)

6. The slope of the tangent to a curve S at any point (x,y) on S is given by $\frac{dy}{dx}=\left(2x-\frac{1}{x}\right)^3$, where $x>0$.
   A point $P(1,5)$ lies on S.

   (a) Find the equation of the tangent to S at P.

   (b) (i) Expand $\left(2x-\frac{1}{x}\right)^3$.

   (ii) Find the equation of S for $x>0$.

   (7 marks) (2014 DSE-MATH-M1 Q3)

7. (a) Expand $\left(u+\frac{1}{u}\right)^4$ in descending powers of u.

   (b) Express $(e^{ax}+e^{-ax})^4$ in ascending powers of x up to the term in x^2.

   (c) Suppose the coefficient of x^2 in the result of (b) is 2. Find all possible values of a.

   (5 marks) (2013 DSE-MATH-M1 Q1)

8. Let n be a positive integer.

   (a) Expand $(1+3x)^n$ in ascending powers of x up to the term x^2.

   (b) It is given that the coefficient of x^2 in the expansion of $e^{-2x}(1+3x)^n$ is 62. Find the value of n.

   (4 marks) (2012 DSE-MATH-M1 Q1)

9. (a) Expand $(2x+1)^3$.

   (b) Expand e^{-ax} in ascending powers of x as far as the term in x^2, where a is a constant.

   (c) If the coefficient of x^2 in the expansion of $\frac{(2x+1)^3}{e^{ax}}$ is -4, find the value(s) of a.

   (5 marks) (PP DSE-MATH-M1 Q1)

10. Expand the following in ascending powers of x as far as the term in x^2:

(a) e^{-2x};

(b) $\frac{(1+2x)^6}{e^{2x}}$.

(4 marks) (SAMPLE DSE-MATH-M1 Q1)

Math & Stat

11. (a) (i) Expand $(x+y+z)^2$.

(ii) Find the coefficients of x^3y , x^3z , xy^3 , y^3z , xz^3 and yz^3 in the expansion of $(x+y+z)^4$.

(b) If a cup is randomly selected from a box containing red cups, blue cups and green cups, the probabilities of getting a red cup, a blue cup and a green cup are p, q and r respectively.
If 4 cups are randomly selected from the box one by one with replacement, find, in terms of p, q and r ,

(i) the probability that at least 2 cups of different colours are selected;

(ii) the probability that exactly 3 cups of the same colour are selected.

(7 marks)
(2004 ASL-M&S Q4)

Out of Syllabus

12. (a) Expand e^{-2x} in ascending powers of x as far as the term in x^3.

(b) Using (a), expand $\frac{(1+x)^{\frac{1}{2}}}{e^{2x}}$ in ascending powers of x as far as the term in x^3 .

State the range of values of x for which the expansion is valid.

(6 marks) (1999 ASL-M&S Q2)

13. (a) Prove that $\frac{1}{1+\sqrt{1-x}}=\frac{1}{x}\left(1-\sqrt{1-x}\right)$ for $x<1$ and $x\neq 0$.

(b) Let $I=\int_{\frac{1}{4}}^{\frac{1}{2}}\frac{1}{1+\sqrt{1-x}}dx$. By considering the expansion of $\frac{1}{1+\sqrt{1-x}}$ in ascending powers of x as far as the term in x^2, estimate the value of I .

(c) Let $J=\int_{-2}^{-1}\frac{1}{1+\sqrt{1-x}}dx$. Can we use the same method in (b) to estimate the value of J ? Explain your answer.

(7 marks)
(2011 ASL-M&S Q1)

Suggested modification

14. (a) Expand $\frac{(1+x)^{10}-1}{x}$ in ascending powers of x as far as the term in x^2.

(b) Let $I=\int_{0.1}^{0.2}\frac{(1+x)^{10}-1}{x}dx$. By using (a), estimate the value of I .

(c) Determine whether the estimate in (b) is an over-estimate or an under-estimate.

1. Binomial Expansion

1. (2019 DSE-MATH-M1 Q6)

Solution	Marks	Remarks
(a) e^{-18x}		
$=1+(-18x)+\frac{(-18x)^2}{2!}+\cdots$	1M	
$=1-18x+162x^2+\cdots$	1A	
(b) $(1+4x)^n$		
$=1+C_1^n(4x)+C_2^n(4x)^2+\cdots+C_n^n(4x)^n$	1M	
$=1+4C_1^n x+16C_2^n x^2+\cdots+4^n x^n$		
$16C_2^n-72C_1^n+162=-38$	1M	
$16\left(\frac{n(n-1)}{2}\right)-72n+162=-38$		
$n^2-10n+25=0$	1M	
$n=5$	1A	
	----------(6)	

2. (2018 DSE-MATH-M1 Q6)

Solution	Marks	Remarks
(a) $e^{kx}+e^{2x}$		
$=\left(1+kx+\frac{(kx)^2}{2!}+\cdots\right)+\left(1+2x+\frac{(2x)^2}{2!}+\cdots\right)$	1M	for expanding e^{kx} or e^{2x}
$=2+(k+2)x+\frac{(k^2+4)}{2}x^2+\cdots$	1A	
(b) $(1-3x)^8$		
$=1+C_1^8(-3x)+C_2^8(-3x)^2+\cdots$	1M	
$=1-24x+252x^2+\cdots$		
$e^{kx}+e^{2x}-1$		
$=1+(k+2)x+\frac{(k^2+4)}{2}x^2+\cdots$		
$(1)(k+2)+(-24)(1)=(1)\left(\frac{k^2+4}{2}\right)+(-24)(k+2)+(252)(1)$	1M+1M	
$k^2-50k+456=0$		
$k=12$ or $k=38$	1A	
	----------(6)	

3. (2017 DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) $(1+e^{3x})^2$		
$=1+2e^{3x}+e^{6x}$	1M	
$=1+2\left(1+3x+\frac{(3x)^2}{2!}+\cdots\right)+\left(1+6x+\frac{(6x)^2}{2!}+\cdots\right)$	1M	for expanding e^{3x} or e^{6x}
$=4+12x+27x^2+\cdots$	1A	
$(1+e^{3x})^2$		
$=\left(1+1+3x+\frac{(3x)^2}{2!}+\cdots\right)^2$	1M	for expanding e^{3x}
$=(2)(2)+(2)(2)(3x)+(3x)(3x)+(2)(2)\left(\frac{9x^2}{2}\right)+\cdots$	1M	
$=4+12x+27x^2+\cdots$	1A	
(b) $(5-x)^4$		
$=5^4-C_1^4(5^3)x+C_2^4(5^2)x^2-C_3^4(5)x^3+x^4$	1M	
$=625-500x+150x^2-20x^3+x^4$		
The required coefficient		
$=(625)(27)+(-500)(12)+(150)(4)$	1M	withhold 1M if the step is skipped
$=11475$	1A	
	----------(6)	

(a)	Very good. Most candidates were able to expand $(1+e^{3x})^2$.
(b)	Very good. Most candidates were able to find the coefficient of x^2 .

4. (2016 DSE-MATH-M1 Q5)

(a) e^{kx}

$=1+kx+\frac{(kx)^2}{2!}+\cdots$

$=1+kx+\frac{k^2x^2}{2}+\cdots$ 1A

(b) $(1+2x)^7e^{kx}$

$=\left(1+C_1^7(2x)+C_2^7(2x)^2+\cdots+(2x)^7\right)\left(1+kx+\frac{k^2x^2}{2}+\cdots\right)$ 1M

$=\left(1+14x+84x^2+\cdots+(2x)^7\right)\left(1+kx+\frac{k^2x^2}{2}+\cdots\right)$

$\therefore\ 14+k=8$ 1M

$k=-6$

The coefficient of x^2

$=(1)\left(\frac{(-6)^2}{2}\right)+14(-6)+(84)(1)$ 1M

$=18$ 1A

----------(5)

(a)	Very good. A very high proportion of the candidates were able to expand e^{kx} while some candidates were unable to simplify the coefficient of x^2.
(b)	Very good. More than 70% of the candidates were able to find the coefficient of x^2 while a small number of candidates made careless mistakes in expanding $(1+2x)^7$.

5. (2015 DSE-MATH-M1 Q5)

(a) e^{-4x}

$=1+(-4x)+\frac{(-4x)^2}{2!}+\cdots$ 1M

$=1-4x+8x^2-\cdots$ 1A

(b) $(2+x)^5$

$=2^5+C_1^5(2^4)x+C_2^5(2^3)x^2+\cdots+x^5$ 1M

$=32+80x+80x^2+\cdots+x^5$

The required coefficient

$=(1)(80)+(-4)(80)+(8)(32)$ 1M

$=16$ 1A

----------(5)

(a)	Very good. Most candidates were able to expand e^{-4x} while a few candidates failed to show working steps.
(b)	Very good. Most candidates were able to find the coefficient of x^2 while a few candidates made a carless mistake in expanding $(2+x)^5$ as $2^5+C_1^5(2^4)x+C_2^5(2^3)x^2+\cdots+x^5$.

6. (2014 DSE-MATH-M1 Q3)

(a) $\left.\frac{dy}{dx}\right|_{(1,5)}=\left(2\cdot1-\frac{1}{1}\right)^3$

$=1$ 1A

Hence the equation of tangent is $y-5=1(x-1)$.

i.e. $x-y+4=0$ 1A

(b) (i) $\left(2x-\frac{1}{x}\right)^3=(2x)^3-3(2x)^2\left(\frac{1}{x}\right)+3(2x)\left(\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^3$ 1M

$=8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$ 1A

(ii) $y=\int\left(2x-\frac{1}{x}\right)^3dx$

$=\int\left(8x^3-12x+\frac{6}{x}-\frac{1}{x^3}\right)dx$ by (i)

$=2x^4-6x^2+6\ln|x|+\frac{1}{2x^2}+C$ 1M

Since $P(1,5)$ lies on S, $5=2(1)^4-6(1)^2+6\ln|1|+\frac{1}{2(1)^2}+C$. 1M

i.e. $C=\frac{17}{2}$

Hence the equation of S is $y=2x^4-6x^2+6\ln x+\frac{1}{2x^2}+\frac{17}{2}$ for $x>0$. 1A

(7)

(a)		Very good.
(b)	(i)	Excellent.
	(ii)	Satisfactory. Some candidates did not know $\int\frac{1}{x}dx=\ln|x|+C$, or wrote $\int\frac{1}{x^3}dx=-\frac{2}{x^2}$ or $\frac{1}{2x^2}$.

7. (2013 DSE-MATH-M1 Q1)

(a) $$\left(u+\frac{1}{u}\right)^4 = u^4+4u^3\left(\frac{1}{u}\right)+6u^2\left(\frac{1}{u}\right)^2+4u\left(\frac{1}{u}\right)^3+\left(\frac{1}{u}\right)^4$$

$$= u^4+4u^2+6+\frac{4}{u^2}+\frac{1}{u^4}$$ 1A

(b) $$(e^{ax}+e^{-ax})^4$$

$$= e^{4ax}+4e^{2ax}+6+4e^{-2ax}+e^{-4ax} \quad \text{by (a)}$$ 1M

$$= \left[1+\frac{4ax}{1!}+\frac{(4ax)^2}{2!}+\cdots\right]+4\left[1+\frac{2ax}{1!}+\frac{(2ax)^2}{2!}+\cdots\right]+6$$

$$+4\left[1+\frac{-2ax}{1!}+\frac{(-2ax)^2}{2!}+\cdots\right]+\left[1+\frac{-4ax}{1!}+\frac{(-4ax)^2}{2!}+\cdots\right]$$ 1M

$$= 1+4ax+8a^2x^2+4+8ax+8a^2x^2+6+4-8ax+8a^2x^2+1-4ax+8a^2x^2+\cdots$$

$$= 16+32a^2x^2+\cdots$$ 1A

(c) $$32a^2=2$$

$$a^2=\frac{1}{16}$$

$$a=\pm\frac{1}{4}$$ 1A

(5)

(a)	Excellent. A few candidates neglected the requirement 'in descending powers of u' when expanding $\left(u+\frac{1}{u}\right)^4$.
(b)	Satisfactory. Some candidates repeated steps in (a) because they did not make use of the fact that $e^{-ax}=\frac{1}{e^{ax}}$. Some candidates were not able to use power series of an exponential function, while some others expressed $(e^{ax}+e^{-ax})^4$ in powers of e^{2ax}.
(c)	Poor. Many candidates were not able to get the correct answer of (b), hence failed to get the answer for this part.

8. (2012 DSE-MATH-M1 Q1)

(a) $$(1+3x)^n = 1+C_1^n(3x)+C_2^n(3x)^2+\cdots$$

$$= 1+3nx+\frac{9n(n-1)}{2}x^2+\cdots$$ 1A

(b) $$e^{-2x}(1+3x)^n = \left[1+(-2x)+\frac{(-2x)^2}{2!}+\cdots\right]\left[1+3nx+\frac{9n(n-1)}{2}x^2+\cdots\right]$$ 1A For $1+(-2x)+\frac{(-2x)^2}{2!}+\cdots$

$$= (1-2x+2x^2+\cdots)\left[1+3nx+\frac{9n(n-1)}{2}x^2+\cdots\right]$$

$$\therefore\ 1\cdot\frac{9n(n-1)}{2}+(-2)(3n)+2\cdot 1=62$$ 1M

$$9n^2-21n-120=0$$

$$n=5 \quad \text{or } \frac{-8}{3} \text{ (rejected)}$$ 1A

(4)

(a) (b)	Very good. A minority of candidates, however, did not simplify the results obtained. Very good. A minority of candidates, however, did not reject the negative root $\frac{-8}{3}$.

9. (PP DSE-MATH-M1 Q1)

(a) $$(2x+1)^3 = 8x^3+12x^2+6x+1$$ 1A

(b) $$e^{-ax} = 1-ax+\frac{a^2x^2}{2}-\cdots$$ 1A

(c) $$\frac{(2x+1)^3}{e^{ax}} = (8x^3+12x^2+6x+1)\left(1-ax+\frac{a^2x^2}{2}-\cdots\right)$$ 1M

The coefficient of $x^2 = 12(1)+6(-a)+(1)\frac{a^2}{2}$ 1M

$$\therefore\ \frac{a^2}{2}-6a+12=-4$$

$$a^2-12a+32=0$$

$$a=4 \text{ or } 8$$ 1A

(5)

(a)	甚佳。 很多學生熟識二項式展式。
(b)	甚佳。 少部分學生未能展開指數函數。
(c)	良好。 少部分學生未能正確利用(b) 的結果。

10. (SAMPLE DSE-MATH-M1 Q1)

(a) $$e^{-2x} = 1+(-2x)+\frac{(-2x)^2}{2!}-\cdots$$

$$= 1-2x+2x^2+\cdots$$ 1A

(b) $$\frac{(1+2x)^6}{e^{2x}} = [1+6(2x)+15(2x)^2+\cdots]\cdot e^{-2x}$$ 1A For $1+6(2x)+15(2x)^2+\cdots$

$$= (1+12x+60x^2+\cdots)(1-2x+2x^2+\cdots)$$ 1M For using (a)

$$= 1+10x+38x^2+\cdots$$ 1A (pp-1) if dots were omitted in most cases

(4)

11. (2004 ASL-M&S Q4)

(a) (i) $(x+y+z)^2$	
$= x^2+y^2+z^2+2xy+2yz+2xz$	1A
(ii) Note that	
$(x+y+z)^4$	
$= (x^2+y^2+z^2+2xy+2yz+2xz)(x^2+y^2+z^2+2xy+2yz+2xz)$	
Thus, we have	
the coefficients of x^3y , x^3z , xy^3 , y^3z , xz^3 and yz^3	
$= (1)(2)+(2)(1)$	1M can be absorbed
$= 4$	1A
(b) (i) The required probability	
$= 1-p^4-q^4-r^4$	1M for complementary probability + 1A

The required probability	
$= (p+q+r)^4-p^4-q^4-r^4$	1M
$= 1-p^4-q^4-r^4$	1A

The required probability	
$= (p+q+r)^4-p^4-q^4-r^4$	1M
$= (p^2+q^2+r^2+2pq+2gr+2pr)(p^2+q^2+r^2+2pq+2qr+2pr)-p^4-q^4-r^4$	
$= p^4+q^4+r^4+4p^3q+4p^3r+4pq^3+4q^3r+4pr^3+4qr^3+$	
$6p^2q^2+6p^2r^2+6q^2r^2+12p^2qr+12pq^2r+12pqr^2-p^4-q^4-r^4$	
$=4p^3q+4p^3r+4pq^3+4q^3r+4pr^3+4qr^3+6p^2q^2+6p^2r^2+6q^2r^2+12p^2qr+12pq^2r+12pqr^2$	1A

(ii) The required probability	
$= 4(p^3q+p^3r+pq^3+q^3r+pr^3+qr^3)$	1A for $(p^3q+p^3r+pq^3+q^3r+pr^3+qr^3)$ + 1A for all being correct

The required probability	
$= 4p^3(1-p)+4q^3(1-q)+4r^3(1-r)$	1A for $(p^3(1-p)+q^3(1-q)+r^3(1-r))$ + 1A for all being correct

The required probability	
$= 1-(p^4+q^4+r^4+6p^2q^2+6p^2r^2+6q^2r^2+12p^2qr+12pq^2r+12pqr^2)$	1A for $(p^4+q^4+r^4+6p^2q^2+6p^2r^2+6q^2r^2$ $+12p^2qr+12pq^2r+12pqr^2)$ + 1A for all being correct
	-----------(7)

Fair. Many candidates did not make use of the fact that $p+q+r=1$, which simplifies the expressions.

2. Exponential and Logarithmic Functions

Learning Unit	Learning Objective
Foundation Knowledge Area	
2. Exponential and logarithmic functions	2.1 recognise the definition of the number e and the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ 2.2 recognise exponential functions and logarithmic functions 2.3 use exponential functions and logarithmic functions to solve problems 2.4 transform $y = kx^n$ and $y = ka^x$ to linear relations, where a, n and k are real numbers, $a > 0$ and $a \neq 1$

Section A

1. (a) Expand e^{-18x} in ascending powers of x as far as the term in x^2 .

   (b) Let n be a positive integer. If the coefficient of x^2 in the expansion of $e^{-18x}(1+4x)^n$ is -38 . Find n .

(6 marks) (2019 DSE-MATH-M1 Q6)

2. Let k be a constant.

   (a) Expand $e^{kx} + e^{2x}$ in ascending powers of x as far as the term in x^2 .

   (b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8(e^{kx}+e^{2x}-1)$ are equal, find k .

(6 marks) (2018 DSE-MATH-M1 Q6)

3. Let k be a constant.

(a) Expand $e^{kx}+e^{2x}$ in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x and the coefficient of x^2 in the expansion of $(1-3x)^8\left(e^{kx}+e^{2x}-1\right)$ are equal, find k.

(6 marks) (2018 DSE-MATH-M1 Q6)

4. (a) Expand $\left(1+e^{3x}\right)^2$ in ascending powers of x as far as the term in x^2.

(b) Find the coefficient of x^2 in the expansion of $(5-x)^4\left(1+e^{3x}\right)^2$.

(6 marks) (2017 DSE-MATH-M1 Q5)

5. Let k be a constant.

(a) Expand e^{kx} in ascending powers of x as far as the term in x^2.

(b) If the coefficient of x in the expansion of $(1+2x)^7 e^{kx}$ is 8, find the coefficient of x^2.

(5 marks) (2016 DSE-MATH-M1 Q5)

6. (a) Expand e^{-4x} in ascending powers of x as far as the term in x^2.

(b) Find the coefficient of x^2 in the expansion of $\dfrac{(2+x)^5}{e^{4x}}$.

(5 marks) (2015 DSE-MATH-M1 Q5)

7. After launching an advertisement for x weeks, the number y (in thousand) of members of a club can be modeled by

$$y=\frac{8\left(1-ae^{-bx}\right)}{1+ae^{-bx}}, \text{ where } a \text{ and } b \text{ are positive integers and } x\geq 0 .$$

The values of y when $x=2,4,6,8,10$ were recorded as follows:

x	2	4	6	8	10
y	5.97	6.26	6.75	7.11	7.37

(a) Let $u=ae^{-bx}$.

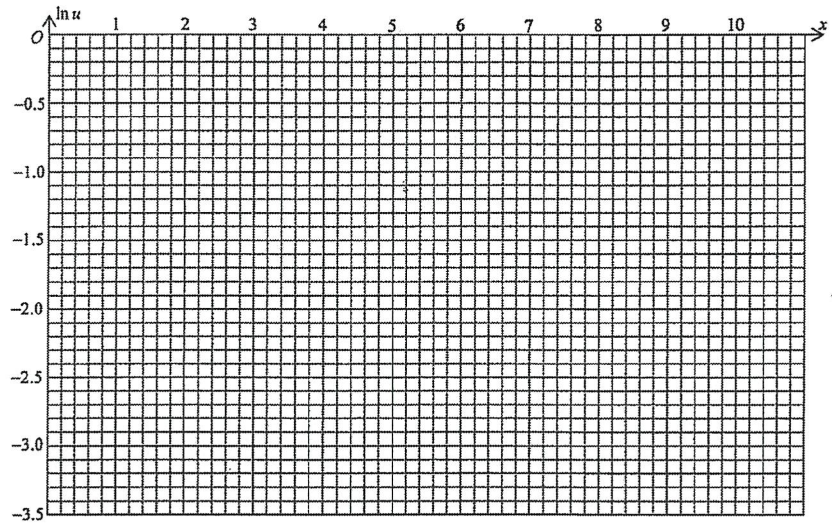
(i) Express $\ln u$ as a linear function of x.

(ii) Find u in terms of y.

(b) It is known that one of the values of y in the above table is incorrect.

(i) Using the graph paper on page 9 to determine which value of y is incorrect.

(ii) By removing the incorrect value of y, estimate the values of a and b. Correct your answers to 2 decimal places.



(7 marks)

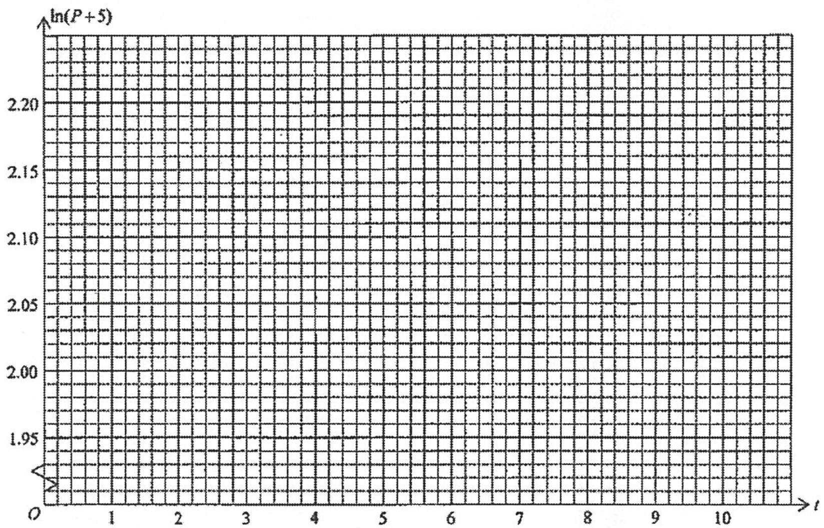
(2013 DSE-MATH-M1 Q4)

8. The population P (in millions) of a city can be modelled by $P = ae^{\frac{kt}{40}} - 5$ where a and k are constants and t is the number of years since the beginning of a certain year. The population of the city is recorded as follows.

t	2	4	6	8	10
P	2.36	2.81	3.23	3.55	4.01

(a) Express $\ln(P+5)$ as a linear function of t.

(b) Using the graph paper below, estimate the values of a and k. Correct your answers to the nearest integers.



(5 marks)

(2012 DSE-MATH-M1 Q3)

9. The number $N(t)$ of fish, which are infected by a certain disease in a pool, can be modelled by

$$N(t) = \frac{500}{1 + ae^{-kt}},$$

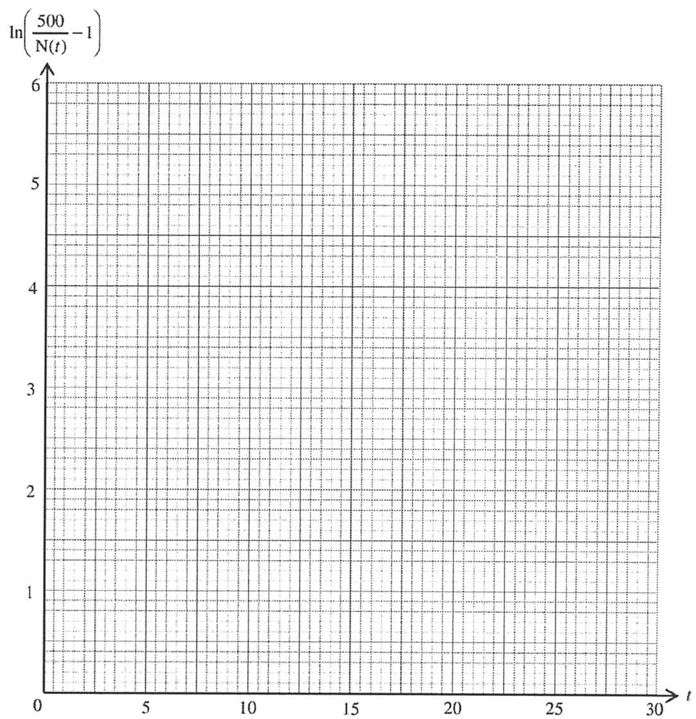
where a, k are positive constants and t is the number of days elapsed since the outbreak of the disease.

t	5	10	15	20
$N(t)$	13	34	83	175

(a) Express $\ln\left(\frac{500}{N(t)} - 1\right)$ as a linear function of t.

(b) Using the graph paper on page 10, estimate graphically the values of a and k (correct your answers to 1 decimal place).

(c) How many days after the outbreak of the disease will the number of fish infected by the disease reach 270?



(6 marks)

(SAMPLE DSE-MATH-M1 Q10)

10. After adding a chemical into a bottle of solution, the temperature S(t) of the surface of the bottle can be modeled by

$$S(t)=2(t+1)^2e^{-\lambda t}+15\ ,$$

where S(t) is measured in $°C$, $t\ (\geq 0)$ is the time measured in seconds after the chemical has been added and λ is a positive constant. It is given that S(9) = S(19) .

(a) Find the exact value of λ .

(b) Will the temperature of the surface of the bottle get higher than 90 $°C$? Explain your answer.

(6 marks) (2006 ASL-M&S Q2)

11. Let $y=\dfrac{1-e^{4x}}{1+e^{8x}}$.

(a) Find the value of $\dfrac{dy}{dx}$ when $x=0$.

(b) Let $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, where α and β are constants.

(i) Express $\ln(z^2+1)+3z$ as a liner function of x .

(ii) It is given that the graph of the linear function obtained in (b)(i) passes through the origin and the slope of the graph is 2. Find the values of α and β .

(iii) Using the values of α and β obtained in (b)(ii), find the value of $\dfrac{dy}{dz}$ when $z=0$.

(7 marks) (2007 ASL-M&S Q3)

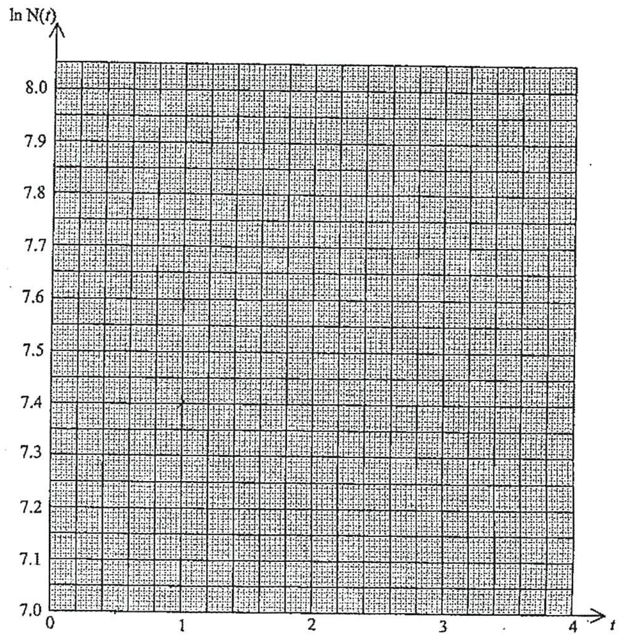
12. A researcher modeled the number of bacteria N(t) in a sample t hours after the beginning of his observation by $N(t)=900a^{kt}$, where a (>0) and k are constants. He observed and recorded the following data:

t (in hours)	0.5	1.0	2.0	3.0
$N(t)$	1100	1630	2010	2980

The researcher made one mistake when writing down the data for N(t) .

Express $\ln N(t)$ as a linear function of t and use a graph paper to determine which one of the data was incorrect, and estimate the value of N(2.5) correct to 3 significant figures.

(4 marks) (2002 ASL-M&S Q3)



Section B

13. In an experiment, the temperature (in °C) of a certain liquid can be modelled by

$$S=\frac{200}{1+a2^{bt}}\ ,$$

where a and b are constants and t is the number of hours elapsed since the start of the experiment.

(a) Express $\ln\left(\dfrac{200}{S}-1\right)$ as a linear function of t .

(2 marks)

(b) It is found that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function obtained in (a) are $\ln 4$ and 4 respectively.

(i) Find a and b .

(ii) Find $\dfrac{dS}{dt}$ and $\dfrac{d^2S}{dt^2}$.

(iii) Describe how S and $\dfrac{dS}{dt}$ vary during the first 48 hours after the start of the experiment. Explain your answer.

(11 marks)

(2015 DSE-MATH-M1 Q12)

14. Let y be the amount (in suitable units) of suspended particulate in a laboratory. It is given that

$$(E):\qquad y=\frac{340}{2+e^{-t}-2e^{-2t}}\qquad (t\geq 0)\ ,$$

where t is the time (in hours) which has elapsed since an experiment started.

(a) Will the value of y exceed 171 in the long run? Justify your answer.

(2 marks)

(b) Find the greatest value and least value of y.

(6 marks)

(c) (i) Rewrite (E) as a quadratic equation in e^{-t}.

(ii) It is known that the amounts of suspended particulate are the same at the time $t=\alpha$ and $t=3-\alpha$. Given that $0\leq\alpha<3-\alpha$, find α.

(4 marks)

(2014 DSE-MATH-M1 Q11)

15. A researcher models the rate of change of the population size of a kind of insects in a forest by

$$P'(t)=kte^{\frac{a}{20}t}\ ,$$

where $P(t)$, in thousands, is the population size, t (≥ 0) is the time measured in weeks since the start of the research, and a, k are integers.

The following table shows some values of t and $P'(t)$.

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60

(a) Express $\ln\frac{P'(t)}{t}$ as a linear function of t.

(1 mark)

(b) By plotting a suitable straight line on the graph paper on next page, estimate the integers a and k.

(5 marks)

(c) Suppose that $P(0)=30$. Using the estimates in (b),

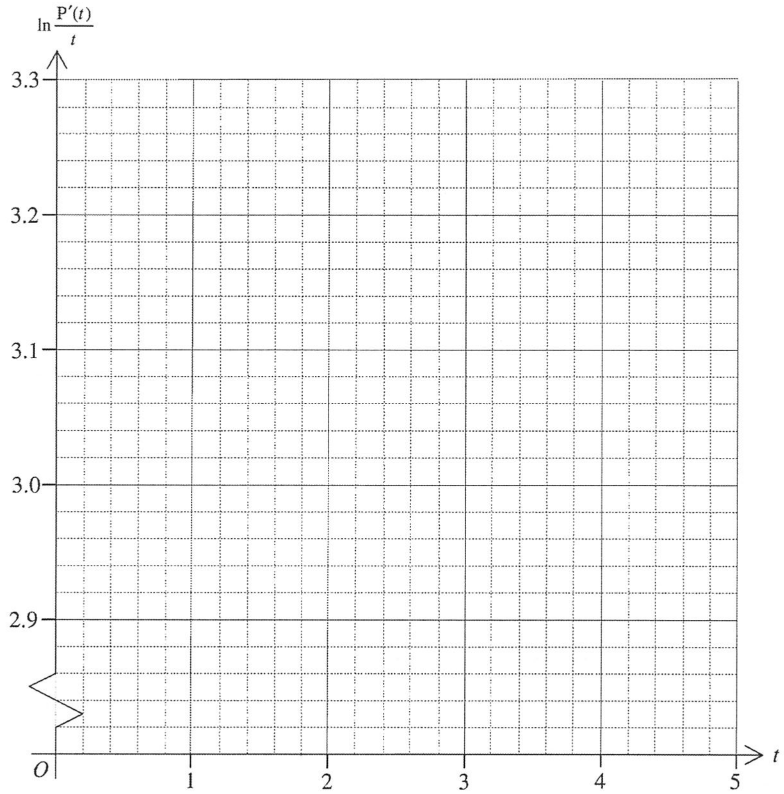
(i) find the value of t such that the rate of change of the population size of the insect is the greatest;

(ii) find $\frac{d}{dt}\left(te^{\frac{a}{20}t}\right)$ and hence, or otherwise, find $P(t)$;

(iii) estimate the population size after a very long time.

[Hint: You may use the fact that $\lim_{t\to\infty}\frac{t}{e^{mt}}=0$ for any positive constant m.]

(9 marks)



(PP DSE-MATH-M1 Q11)

16. A researcher studies the growth of the population size and the electricity consumption of a certain city. Suppose that the population size P (in hundred thousand) of the city can be modelled by

$$P=\frac{ke^{-\lambda t}}{t^2},\quad 0<t<6,$$

where k and λ are constants and t is the time in years elapsed since the start of the research.

(a) (i) Express $\ln P+2\ln t$ as a linear function of t.

(ii) Given that the intercepts on the horizontal and vertical axes of the graph of the linear function in (a)(i) above are -1.15 and 2.3 respectively, find the values of k and λ correct to the nearest integer.

Hence find the minimum population size correct to the nearest hundred thousand.

(6 marks)

(b) The annual electricity consumption E (in thousand terajoules per year) of the city can be modelled by

$$\frac{dE}{dt}=hte^{ht}-1.2e^{ht}+4.214,\quad t\ge 0,$$

where h is a non-zero constant and t is the time in years elapsed since the start of the research. It is known that the population size and the rate of change of annual electricity consumption both attain minimum at the same time t_0, and when $t=0$, $E=1$.

(i) Find the value of h.

(ii) By considering $\frac{d}{dt}\left(te^{ht}\right)$, find $\int te^{ht}\,dt$.

Hence find the annual electricity consumption of the city at t_0 correct to the nearest thousand terajoules per year.

(iii) A green campaign is launched to save the annual electricity consumption immediately after t_0. The new annual electricity consumption F (in thousand terajoules per year) of the city can then be modelled by

$$F=\frac{6}{1-5e^{rt}+3e^{2rt}}+2,\quad t\ge t_0.$$

If the new annual electricity consumption is the same as the original annual electricity consumption at $t=t_0$, find the value of r.

(9 marks)

(2011 ASL-M&S Q9)

17. A researcher models the population size R, in hundreds, of a certain species of fish in a lake by

$$R=kt^{1.2}e^{\frac{\lambda t}{20}}\quad (0\le t\le 30),$$

where t is the number of months elapsed since the beginning of the study and k and λ are constants.

(a) (i) Express $\ln R-1.2\ln t$ as a linear function of t.

(ii) It is given that the graph of $\ln R-1.2\ln t$ against t has intercept 2.89 on the vertical axis and slope -0.05. Find the values of k and λ correct to the nearest integer.

(iii) Using the approximate integral values of k and λ obtained in (a)(ii), find the maximum population size of the species of fish correct to the nearest hundreds. When will this take place?

(7 marks)

(b) In order to stimulate the growth of this species of fish, more food is added immediately when the population size of the fish attains 240 hundreds. The population size of the species of fish can then be modelled by

$$Q=L-20(6e^{-t}+t^3)\quad (0\le t\le 2),$$

where Q is the population size (in hundreds) of the species of fish, t is the number of months elapsed since more food has been added and L is a constant.

(i) Find the value of L.

(ii) Expand e^{-t} in ascending powers of t as far as the term in t^3.

Hence, find a quadratic polynomial which approximates Q.

(iii) Using the result obtained in (b)(ii), check whether the species of fish will reach a population size of 300 hundreds.

(iv) Do you think that the conclusion in (b)(iii) is still valid if terms up to and including t^7 in the expansion of e^{-t} in (b)(ii) are used? Explain your answer briefly.

(8 marks)

(2009 ASL-M&S Q8)

18. A biologist studied the population of fruit fly A under limited food supply. Let t be the number of days since the beginning of the experiment and $N'(t)$ be the number of fruit fly A at time t. The biologist modelled the rate of change of the number of fruit fly A by

$$N'(t) = \frac{20}{1+he^{-kt}} \quad (t \geq 0) \ ,$$

where h and k are positive constants.

(a) (i) Express $\ln\left(\frac{20}{N'(t)}-1\right)$ as a linear function of t.

(ii) It is given that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function in (i) are 1.5 and 7.6 respectively. Find the values of h and k.

(4 marks)

(b) Take $h = 4.5$ and $k = 0.2$, and assume that $N(0) = 50$.

(i) Let $v = h + e^{kt}$, find $\frac{dv}{dt}$.

Hence, or otherwise, find $N(t)$.

(ii) The population of fruit fly B can be modelled by

$$M(t) = 21\left(t + \frac{h}{k}e^{-kt}\right) + b \ ,$$

where b is a constant. It is known that $M(20) = N(20)$.

(1) Find the value of b.

(2) The biologist claims that the number of fruit fly A will be smaller than that of fruit fly B for $t > 20$. Do you agree? Explain your answer.

[Hint: Consider the difference between the rates of change of the two populations.]

(11 marks)

(2008 ASL-M&S Q8)

19. After upgrading the production line of a cloth factory, two engineers, John and Mary, model the rate of change of the amount of cloth production in thousand metres respectively by

$$f(t) = 25t^2(t+10)^{\frac{-1}{3}} \quad \text{and} \quad g(t) = 28 + ke^{ht^2} \ ,$$

where h and k are positive constants and $t\ (\geq 0)$ is the time measured in months since the upgrading of the production line.

(a) Using the substitution $u = t + 10$, or otherwise, find the total amount of cloth production from $t = 0$ to $t = 3$ under John's model.

(5 mark)

(b) Express $\ln(g(t) - 28)$ as a linear function of t^2.

(1 mark)

(c) Given that the slope and the intercept on the vertical axis of the graph of the linear function in (b) are measured to be 0.3 and 1.0 respectively, estimate the values of h and k correct to 1 decimal place.

(2 marks)

(d) Using the estimated values of h and k obtained in (c) correct to 1 decimal place.

(i) expand $g(t)$ in ascending powers of t as fact as t^6, and hence estimate the total amount of cloth production from $t = 0$ to $t = 3$ under Mary's model;

(ii) determine whether the estimate in (d)(i) is an over-estimate or an under-estimate;

(iii) determine whether the total amount of cloth production from $t = 0$ to $t = 3$ under Mary's model is greater than that under John's model.

(7 marks)

(2006 ASL-M&S Q9)

20. A researcher studied the soot reduction effect of a petrol additive on soot emission of a car. Let t be the number of hours elapsed after the petrol additive has been used and $r(t)$, measured in ppm per hour, be the rate of change of the amount of soot reduced. The researcher suggested that $r(t)$ can be modeled by $r(t) = \alpha te^{-\beta t}$, where α and β are positive constants.

(a) Express $\ln \frac{r(t)}{t}$ as a linear function of t.

(1 mark)

(b) It is given that the slope and the intercept on the vertical axis of the graph of the linear function obtained in (a) are -0.50 and 2.3 respectively. Find the values of α and β correct to 1 significant figure.

Hence find the greatest rate of change of the amount of soot reduced after the petrol additive has been used. Give your answer correct to 1 significant figure.

(6 marks)

(c) Using the values of α and β obtained in (b) correct to 1 significant figure,

(i) find $\frac{d}{dt}\left((t+\frac{1}{\beta})e^{-\beta t}\right)$ and hence find, in terms of T, the total amount of soot reduced when the petrol additive has been used for T hours;

(ii) estimate the total amount of soot reduced when the petrol additive has been used for a very long time.

[Note: Candidates may use $\lim_{T\to\infty}(Te^{-\beta T}) = 0$ without proof.]

(8 marks)

(2005 ASL-M&S Q8)

21. A researcher modeled the relationship between the atmospheric pressure y (in cmHg) and the altitude x (in km) above sea-level by

$$\frac{dy}{dx} = -\alpha\beta^{-x} \qquad (x \ge 0) \ ,$$

where α and β are positive constants.

(a) It is known that $\ln\left(-\frac{dy}{dx}\right)$ can be expressed as a linear function of x. The slope of the graph of the linear function is -0.125.

(i) Find the value of β correct to 3 decimal places.

(ii) The researcher found that the atmospheric pressures at sea-level (i.e. $x = 0$) and at an altitude of 2 km above sea-level were 76 cmHg and 59.2 cmHg respectively. If $\beta^{-x} = e^{-\lambda x}$ for all $x \ge 0$, find the value of λ.

Hence or otherwise, find the value of α correct to 1 decimal place.

(8 marks)

(b) A balloon filled with helium gas is released from a point on a mountain. The altitude of the point is h km above sea-level. The balloon bursts when it reaches an altitude of $2h$ km above sea-level. The difference in the atmospheric pressures between the two altitudes is 13 cmHg. It is also known that the atmospheric pressure at the top of the mountain is 25.2 cmHg. Using the values of α and β obtained in (a),

(i) find the altitude of the mountain above sea-level correct to the nearest 0.1 km.

(ii) find the value(s) of h correct to 1 decimal place.

(7 marks)

(2004 ASL-M&S Q9)

22. The spread of an epidemic in a town can be measured by the value of PPI (the proportion of population infected). The value of PPI will increase when the epidemic breaks out and will stabilize when it dies out.

The spread of the epidemic in town A last year could be modelled by the equation

$P'(t) = \frac{0.04ake^{-kt}}{1-a}$, where a, $k > 0$ and $P(t)$ was the PPI t days after the outbreak of the epidemic. The figure shows the graph of $\ln P'(t)$ against t, which was plotted based on some observed data obtained last year. The initial value of PPI is 0.09 (i.e. $P(0) = 0.09$).

(a) (i) Express $\ln P'(t)$ as a linear function of t and use the figure to estimate the values of a and k correct to 2 decimal places.

Hence find $P(t)$.

(ii) Let μ be the PPI 3 days after the outbreak of the epidemic. Find μ.

(iii) Find the stabilized PPI.

(8 marks)

(b) In another town B, the health department took precautions so as to reduce the PPI of the epidemic. It is predicted that the rate of spread of the epidemic will follow the equation

$Q'(t) = 6(b-0.05)(3t+4)^{\frac{-3}{2}}$, where $Q(t)$ is the PPI t days after the outbreak of the epidemic in town B and b is the initial value of PPI.

(i) Suppose $b = 0.09$.

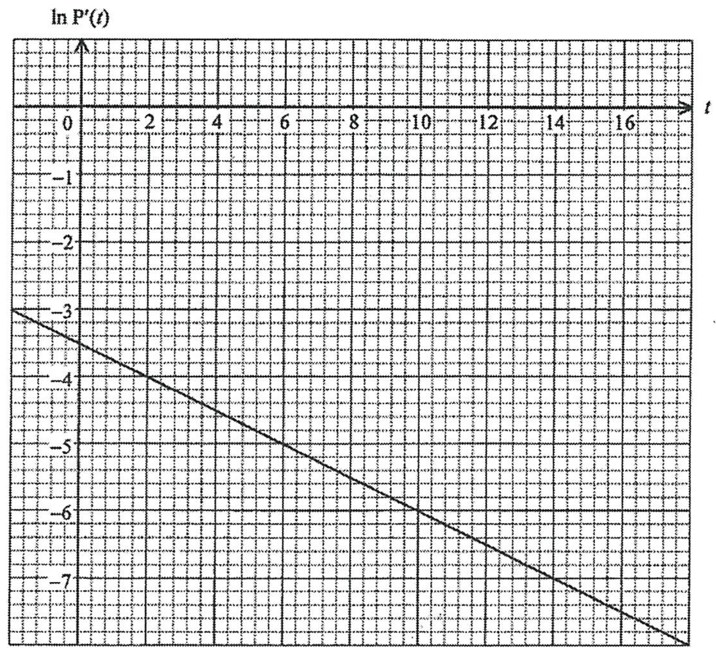
(I) Determine whether the PPI in town B will reach the value of μ in (a)(ii).

(II) How much is the stabilized PPI reduced in town B as compared with that in town A?

(ii) Find the range of possible values of b if the epidemic breaks out in town B. Explain your answer briefly.

(7 marks)

The graph of $\ln P'(t)$ against t



(2001 ASL-M&S Q9)

23. A researcher studied the growth of a certain kind of bacteria. 100 000 such bacteria were put into a beaker for cultivation. Let t be the number of days elapsed after the cultivation has started and $r(t)$, in thousands per day, be the growth rate of the bacteria. The researcher obtained the following data:

t	1	2	3	4
$r(t)$	7.9	12.3	15.3	17.5

(a) The researcher suggested that $r(t)$ can be modelled by $r(t) = at^b$, where a and b are positive constants.

(i) Express $\ln r(t)$ as a linear function of $\ln t$.

(ii) Using the graph paper, estimate graphically the value of $r(5)$ to 1 decimal place without finding the values of a and b.

(5 marks)

(b) The researcher later observed that $r(5)$ was 18.5 and considered the model in (a) unsuitable. After reviewing some literature, he used the model $r(t) = 20 - pe^{-qt}$, where p and q are positive constants.

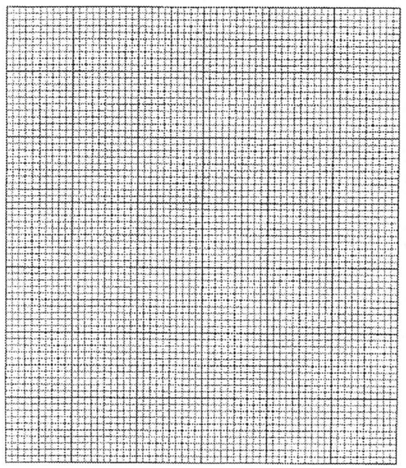
(i) Express $\ln[20 - r(t)]$ as a linear function of t.

(ii) Using the graph paper on next page, estimate graphically the values of p and q to 3 significant figures.

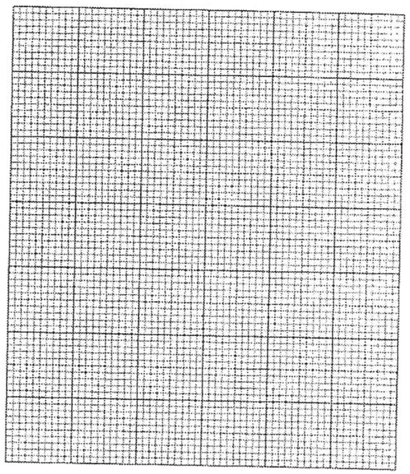
(iii) Estimate the total number of bacteria, to the nearest thousand, after 15 days of cultivation.

(10 marks)

Graph paper for part (a)(ii)



Graph paper for part (b)(ii)



(2000 ASL-M&S Q10)

24. An ecologist studies the birds at Mai Po Nature Reserve. Only 21% of the birds are "residents", i.e. found throughout the year. The remaining birds are migrants. The ecologist suggests that the number $N(t)$ of a certain species of migrants can be modelled by the function

$$N(t) = \frac{3000}{1 + ae^{-bt}},$$

where a, b are positive constants and t is the number of days elapsed since the first one of that species of migrants was found at Mai Po in that year.

(a) This year, the ecologist obtained the following data:

t	5	10	15	20
$N(t)$	250	870	1940	2670

(i) Express $\ln(\frac{3000}{N(t)} - 1)$ as a linear function of t.

(ii) Use the graph paper on next page to estimate graphically the values of a and b correct to 1 decimal place.

(5 marks)

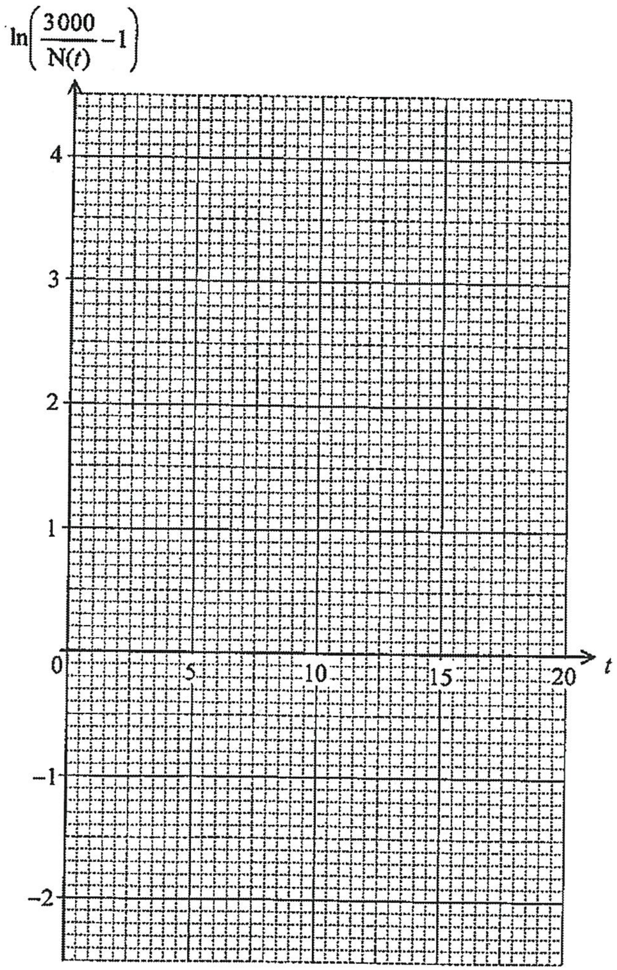
(b) Basing on previous observations, the migrants of that species start to leave Mai Po when the rate of change of $N(t)$ is equal to one hundredth of $N(t)$. Once they start to leave, the original model will not be valid and no more migrants will arrive. It is known that the migrants will leave at the rate $r(s)$ per day where $r(s) = 60\sqrt{s}$ and s is the number of days elapsed since they started to leave Mai Po. Using the values of a and b obtained in (a)(ii),

(i) find $N'(t)$, and show that $N(t)$ is increasing;

(ii) find the greatest number of the migrants which can be found at Mai Po this year;

(iii) find the number of days in which the migrants can be found at Mai Po this year.

(10 marks)



Q9)

25. A forest fire has started in a country. An official of the Department of Environmental Protection wants to estimate the number of trees destroyed in the fire when the fire is out of control. Let t be the number of days after the fire has started and $r(t)$, in hundred trees per day, be the rate of trees destroyed. The official obtained the following data:

t	2	3	4	5	6	7
$r(t)$	6.4	15.7	29.5	48.3	72.2	101.2

(a) It is suggested that $r(t)$ can be modelled by either one of the following functions

(I): $r(t)=\alpha t^{\beta}$ or

(II): $r(t)=\gamma e^{\lambda t}$,

where α, β, γ and λ are constants.

(i) Express $\ln r(t)$ in terms of $\ln t$ and t in (I) and (II) respectively.

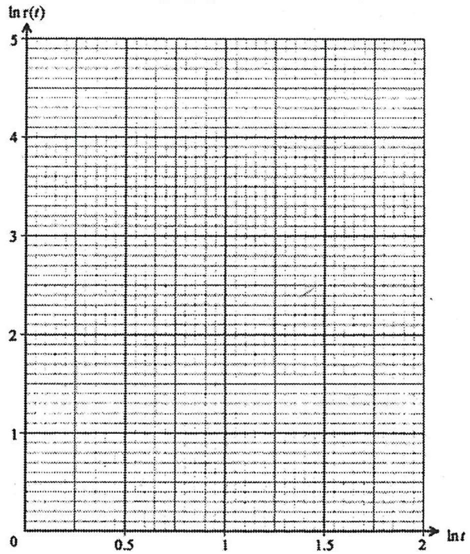
(ii) Use the graph papers to determine which function can better describe $r(t)$. Hence estimate graphically the two unknown constants in that function. Give your answers correct to 1 decimal place.

(10 marks)

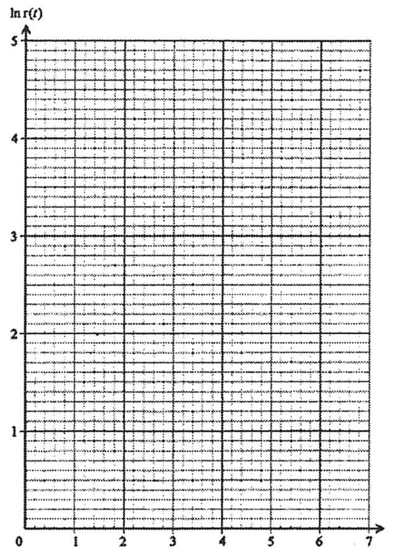
(b) Assume the fire is out of control and the function in (a) which describes $r(t)$ better is used. Estimate the total number, correct to the nearest hundred, of trees destroyed in the first 14 days of the fire. How many days more will it take for the total number of trees destroyed to be doubled?

(5 marks)

Graph paper for function (I)



Graph paper for function (II)



(1998 ASL-M&S Q10)

26. A stall sells clams only. The relationship between the selling price $\$x$ of each clam and the number $N(x)$ of clams sold per day can be modelled by

$$\ln N(x) = bx + \ln a$$

where a and b are constants. This relationship is represented by the straight line shown in the figure.

(a) Use the graph in the figure to estimate the values of a and b correct to 1 significant figure.

(3 marks)

(b) Suppose the daily running cost of the stall is \$ 5 000 and the cost of each clam is \$ 2. Using the values of a and b estimated in (a),

(i) express the daily profit of selling $N(x)$ clams in terms of x, and

(ii) determine the selling price of each clam so that the daily profit of selling $N(x)$ clams will attain its maximum. What is then the number of clams sold per day? Give the answer correct to the nearest integer.

(7 marks)

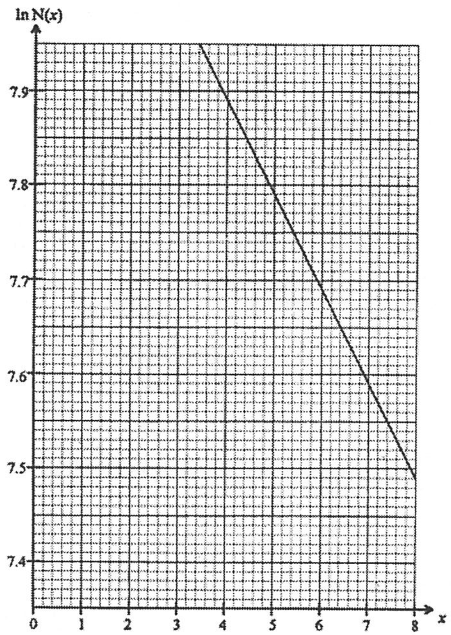
(c) The stall has been running a promotion programme every day from April 15, 1997. The number $M(n)$ of clams sold on the n-th day of the programme is given by

$$M(n) = 1500 + 1000(1 - e^{-0.1n}) \quad .$$

The stall will stop running the programme once the increase in the number of clams sold between two consecutive days falls below 15 . Determine how many days the programme should be run. Give the answer correct to the nearest integer.

(5 marks)

(1997 ASL-M&S Q9)



27. A textile factory plans to install a weaving machine on 1st January 1995 to increase its production of cloth. The monthly output x (in km) of the machine, after t months, can be modelled by the function

$$x = 100e^{-0.01t} - 65e^{-0.02t} - 35 \quad .$$

(a) (i) In which month and year will the machine cease producing any more cloth?

(ii) Estimate the total amount of cloth, to the nearest km, produced during the lifespan of the machine.

(5 marks)

(b) Suppose the cost of producing 1 km of cloth is US\$300; the monthly maintenance fee of the machine is U5\$300 and the selling price of 1 km of cloth is US\$800. In which month and year will the greatest monthly profit be obtained? Find also the profit, to the nearest US\$, in that month.

(6 marks)

(c) The machine is regarded as 'inefficient' when the monthly profit falls below US\$500 and it should then be discarded. Find the month and year when the machine should be discarded. Explain your answer briefly.

(4 marks)

(1994 ASL-M&S Q9)

Questions involve other topics

28. The chickens in a farm are infected by a certain bird flu. The number of chickens (in thousand) in the farm is modelled by

$$N = \frac{27}{2 + \alpha t e^{\beta t}} ,$$

where $t\ (\geq 0)$ is the number of days elapsed since the start of the spread of the bird flu and α and β are constants.

(a) Express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t.

(2 marks)

(b) It is given that the slope and the intercept on the horizontal axis of the graph of the linear function obtained in (a) are –0.1 and 10ln0.03 respectively.

(i) Find α and β.

(ii) Will the number of chickens in the farm be less than 12 thousand on a certain day after the start of the spread of the bird flu? Explain your answer.

(iii) Describe how the rate of change of the number of chickens in the farm varies during the first 20 days after the start of the spread of the bird flu. Explain your answer.

(10 marks)

(2016 DSE-MATH-M1 Q12)

29. Let $y = \frac{1-e^{4x}}{1+e^{8x}}$.

(a) Find the value of $\frac{dy}{dx}$ when $x = 0$.

(b) Let $(z^2+1)e^{3z} = e^{\alpha+\beta x}$, where α and β are constants.

(i) Express $\ln(z^2+1)+3z$ as a liner function of x.

(ii) It is given that the graph of the linear function obtained in (b)(i) passes through the origin and the slope of the graph is 2. Find the values of α and β.

(iii) Using the values of α and β obtained in (b)(ii), find the value of $\frac{dy}{dz}$ when $z = 0$.

(7 marks) (2007 ASL-M&S Q3)

30. The population of a kind of bacterium $p(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$p(t) = \frac{a}{b+e^{-t}} + c , \quad -\infty < t < \infty$$

where a, b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a, b and c,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population*.

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[p(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a, b and c.

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the primordial population and ultimate population. Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a, b, c and $p(t)$, express $p'(t)$ in the form of $\frac{-b}{a}[p(t)-\alpha][p(t)-\beta]$, where $\alpha < \beta$.

Hence express α and β in terms of a, b and c.

Sketch $p'(t)$ against $p(t)$ for $\alpha < p(t) < \beta$ and hence verify your answer in (c).

(5 marks)

(2010 ASL-M&S Q9)

31. A merchant sells compact discs (CDs). A market researcher suggests that if each CD is sold for \$$x$, the number N($x$) of CDs sold per week can be modeled by

$$N(x) = ae^{-bx}$$

where a and b are constants.

The merchant wants to determine the values of a and b based on the following results obtained from a survey:

x	20	30	40	50
N(x)	450	301	202	136

(a) (i) Express $\ln N(x)$ as a linear function of x .

(ii) Use a graph paper to estimate graphically the values of a and b correct to 2 decimal places.

(7 marks)

(b) Suppose the merchant wishes to sell 400 CDs in the next week. Use the values of a and b estimated in (a) to determine the price of each CD. Give your answer correct to 1 decimal place.

(2 marks)

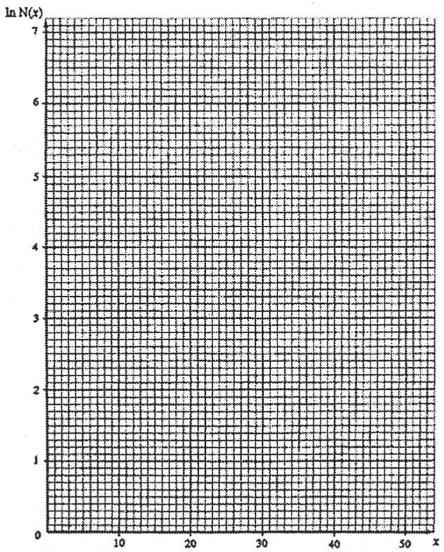
(c) It is known that the merchant obtains CDs at a cost of \$10 each. Let G(x) dollars denote the weekly profit. Using the values of a and b estimated in (a),

(i) express G(x) in terms of x.

(ii) find $G'(x)$ and hence determine the selling price for each CD in order to maximize the profit.

(6 marks)

(1995 ASL-M&S Q8)



32. A biologist studied the population of fruit fly A under limited food supply. Let t be the number of days since the beginning of the experiment and N'(t) be the number of fruit fly A at time t. The biologist modelled the rate of change of the number of fruit fly A by

$$N'(t) = \frac{20}{1+he^{-kt}} \qquad (t \geq 0)$$

where h and k are positive constants.

(a) (i) Express $\ln\left(\frac{20}{N'(t)}-1\right)$ as a linear function of t .

(ii) It is given that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function in (i) are 1.5 and 7.6 respectively. Find the values of h and k .

(4 marks)

(b) Take $h = 4.5$ and $k = 0.2$, and assume that N(0) = 50 .

(i) Let $v = h + e^{kt}$, find $\frac{dv}{dt}$.

Hence, or otherwise, find N(t) .

(ii) The population of fruit fly B can be modelled by

$$M(t) = 21\left(t + \frac{h}{k}e^{-kt}\right) + b \ ,$$

where b is a constant. It is known that $M(20) = N(20)$.

(1) Find the value of b .

(2) The biologist claims that the number of fruit fly A will be smaller than that of fruit fly B for $t > 20$. Do you agree? Explain your answer.

[Hint: Consider the difference between the rates of change of the two populations.]

(11 marks)

(2008 ASL-M&S Q8)

33. In a certain country, the daily rate of change of the amount of oil production P , in million barrels per day, can be modelled by

$$\frac{dP}{dt}=\frac{k-3t}{1+ae^{-bt}}$$

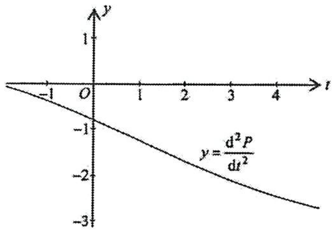
where t (≥ 0) is the time measured in days. When $\ln\left(\frac{k-3t}{\frac{dP}{dt}}-1\right)$ is plotted against t , the graph is a straight line with slope -0.3 and the intercept on the horizontal axis 0.32 . Moreover, P attains its maximum when $t=3$.

(a) Find the values of a , b and k .

(5 marks)

(b) (i) Using trapezoidal rule with 6 subintervals, estimate the total amount of oil production from $t=0$ to $t=3$.

(ii)



The figure shows the graph of $y=\frac{d^2P}{dt^2}$. Using the graph, determine whether the estimation in (i) is an under-estimate or an over-estimate.

(4 marks)

(c) The daily rate of change of the demand for oil D , in million barrels per day, can be modelled by

$$\frac{dD}{dt}=1.63^{2-0.1t}$$

where t (≥ 0) is the time measured in days.

(i) Let $y=\alpha^{\beta x}$, where α , β ($\alpha>0$, $\alpha\neq 1$ and $\beta\neq 0$) are constants. Find $\frac{dy}{dx}$ in terms of x .

(ii) Find the demand of oil from $t=0$ to $t=3$.

(iii) Does the overall oil production meet the overall demand of oil from $t=0$ to $t=3$? Explain your answer.

(6 marks)

(part (c)(i) is out of syllabus) (2013 ASL-M&S Q8)

34. A textile factory has bought two new dyeing machines P and Q . The two machines start to operate at the same time and will emit sewage into a lake near the factory. The manager of the factory estimates the amount of sewage emitted (in tonnes) by the two machines and finds that the rates of emission of sewage by the two machines P and Q can be respectively modelled by

$$p'(t)=4.5+2t(1+6t)^{\frac{-2}{3}} \text{ and}$$

$$q'(t)=3+\ln(2t+1) ,$$

where t (≥ 0) is the number of months that the machines have been in operation.

(a) By using a suitable substitution, find the total amount of sewage emitted by machine P in the first year of operation.

(4 marks)

(b) (i) By using the trapezoidal rule with 5 sub-intervals, estimate the total amount of sewage emitted by machine Q in the first year of operation.

(ii) The manager thinks that the amount of sewage emitted by machine Q will be less than that emitted by machine P in the first year of operation. Do you agree? Explain your answer.

(5 marks)

(c) The manager studies the relationship between the environmental protection tax R (in million dollars) paid by the factory and the amount of sewage x (in tonnes) emitted by the factory. He uses the following model:

$$R=16-ae^{-bx} ,$$

where a and b are constants.

(i) Express $\ln(16-R)$ as a linear function of x .

(ii) Given that the graph of the linear function in (c)(i) passes through the point $(-10,1)$ and the x-intercept of the graph is 90 , find the values of a and b .

(iii) In addition to the sewage emitted by the machines P and Q , the other operations of the factory emit 80 tonnes of sewage annually. Using the model suggested by the manager and the values of a and b found in (c)(ii), estimate the tax paid by the factory in the first year of the operation of machines P and Q .

(6 marks)

(2012 ASL-M&S Q8)

35. According to the past production record, an oil company manager modelled the rate of change of the amount of oil production in thousand barrels by

$$f(t)=5+2^{-kt+h} ,$$

where h and k are positive constants and $t(\geq 0)$ is the time measured in months.

(a) Express $\ln(f(t)-5)$ as a linear function of t .

(1 marks)

(b) Given that the slope and the intercept on the vertical axis of the graph of the linear function in (a) are –0.35 and 1.39 respectively, find the values of h and k correct to 1 decimal place.

(2 marks)

(c) The manager decides to start a production improvement plan and predicts the rate of change of the amount of oil production in thousand barrels by

$$g(t)=5+\ln(t+1)+2^{-kt+h} ,$$

where h and k are the values obtained in (b) correct to 1 decimal place, and $t(\geq 0)$ is the time measured in months from the start of the plan.

Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of oil production in thousand barrels from $t=2$ to $t=12$.

(2 marks)

(d) It is known that $g(t)$ in (c) satisfies

$$\frac{d^2g(t)}{dt^2}=p(t)-q(t), \text{ where } q(t)=\frac{1}{(t+1)^2} .$$

(i) If $2^t=e^{at}$ for all $t\geq 0$, find a .

(ii) Find $p(t)$.

(iii) It is known that there is no intersection between the curve $y=p(t)$ and the curve $y=q(t)$, where $2\leq t\leq 12$. Determine whether the estimate in (c) is an over-estimate or under -estimate.

(10 marks)

(2003 ASL-M&S Q8)

36. The monthly cost $C(t)$ at time t of operating a certain machine in a factory can be modelled by

$$C(t)=ae^{bt}-1 \qquad (0<t\leq 36),$$

where t is in month and $C(t)$ is in thousand dollars.

Table 2 shows the values of $C(t)$ when $t=1, 2, 3, 4$.

Table

t	1	2	3	4
$C(t)$	1.21	1.44	1.70	1.98

(a) (i) Express $\ln[C(t)+1]$ as a linear function of t .

(ii) Use the table and a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(iii) Using the values of a and b found in (a)(ii), estimate the monthly cost of operating this machine when $t=36$.

(8 marks)

(b) The monthly income $P(t)$ generated by this machine at time t can be modelled by

$$P(t)=439-e^{0.2t} \qquad (0<t\leq 36) ,$$

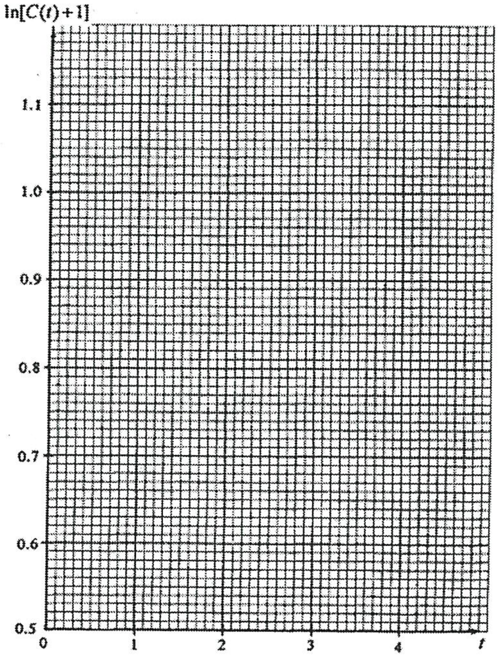
where t is in month and $P(t)$ is in thousand dollars.

The factory will stop using this machine when the monthly cost of operation exceeds the monthly income.

(i) Find the value of t when the factory stops using this machine. Give the answer correct to the nearest integer.

(ii) What is the total profit generated by this machine? Give the answer correct to the nearest thousand dollars.

(7 marks)



(1996 ASL-M&S Q10)

37. A chemical plant discharges pollutant to be a lake at an unknown rate of $r(t)$ units per month, where t is the number of months that the plant has been in operation.

Suppose that $r(0) = 0$.

The government measured $r(t)$ once every two months and reported the following figures:

t	2	4	6	8
$r(t)$	11	32	59	90

(a) Use the trapezoidal rule to estimate the total amount of pollutant which entered the lake in the first 8 months of the plant's operation.

(2 mark)

(b) An environmental scientist suggests that

$$r(t) = at^b,$$

where a and b are constants.

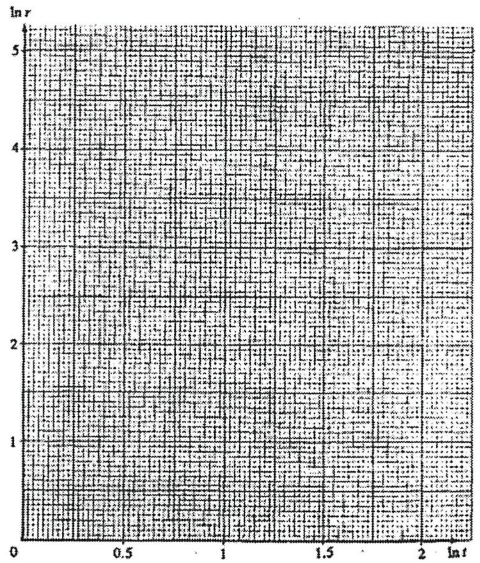
(i) Use a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(ii) Based on this scientist's model, estimate the total amount of pollutant, correct to 1 decimal place, which entered the lake in the first 8 months of the plant's operation.

(8 mark)

(c) It is known that no life can survive when 1000 units of pollutant have entered the lake. Adopting the scientist's model in (b), how long does it take for the pollutant from the plant to destroy all life in the lake? Give your answer correct to the nearest month.

(5 mark)



(1994 ASL-M&S Q10)

2021 DSE Q6

(a) Expand e^{-6x} in ascending powers of x as far as the term in x^4.

(b) Find the constant k such that the coefficient of x^4 in the expansion of $e^{-6x}(1-kx^2)^5$ is -26.

(5 marks)

Out of Syllabus

1. (a) Expand e^{-2x} in ascending powers of x as far as the term in x^3.

   (b) Using (a), expand $\frac{(1+x)^{\frac{1}{2}}}{e^{2x}}$ in ascending powers of x as far as the term in x^3.

   State the range of values of x for which the expansion is valid.

(6 marks) (1999 ASL-M&S Q2)

2. Exponential and Logarithmic Functions

1. (2019 DSE-MATH-M1 Q6)

(a) e^{-18x}		
$=1+(-18x)+\dfrac{(-18x)^2}{2!}+\cdots$	1M	
$=1-18x+162x^2+\cdots$	1A	
(b) $(1+4x)^n$		
$=1+C_1^n(4x)+C_2^n(4x)^2+\cdots+C_n^n(4x)^n$	1M	
$=1+4C_1^n x+16C_2^n x^2+\cdots+4^n x^n$		
$16C_2^n-72C_1^n+162=-38$	1M	
$16\left(\dfrac{n(n-1)}{2}\right)-72n+162=-38$		
$n^2-10n+25=0$	1M	
$n=5$	1A	
	----------(6)	

2. (2018 DSE-MATH-M1 Q6)

(a) $e^{kx}+e^{2x}$		
$=\left(1+kx+\dfrac{(kx)^2}{2!}+\cdots\right)+\left(1+2x+\dfrac{(2x)^2}{2!}+\cdots\right)$	1M	for expanding e^{kx} or e^{2x}
$=2+(k+2)x+\dfrac{(k^2+4)}{2}x^2+\cdots$	1A	
(b) $(1-3x)^8$		
$=1+C_1^8(-3x)+C_2^8(-3x)^2+\cdots$	1M	
$=1-24x+252x^2+\cdots$		
$e^{kx}+e^{2x}-1$		
$=1+(k+2)x+\dfrac{(k^2+4)}{2}x^2+\cdots$		
$(1)(k+2)+(-24)(1)=(1)\left(\dfrac{k^2+4}{2}\right)+(-24)(k+2)+(252)(1)$	1M+1M	
$k^2-50k+456=0$		
$k=12$ or $k=38$	1A	
	----------(6)	

3. (2017 DSE-MATH-M1 Q5)

(a) $(1+e^{3x})^2$		
$=1+2e^{3x}+e^{6x}$	1M	
$=1+2\left(1+3x+\dfrac{(3x)^2}{2!}+\cdots\right)+\left(1+6x+\dfrac{(6x)^2}{2!}+\cdots\right)$	1M	for expanding e^{3x} or e^{6x}
$=4+12x+27x^2+\cdots$	1A	
$(1+e^{3x})^2$		
$=\left(1+1+3x+\dfrac{(3x)^2}{2!}+\cdots\right)^2$	1M	for expanding e^{3x}
$=(2)(2)+(2)(2)(3x)+(3x)(3x)+(2)(2)\left(\dfrac{9x^2}{2}\right)+\cdots$	1M	
$=4+12x+27x^2+\cdots$	1A	
(b) $(5-x)^4$		
$=5^4-C_1^4(5^3)x+C_2^4(5^2)x^2-C_3^4(5)x^3+x^4$	1M	
$=625-500x+150x^2-20x^3+x^4$		
The required coefficient		
$=(625)(27)+(-500)(12)+(150)(4)$	1M	withhold 1M if the step is skipped
$=11475$	1A	
	----------(6)	

(a)	Very good. Most candidates were able to expand $(1+e^{3x})^2$.
(b)	Very good. Most candidates were able to find the coefficient of x^2 .

4. (2016 DSE-MATH-M1 Q5)

(a) e^{kx}

$= 1 + kx + \frac{(kx)^2}{2!} + \cdots$

$= 1 + kx + \frac{k^2x^2}{2} + \cdots$ — 1A

(b) $(1+2x)^7 e^{kx}$

$= \left(1 + C_1^7(2x) + C_2^7(2x)^2 + \cdots + (2x)^7\right)\left(1 + kx + \frac{k^2x^2}{2} + \cdots\right)$ — 1M

$= \left(1 + 14x + 84x^2 + \cdots + (2x)^7\right)\left(1 + kx + \frac{k^2x^2}{2} + \cdots\right)$

$\therefore\ 14 + k = 8$ — 1M

$k = -6$

The coefficient of x^2

$= (1)\left(\frac{(-6)^2}{2}\right) + 14(-6) + (84)(1)$ — 1M

$= 18$ — 1A

----------(5)

(a)	Very good. A very high proportion of the candidates were able to expand e^{kx} while some candidates were unable to simplify the coefficient of x^2.
(b)	Very good. More than 70% of the candidates were able to find the coefficient of x^2 while a small number of candidates made careless mistakes in expanding $(1+2x)^7$.

5. (2015 DSE-MATH-M1 Q5)

(a) e^{-4x}

$= 1 + (-4x) + \frac{(-4x)^2}{2!} + \cdots$ — 1M

$= 1 - 4x + 8x^2 - \cdots$ — 1A

(b) $(2+x)^5$

$= 2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$ — 1M

$= 32 + 80x + 80x^2 + \cdots + x^5$

The required coefficient

$= (1)(80) + (-4)(80) + (8)(32)$ — 1M

$= 16$ — 1A

----------(5)

(a)	Very good. Most candidates were able to expand e^{-4x} while a few candidates failed to show working steps.
(b)	Very good. Most candidates were able to find the coefficient of x^2 while a few candidates made a carless mistake in expanding $(2+x)^5$ as $2^5 + C_1^5(2^4)x + C_2^5(2^3)x^2 + \cdots + x^5$.

6. (2013 DSE-MATH-M1 Q4)

(a) (i) $u = ae^{-bx}$

$\ln u = \ln a - bx$ — 1A

(ii) $y = \frac{8(1-ae^{-bx})}{1+ae^{-bx}}$

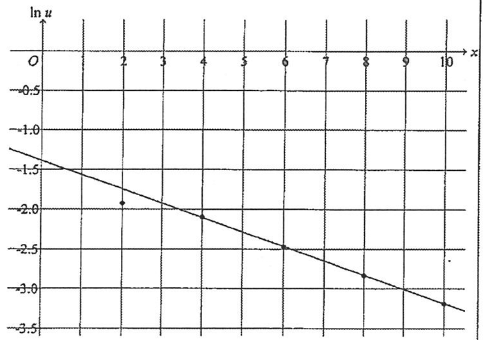
$= \frac{8-8u}{1+u}$

$u = \frac{8-y}{8+y}$ — 1A

(b) (i) By (a), $\ln\frac{8-y}{8+y} = \ln a - bx$

x	2	4	6	8	10
$\ln\frac{8-y}{8+y}$	−1.93	−2.10	−2.47	−2.83	−3.19

1A — For any two pairs of values



1A

From the graph, we see that the value $y = 5.97$ is incorrect. — 1A

(ii) The y-intercept $= \ln a \approx -1.4$ — 1M (For either one)

$\therefore\ a \approx 0.25$

The slope $= -b \approx \frac{-3.19-(-2.10)}{10-4}$

$\therefore\ b \approx 0.18$ — 1A (For both a and b)

(7)

(a)	(i) (ii)	Excellent. Very good. A few candidates found $\frac{dy}{du}$ or $\ln y$ which was not required.
(b)	(i)	Satisfactory. Many candidates used values of $\ln u$ with only one decimal place to plot graphs, which made it difficult to determine which value of y should be incorrect. Some others made mistakes in plotting the graph although using more accurate values of $\ln u$.
	(ii)	Fair. Some candidates used correct algebraic method but with values of $\ln u$ not accurate enough. Some others used graphs plotted in (i), but were not able to get the value of the ($\ln u$)-intercept of the straight line accurate enough.

7. (2012 DSE-MATH-M1 Q3)

(a) $P = ae^{\frac{kt}{40}} - 5$

$\ln(P+5) = \frac{k}{40}t + \ln a$ — 1A

(b)

t	2	4	6	8	10
P	2.36	2.81	3.23	3.55	4.01
$\ln(P+5)$	2.00	2.06	2.11	2.15	2.20

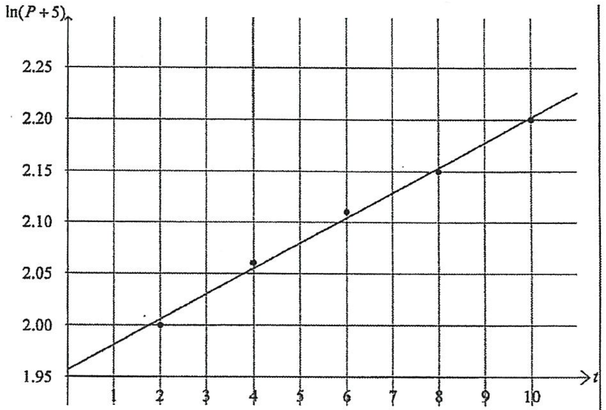
1M

From the graph on the next page, $\ln a \approx 1.96$ — 1M (Either one)

$a \approx 7$

$\frac{k}{40} \approx \frac{2.21-1.96}{10-0}$ — (Either one)

$k \approx 1$ — 1A — For both a and k



1A

(5)

(a)	Very good.
(b)	Very good. Candidates performed well in plotting graphs, but a small number of them did not use the plotting to estimate the values of a and k.

8. (SAMPLE DSE-MATH-M1 Q10)

(a) $N(t) = \frac{500}{1+ae^{-kt}}$

$\ln\left(\frac{500}{N(t)} - 1\right) = \ln(ae^{-kt})$

$= -kt + \ln a$ — 1A

(b)

t	5	10	15	20
$\ln\left(\frac{500}{N(t)}-1\right)$	3.6	2.6	1.6	0.6

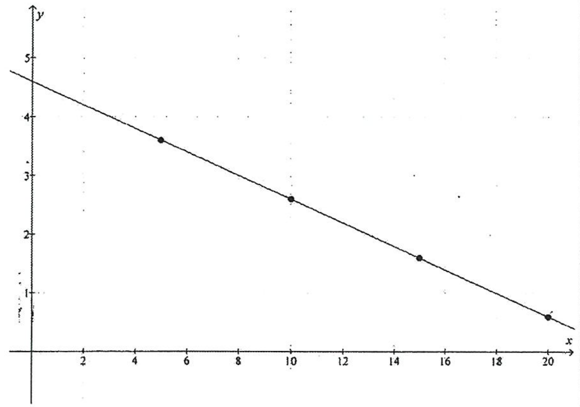
From the graph below, $\ln a \approx 4.6$ — Accept $4.5 \le \ln a \le 4.7$

$a \approx 99.48431564$

≈ 99.5 (correct to 1 d.p.) — 1A — Accept $90.0 \le a \le 110.0$

$-k = \frac{0.6-3.6}{20-5}$ — 1M

$k = 0.2$ — 1A



1A

(c) $270 = \frac{500}{1+99.48431564e^{-0.2t}}$

$t = 23.80171325$ — 1A — Accept $23.3 \le t \le 24.3$

24 days after the outbreak of the disease, the number of fish infected by the disease will reach 270 .

(6)

9. (2006 ASL-M&S Q2)

(a) $S(9)=S(19)$

$2(10^2)e^{-9\lambda}+15=2(20^2)e^{-19\lambda}+15$

$e^{10\lambda}=4$

$\lambda=\frac{\ln 4}{10}$ — 1A

Thus, we have $\lambda=\frac{\ln 2}{5}$.

(b) $S(t)=2(t+1)^2e^{-\lambda t}+15$

$\frac{dS(t)}{dt}=2\left(2(t+1)e^{-\lambda t}-\lambda(t+1)^2e^{-\lambda t}\right)$ — 1A

$=2(t+1)(2-\lambda-\lambda t)e^{-\lambda t}$

$\frac{dS(t)}{dt}=0$ when $t=\frac{2-\lambda}{\lambda}=\frac{2-\frac{\ln 2}{5}}{\frac{\ln 2}{5}}=\frac{10-\ln 2}{\ln 2}=T\approx 13.42695041$ — 1M

$\frac{dS(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<T\\ =0 & \text{if } t=T\\ <0 & \text{if } t>T\end{cases}$ — 1M for testing + 1A

Therefore, $S(t)$ attains its greatest value when $t=T$.

The greatest value of $S(t)$

$=2\left(\frac{10-\ln 2}{\ln 2}+1\right)^2 e^{\frac{\ln 2}{5}-2}+15$

≈ 79.71368176

<90

Thus, the temperature will not get higher than 90°C . — 1A f.t.

$S(t)=2(t+1)^2e^{-\lambda t}+15$	
$\frac{dS(t)}{dt}=2\left(2(t+1)e^{-\lambda t}-\lambda(t+1)^2e^{-\lambda t}\right)$	1A
$=2(t+1)(2-\lambda-\lambda t)e^{-\lambda t}$	
$\frac{d^2S(t)}{dt^2}=2\left(2e^{-\lambda t}-2\lambda(t+1)e^{-\lambda t}-2\lambda(t+1)e^{-\lambda t}+\lambda^2(t+1)^2e^{-\lambda t}\right)$	
$=2\left((\lambda^2-4\lambda+2)+(2\lambda^2-4\lambda)t+\lambda^2t^2\right)e^{-\lambda t}$	
$\frac{dS(t)}{dt}=0$ when $t=\frac{2-\lambda}{\lambda}=\frac{2-\frac{\ln 2}{5}}{\frac{\ln 2}{5}}=\frac{10-\ln 2}{\ln 2}=T\approx 13.42695041$	1M
$\left.\frac{d^2S(t)}{dt^2}\right\|_{t=T}=-4e^{-\lambda T}<0$	1M for testing + 1A
Note that there is only one local maximum.	
So, $S(t)$ attains its greatest value when $t=T$.	
The greatest value of $S(t)$	
$=2\left(\frac{10-\ln 2}{\ln 2}+1\right)^2 e^{\frac{\ln 2}{5}-2}+15$	
≈ 79.71368176	
<90	
Thus, the temperature will not get higher than 90°C .	1A f.t.

----------(6)

Fair. Many candidates did not get marks in (a) because they did not give the 'exact value' as required.

10. (2007 ASL-M&S Q3)

(a) $y=\frac{1-e^{4x}}{1+e^{8x}}$

$\frac{dy}{dx}=\frac{(1+e^{8x})(-4e^{4x})-(1-e^{4x})(8e^{8x})}{(1+e^{8x})^2}$ — 1M for quotient rule or product rule

When $x=0$, we have $\frac{dy}{dx}=-2$. — 1A

(b) (i) Since $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, we have $\ln(z^2+1)+3z=\alpha+\beta x$. — 1A

(ii) Since the graph of the linear function passes through the origin and the slope of the graph is 2 , we have $\alpha=0$ and $\beta=2$. — 1A for both correct

(iii) $\ln(z^2+1)+3z=2x$

$\frac{2z}{z^2+1}+3=2\frac{dx}{dz}$

Therefore, we have $\left.\frac{dx}{dz}\right|_{z=0}=\frac{3}{2}$. — 1A

Note that $x=0$ when $z=0$.

Also note that $\left.\frac{dy}{dx}\right|_{x=0}=-2$.

$\left.\frac{dy}{dz}\right|_{z=0}$

$=\left(\left.\frac{dy}{dx}\right|_{x=0}\right)\left(\left.\frac{dx}{dz}\right|_{z=0}\right)$

$=(-2)\left(\frac{3}{2}\right)$ — 1M for chain rule

$=-3$ — 1A

$y=\frac{1-e^{6z+2\ln(z^2+1)}}{1+e^{12z+4\ln(z^2+1)}}$	1A
$y=\frac{1-(z^2+1)^2e^{6z}}{1+(z^2+1)^4e^{12z}}$	
$\frac{dy}{dz}=\frac{\left(1+(z^2+1)^4e^{12z}\right)\left(-6(z^2+1)^2e^{6z}-2(z^2+1)(2z)e^{6z}\right)-\left(1-(z^2+1)^2e^{6z}\right)\left(12(z^2+1)^4e^{12z}+4(z^2+1)^3(2z)e^{12z}\right)}{\left(1+(z^2+1)^4e^{12z}\right)^2}$	1M for quotient rule or product rule
$\left.\frac{dy}{dz}\right\|_{z=0}=-3$	1A

----------(7)

Good. Most candidates could handle quotient rule and product rule. It is more efficient to apply $\frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}$ but some candidates went through the tedious way by expressing the function of y in terms of z .

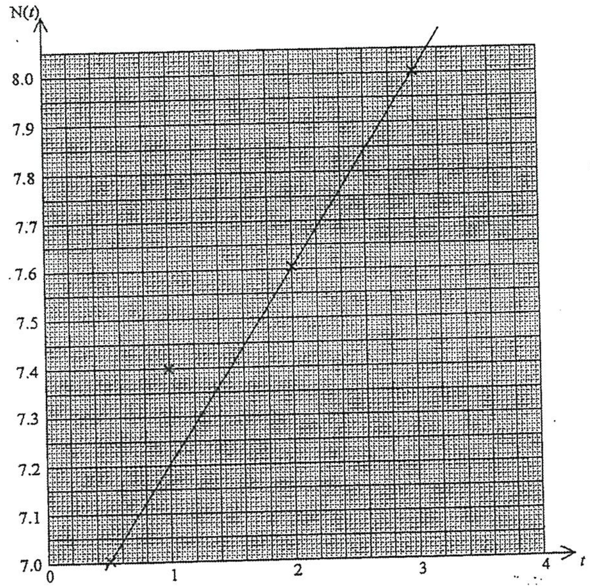
11. (2002 ASL-M&S Q3)

$N(t) = 900a^{kt}$

$\ln N(t) = (k\ln a)t + \ln 900$ 1A

t	0.5	1.0	2.0	3.0
$N(t)$	1100	1630	2010	2980
$\ln N(t)$	7.0031	7.3963	7.6059	7.9997

1M



At $t = 1.0$, $N(t) = 1630$ is incorrect, 1A

$\ln N(2.5) \approx 7.8$

$\therefore$ $N(2.5) \approx 2440$ 1A a–1 for more than 3 s.f.

(Accept : $N(2.5) \in [2420, 2470]$)

----------(4)

Section B

12. (2015 DSE-MATH-M1 Q12)

(a) $S = \dfrac{200}{1 + a2^{bt}}$

$\dfrac{200}{S} - 1 = a2^{bt}$ 1M

$\ln\left(\dfrac{200}{S} - 1\right) = (b\ln 2)t + \ln a$ 1A

----------(2)

(b) (i) $\ln a = \ln 4$

$a = 4$ 1A

$b\ln 2 = \dfrac{0 - \ln 4}{4 - 0}$

$b = -0.5$ 1A

(ii) $\dfrac{dS}{dt}$

$= \dfrac{-200(4)2^{-0.5t}(-0.5)\ln 2}{(1 + 4(2^{-0.5t}))^2}$ 1M for $\dfrac{d}{dt}2^{bt}$

$= \dfrac{(400\ln 2)2^{-0.5t}}{(1 + 4(2^{-0.5t}))^2}$ 1A

$\dfrac{d^2S}{dt^2}$

$= \dfrac{-200(\ln 2)^2(1 + 4(2^{-0.5t}))^2 2^{-0.5t} + 1600(\ln 2)^2(1 + 4(2^{-0.5t}))2^{-t}}{(1 + 4(2^{-0.5t}))^4}$ 1M for quotient rule

$= \dfrac{-200(\ln 2)^2 2^{-0.5t}(1 - 4(2^{-0.5t}))}{(1 + 4(2^{-0.5t}))^3}$ 1A

(iii) Note that $\dfrac{dS}{dt} > 0$ for $0 \le t \le 48$. 1M

Therefore, S increases for $0 \le t \le 48$. 1A f.t.

For $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right) = 0$, we have $4(2^{-0.5t}) = 1$.

Hence, we have $\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right) = 0$ when $t = 4$.

t	$0 \le t < 4$	$t = 4$	$4 < t \le 48$
$\dfrac{d}{dt}\left(\dfrac{dS}{dt}\right)$	+	0	–

1M+1A

Thus, $\dfrac{dS}{dt}$ increases for $0 \le t \le 4$ and

$\dfrac{dS}{dt}$ decreases for $4 \le t \le 48$. 1A f.t.

----------(11)

(a)	Very good. Most candidates were able to express $\ln\left(\dfrac{200}{S} - 1\right)$ as a linear function of t .
(b) (i)	Very good. A few candidates failed to use the slope of the linear function as a means to find the value of the unknown b .
(ii)	Fair. Many candidates failed to differentiate $2^{-0.5t}$ with respect to t correctly when finding the required derivatives $\dfrac{dS}{dt}$ and $\dfrac{d^2S}{dt^2}$.
(iii)	Poor. Only a few candidates were able to make use of the sign of $\dfrac{dS}{dt}$ to discuss the behaviour of S . Most candidates failed to determine the change of the sign of $\dfrac{d^2S}{dt^2}$ correctly.

13. (2014 DSE-MATH-M1 Q11)

(a) $\lim_{t\to\infty}\dfrac{340}{2+e^{-t}-2e^{-2t}}=\dfrac{340}{2+0-2\cdot 0}$ 1M

$=170$

Hence the value of y will not exceed 171 in the long run. 1A

(2)

(b) $\dfrac{dy}{dt}=340[-(2+e^{-t}-2e^{-2t})^{-2}](-e^{-t}+4e^{-2t})$ 1A

$=\dfrac{340(e^{-t}-4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$

$\therefore\ \dfrac{dy}{dt}=0$ when $e^{-t}-4e^{-2t}=0$ 1M (For $\dfrac{dy}{dt}=0$)

i.e. $t=\ln 4$ 1A

t	$0\le t<\ln 4$	$t=\ln 4$	$t>\ln 4$
$\dfrac{dy}{dt}$	-ve	0	+ve

1M

Hence y is minimum when $t=\ln 4$.

The minimum value of $y=\dfrac{340}{2+e^{-\ln 4}-2e^{-2\ln 4}}$

$=160$ 1A

When $t=0$, $y=\dfrac{340}{2+e^{0}-2e^{0}}$

$=340$

As the graph of y is continuous, and by (a), the greatest value of y is 340 and the least value of y is 160. 1A

(6)

(c) (i) $y=\dfrac{340}{2+e^{-t}-2e^{-2t}}$

$2y+ye^{-t}-2ye^{-2t}=340$

$2y(e^{-t})^2-ye^{-t}+340-2y=0$ 1A

(ii) Since $e^{-\alpha}$ and $e^{\alpha-3}$ are roots of the equation in (i),

$\dfrac{340-2y}{2y}=e^{-\alpha}e^{\alpha-3}$ 1M

$340-2y=2ye^{-3}$

Hence the equation becomes $2y(e^{-t})^2-ye^{-t}+2ye^{-3}=0$

i.e. $2(e^{-t})^2-e^{-t}+2e^{-3}=0$ 1A

$\therefore\ e^{-\alpha}=\dfrac{1+\sqrt{1-16e^{-3}}}{4}$ or $\dfrac{1-\sqrt{1-16e^{-3}}}{4}$ (rejected as $e^{-\alpha}$ is the greater root)

i.e. $\alpha=-\ln\dfrac{1+\sqrt{1-16e^{-3}}}{4}$ 1A (OR $\ln\dfrac{1-\sqrt{1-16e^{-3}}}{4}+3$ OR 1.0140)

(4)

(a)	Good. Some candidates thought that $\lim_{t\to+\infty}e^{-t}=1$.
(b)	Fair. Quite a lot of candidates failed to consider both the value of y at $t=0$ and the limit found in (a).
(c)	Very poor. Most candidates wrote wrongly the equation required in (i).

14. (PP DSE-MATH-M1 Q11)

(a) $P'(t)=kte^{\frac{a}{20}t}$

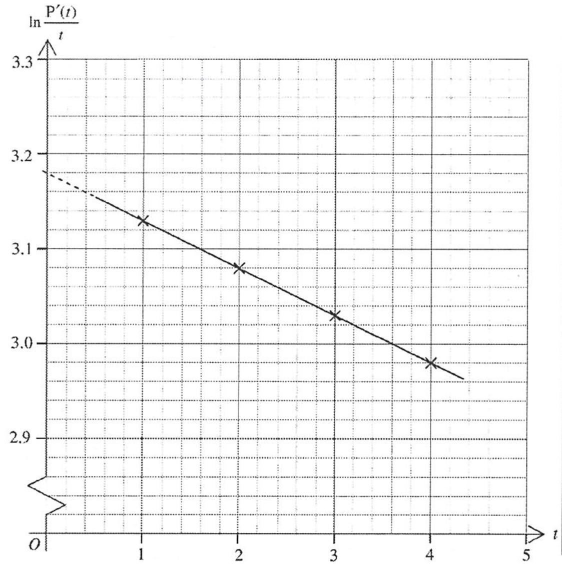
$\ln\dfrac{P'(t)}{t}=\dfrac{a}{20}t+\ln k$ 1A

(1)

(b)

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60
$\ln\dfrac{P'(t)}{t}$	3.13	3.08	3.03	2.98

1A



1A

From the graph, $\dfrac{a}{20}\approx\dfrac{2.98-3.13}{4-1}$ 1M

$a\approx -1$ 1A

From the graph, $\ln k\approx 3.18$ 1M (Either one)

$k\approx 24$ 1A

(5)

(c) (i) $\frac{d}{dt}P'(t)=\frac{d}{dt}\left(24te^{\frac{-t}{20}}\right)$

$=24e^{\frac{-t}{20}}\left(1-\frac{t}{20}\right)$ 1A

$\therefore\ \frac{d}{dt}P'(t)=0$ when $t=20$

t	<20	20	>20
$\frac{d}{dt}P'(t)$	+ve	0	-ve

1M

Alternative Solution

$\frac{d^2}{dt^2}P'(t)=24e^{\frac{-t}{20}}\left[\frac{-1}{20}\left(1-\frac{t}{20}\right)+\frac{-1}{20}\right]$

$=\frac{6}{5}e^{\frac{-t}{20}}\left(\frac{t}{20}-2\right)$

$\therefore\ \frac{d^2}{dt^2}P'(t)<0$ when $t=20$ 1M

Hence the rate of change of the population size is greatest when $t=20$. 1A

(ii) $\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)=e^{\frac{-t}{20}}-\frac{1}{20}te^{\frac{-t}{20}}$ 1A

$24te^{\frac{-t}{20}}=480e^{\frac{-t}{20}}-480\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)$

$\int 24te^{\frac{-t}{20}}\,dt=-9600e^{\frac{-t}{20}}-480te^{\frac{-t}{20}}+C$ 1M

$P(t)=C-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ 1A

Since $P(0)=30$, we have

$C-480(0)e^0-9600e^0=30$ 1M

$C=9630$

$\therefore\ P(t)=9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ 1A

(iii) $\lim_{t\to\infty}P(t)=\lim_{t\to\infty}\left(9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}\right)$

$=9630$ 1A

$\therefore$ the population size after a very long time is estimated to be 9630 thousands.

(9)

(a)	良好。 大部分學生能正確表達線性函數。
(b)	良好。 部分學生看漏了精確度的要求。
(c) (i)	平平。 部分學生在取得第一導數並使該導數相等於零之後，沒有作出極大測試。
(ii)	平平。 大部分學生不懂如何利用 $\frac{d}{dx}\left(te^{\frac{a}{20}t}\right)$ 的結果求 P(t)。
(iii)	甚差。 大部分學生因爲在前部分出錯而得出錯誤的結論。

15. (2011 ASL-M&S Q9)

(a) (i) $P=\frac{ke^{-\lambda t}}{t^2}$

$\ln P+2\ln t=\ln k-\lambda t$ 1A

(ii) Intercept of vertical axis $=2.3=\ln k$

$\therefore\ k\approx 10$ correct to the nearest integer 1A

slope $=\frac{2.3-0}{0-(-1.15)}=-\lambda$

$\therefore\ \lambda=-2$ 1A

$P=\frac{10e^{2t}}{t^2}$

$\frac{dP}{dt}=10\frac{t^2 2e^{2t}-e^{2t}2t}{t^4}$ 1A

$=\frac{20e^{2t}(t-1)}{t^3}$

t	$0<t<1$	$t=1$	$t>1$
$\frac{dP}{dt}$	-ve	0	+ ve

1M

Hence the minimum population size is attained when $t=1$.

$P=\frac{10e^{2(1)}}{(1)^2}$

$=74$ 1A

Hence the minimum population size is 74 hundred thousands.

(6)

(b) (i) $\frac{dE}{dt}=hte^{ht}-1.2e^{ht}+4.214$

$\frac{d^2E}{dt^2}=he^{ht}+h^2te^{ht}-1.2he^{ht}$ 1A

$=he^{ht}(ht-0.2)$

When $t=1$, $\frac{dE}{dt}$ is minimum and hence $\frac{d^2E}{dt^2}=0$.

Thus, $h=0.2$. 1A

(ii) $\frac{d}{dt}(te^{0.2t}) = 0.2te^{0.2t} + e^{0.2t}$ [1A]

$\therefore\ te^{0.2t} = 5\frac{d}{dt}(te^{0.2t}) - 5e^{0.2t}$

$\int te^{0.2t}dt = 5te^{0.2t} - 5\int e^{0.2t}dt$ [1M]

$= 5te^{0.2t} - 25e^{0.2t} + C_1$ [1A]

$\frac{dE}{dt} = 0.2te^{0.2t} - 1.2e^{0.2t} + 4.214$

$E = 0.2\int te^{0.2t}\,dt - 1.2\int e^{0.2t}\,dt + \int 4.214\,dt$

$= te^{0.2t} - 5e^{0.2t} - 6e^{0.2t} + 4.214t + C$ [1M]

$= te^{0.2t} - 11e^{0.2t} + 4.214t + C$

When $t = 0$, $E = 1$.

Hence $1 = 0 - 11 + 0 + C$ which gives $C = 12$.

i.e. $E = te^{0.2t} - 11e^{0.2t} + 4.214t + 12$

When $t = 1$, $E = e^{0.2} - 11e^{0.2} + 4.214 + 12$

≈ 4 [1A]

Thus the annual electricity consumption is 4 thousand terajoules per year.

(iii) $F = \frac{6}{1 - 5e^{rt} + 3e^{2rt}} + 2$

$\frac{6}{1 - 5e^{r} + 3e^{2r}} + 2 \approx 4$ [1M]

$3e^{2r} - 5e^{r} - 2 \approx 0$

$e^{r} \approx 2$ or $\frac{-1}{3}$ (rejected)

$r \approx \ln 2$ [1A] OR 0.6931

(9)

(a)(i)(ii)		Good.
(b)(i)		Fair. Some candidates overlooked that the given condition was for the rate of change of annual electricity consumption. When considering the minimum rate of change, the second derivative $\frac{d^2E}{dt^2}$ should be considered.
(ii)		Poor. Many were unable to proceed after (b)(i).
(iii)		Poor. Many candidates did not attempt this part.

16. (2009 ASL-M&S Q8)

(a) (i) $R = kt^{1.2}e^{\frac{\lambda t}{20}}$

$\ln R = \ln k + 1.2\ln t + \frac{\lambda t}{20}$

$\ln R - 1.2\ln t = \frac{\lambda}{20}t + \ln k$ which is a linear function of t [1A]

(ii) intercept on the vertical axis $= \ln k = 2.89$

$k \approx 18$ (correct to the nearest integer) [1A]

slope $= \frac{\lambda}{20} = -0.05$

$\lambda = -1$ [1A]

(iii) $\therefore\ R = 18t^{1.2}e^{-0.05t}$

$\frac{dR}{dt} = 18[1.2t^{0.2}e^{-0.05t} + t^{1.2}e^{-0.05t}(-0.05)]$ [1M]

$= 0.9t^{0.2}e^{-0.05t}(24 - t)$

	$0 < t < 24$	$t = 24$	$24 < t \le 30$
$\frac{dR}{dt}$	>0	0	<0

[1M]

Hence, R will attain maximum after 24 months. [1A]

$R = 18(24)^{1.2}e^{-0.05(24)}$

≈ 245.6815916

Hence, the maximum population size is 246 hundreds. [1A]

(7)

(b) (i) When $t = 0$, $L - 20(6e^{0} + 0^{3}) = Q = 240$

$\therefore\ L = 360$ [1A]

(ii) $e^{-t} = 1 + (-t) + \frac{(-t)^2}{2!} + \frac{(-t)^3}{3!} + \cdots$

$= 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots$ [1A]

$\therefore\ Q = 360 - 20\left[6\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots\right) + t^3\right]$

$= 360 - 20(6 - 6t + 3t^2 + \cdots)$

$\approx 240 + 120t - 60t^2$ which is a quadratic polynomial [1A]

(iii) Let $300 = Q = 240 + 120t - 60t^2$ (by (b)(ii)) [1M]

i.e. $t^2 - 2t + 1 = 0$

Hence, when $t = 1$, the species of fish will reach a population size of 300. [1A] Follow through

(iv) $Q = L - 20(6e^{-t} + t^3)$

$= 360 - 20\left[6\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \cdots\right) + t^3\right]$

$= 240 + 120t - 60t^2 - 120\left[\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) + \left(\frac{t^6}{6!} - \frac{t^7}{7!}\right) + \cdots\right]$

$= 300 - 60(t-1)^2 - 120\left[\frac{t^4}{5!}(5 - t) + \frac{t^6}{7!}(7 - t) + \cdots\right]$ [1M+1M] 1M for completing square, 1M for factorization

Since $0 \le t \le 2$, $Q < 300$ and so the conclusion in (b)(iii) is no more valid. [1A] Follow through

(8)

(a)	(i)		Very good.
	(ii)		Very good. Some candidates overlooked the precision requirement.
	(iii)		Fair. Some candidates still failed to provide the test for maximum after getting the first derivative and equated it to zero. Algebraic simplification should also be strengthened.
(b)	(i)		Very good.
	(ii)		Good. Familiarity with algebraic simplification would help.
	(iii)		Good. Candidates had little difficulty in attempting this part.
	(iv)		Very poor. Candidates seemed to be unfamiliar in grouping the terms meaningfully for decision making.

17. (2008 ASL-M&S Q8)

(a) (i) $N'(t)=\dfrac{20}{1+he^{-kt}}$ $(t \ge 0)$

$\ln\left[\dfrac{20}{N'(t)}-1\right]=-kt+\ln h$ 1A

(ii) $\ln h = 1.5$ 1M (Either One)

$h=e^{1.5}$

≈ 4.4817 (correct to 4 d.p.) 1A

$-k=\dfrac{1.5-0}{0-7.6}$ (Either One)

$k=\dfrac{15}{76}$

≈ 0.1974 (correct to 4 d.p.) 1A

(4)

(b) (i) $v=4.5+e^{0.2t}$

$\dfrac{dv}{dt}=0.2e^{0.2t}$ 1A

$N(t)=\int \dfrac{20}{1+4.5e^{-0.2t}}\,dt$

$=\int \dfrac{100}{e^{0.2t}+4.5}(0.2e^{0.2t})\,dt$

$=\int \dfrac{100}{v}\,dv$ 1M

$=100\ln|v|+C$

$=100\ln(4.5+e^{0.2t})+C$ $(\because 4.5+e^{0.2t}>0)$ 1A

Since $N(0)=50$, so $C=50-100\ln 5.5$ 1M

i.e. $N(t)=100\ln\dfrac{4.5+e^{0.2t}}{5.5}+50$ 1A

(ii) (1) $M(20)=N(20)$

$21\left[(20)+\dfrac{4.5}{0.2}e^{-0.2(20)}\right]+b=100\ln\dfrac{4.5+e^{0.2(20)}}{5.5}+50$

$b\approx -141.2090$ 1A

(2) Consider $M'(t)-N'(t)$

$=21(1-4.5e^{-0.2t})-\dfrac{20}{1+4.5e^{-0.2t}}$ 1A For the 1st term

$=\dfrac{1-425.25e^{-0.4t}}{1+4.5e^{-0.2t}}$ 1A

$\therefore M'(t)-N'(t)>0$ when $e^{-0.4t}<\dfrac{1}{425.25}$ 1M

i.e. $t>\dfrac{\ln 425.25}{0.4}\approx 15.1317$ 1A

Since $M(20)=N(20)$ and $M(t)-N(t)$ is increasing when $t>20$, so $M(t)>N(t)$ for $t>20$.

Hence the biologist's claim is correct. 1A Follow through

(11)

(a) (i)		Very good.
(a) (ii)		Very good, though some careless mistakes were found.
(b) (i)		Fair. A number of candidates could not apply substitution to do integration.
(ii) (1)		Fair. Some candidates were hindered by failing to complete (b) (i).
(2)		Poor. Not too many candidates attempted and those who attempted could not make use of the given hint.

Marking 2.17

18. (2006 ASL-M&S Q9)

(a) Let $u=t+10$. Then, we have $\dfrac{du}{dt}=1$. 1A can be absorbed

The total amount

$=\int_0^3 f(t)\,dt$

$=\int_0^3 25t^2(t+10)^{\frac{-1}{3}}\,dt$

$=\int_{10}^{13} 25(u-10)^2 u^{\frac{-1}{3}}\,du$ 1A

$=25\int_{10}^{13}(u^{\frac{5}{3}}-20u^{\frac{2}{3}}+100u^{\frac{-1}{3}})\,du$ 1M

$=25\left[\dfrac{3}{8}u^{\frac{8}{3}}-12u^{\frac{5}{3}}+150u^{\frac{2}{3}}\right]_{10}^{13}$ 1M

≈ 97.65521668

≈ 97.6552 thousand metres 1A $a-1$ for r.t. 97.655

--------(5)

(b) $\ln(g(t)-28)=\ln k+ht^2$ 1A

--------(1)

(c) $h\approx 0.3$ (correct to 1 decimal place) 1A

$\ln k\approx 1.0$

$k\approx 2.718281828$

$k\approx 2.7$ (correct to 1 decimal place) 1A

--------(2)

(d) (i) $g(t)\approx 28+2.7e^{0.3t^2}$

$=28+2.7\left(1+0.3t^2+\dfrac{(0.3t^2)^2}{2!}+\dfrac{(0.3t^2)^3}{3!}+\cdots\right)$ 1M

$=30.7+0.81t^2+0.1215t^4+0.01215t^6+\cdots$ 1A pp−1 for omitting ' ⋯ '

The total amount

$=\int_0^3 g(t)\,dt$

$\approx\int_0^3(30.7+0.81t^2+0.1215t^4+0.01215t^6)\,dt$ 1M

$=\left[30.7t+\dfrac{0.81t^3}{3}+\dfrac{0.1215t^5}{5}+\dfrac{0.01215t^7}{7}\right]_0^3$

≈ 109.0909071

≈ 109.0909 thousand metres 1A $a-1$ for r.t. 109.091

(ii) $e^{0.3t^2}=1+0.3t^2+\dfrac{(0.3t^2)^2}{2!}+\dfrac{(0.3t^2)^3}{3!}+r(t)$ and $r(t)>0$ for all $t>0$.

Thus, the estimate in (d)(i) is an under-estimate. 1A f.t.

(iii) Note that the estimate in (d)(i) is greater than the total amount of cloth production under John's model and that the estimate in (d)(i) is an under-estimate. 1M for using (a), (d)(i) and (d)(ii)

Thus, the total amount of cloth production under Mary's model is greater than that under John's model. 1A f.t.

--------(7)

(a)		Good. Some candidates could not handle change of variables in integration.
(b)		Very good.
(c)		Very good.
(d)(i)		Fair. Some candidates could not expand the exponential series.
(ii)		Fair. Many candidates could not state the fact that the neglected part of the exponential series expansion is positive and hence the estimate is an under-estimate.
(iii)		Not satisfactory. A few candidates could develop the logical analysis and arrive at a conclusion.

Marking 2.18

19. (2005 ASL-M&S Q8)

(a) $r(t)=\alpha te^{-\beta t}$

$\frac{r(t)}{t}=\alpha e^{-\beta t}$

$\ln\frac{r(t)}{t}=\ln\alpha-\beta t$ 1A

---------(1)

(b) $\because \ln\alpha=2.3$

$\therefore \alpha\approx10$ (correct to 1 significant figure) 1A

Also, we have $\beta\approx0.5$ (correct to 1 significant figure). 1A

$r(t)=10te^{-0.5t}$

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$

$=10e^{-0.5t}-5te^{-0.5t}$ 1A

$=(10-5t)e^{-0.5t}$

$\frac{dr(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<2\\ =0 & \text{if } t=2\\ <0 & \text{if } t>2\end{cases}$ 1M for testing + 1A

So, $r(t)$ attains its greatest value when $t=2$.

Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx7.357588823$.

Thus, the greatest rate of change is 7 ppm per hour. 1A

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$

$=10e^{-0.5t}-5te^{-0.5t}$ 1A

$=(10-5t)e^{-0.5t}$

$\frac{d^2r(t)}{dt^2}=-5e^{-0.5t}+(-5+2.5t)e^{-0.5t}=(2.5t-10)e^{-0.5t}$

$\frac{dr(t)}{dt}=0$ when $t=2$ only and $\left.\frac{d^2r(t)}{dt^2}\right|_{t=2}=-5e^{-1}<0$ 1M for testing + 1A

So, $r(t)$ attains its greatest value when $t=2$.

Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx7.357588823$.

Thus, the greatest rate of change is 7 ppm per hour. 1A

---------(6)

(c) (i) $\frac{d}{dt}\left((t+\frac{1}{\beta})e^{-\beta t}\right)$

$=\frac{d}{dt}\left((t+2)e^{-0.5t}\right)$

$=e^{-0.5t}-0.5(t+2)e^{-0.5t}$ 1M for product rule or chain rule

$=-0.5te^{-0.5t}$ 1M accept $-\beta te^{-\beta t}$

The required amount

$=\int_0^T r(t)dt$

$=\int_0^T 10te^{-0.5t}dt$ 1M

$=\left[-20(t+2)e^{-0.5t}\right]_0^T$ 1M+1A

$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm 1A

Note that

$\int r(t)dt$

$=\int 10te^{-0.5t}dt$

$=-20(t+2)e^{-0.5t}+C$ 1M+1A

Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours.

Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$. 1M

Since $A(0)=0$, we have $C=40$.

So, we have $A(t)=\left(40-20(t+2)e^{-0.5t}\right)$.

Note that $A(0)=0$.

Thus, the required amount $=A(T)=\left(40-20(T+2)e^{-0.5T}\right)$ ppm 1A

Note that

$\int r(t)dt$

$=\int 10te^{-0.5t}dt$

$=-20(t+2)e^{-0.5t}+C$ 1M+1A

Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours.

Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$. 1M

The required amount

$=A(T)-A(0)$

$=\left(-20(T+2)e^{-0.5T}+C\right)-\left(-40+C\right)$

$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm 1A

(ii) The required amount

$=\lim_{T\to\infty}\left(40-20(T+2)e^{-0.5T}\right)$

$=40-20\lim_{T\to\infty}Te^{-0.5T}-40\lim_{T\to\infty}e^{-0.5T}$

$=40-20(0)-40(0)$ 1M for $\lim_{T\to\infty}e^{-0.5T}=0$ and can be absorbed

$=40$ ppm 1A

---------(8)

(a)		Very Good.
(b)		Good. Some candidates were not able to show that the stationary point is a maximum point.
(c)(i)		Fair. The first part was done well but the later part was less satisfactory. Only some candidates were able to work out the total amount of soot reduced.
(ii)		Poor. Many candidates were not able to complete this part because they failed to solve (c)(i).

20. (2004 ASL-M&S Q9)

Solution	Marks
(a) (i) $\frac{dy}{dx}=-\alpha\beta^{-x}$	
$-\frac{dy}{dx}=\alpha\beta^{-x}$	
$\ln(-\frac{dy}{dx})=\ln\alpha-(\ln\beta)x$	1A do not accept $\ln\alpha-\ln\beta x$
$-0.125=-\ln\beta$	
[illegible]	
$\beta\approx 1.133$ (correct to 3 decimal places)	1A
(ii) $\beta^{-x}=e^{-\lambda x}$ for all $x\ge 0$	
$\lambda=\ln\beta$	
$\lambda=0.125$	1A accept $\lambda\approx 0.1249$ a−1 for r.t. 0.125
$\frac{dy}{dx}=-\alpha\beta^{-x}$	
$\frac{dy}{dx}=-\alpha e^{-\lambda x}$	1M can be absorbed
$y=-\alpha\int e^{-\lambda x}dx$	1M for finding y by integration
$y=\frac{\alpha}{\lambda}e^{-\lambda x}+C$ [illegible]	1A for correct integration
Note that $y(0)=76$ and $y(2)=59.2$. Then, we have	
$\frac{\alpha}{\lambda}+C=76$ and $\frac{\alpha}{\lambda}e^{-2\lambda}+C=59.2$.	1M
So, we have $\frac{\alpha}{\lambda}(1-e^{-2\lambda})=16.8$. Hence, we have	
[illegible]	
$\alpha\approx 9.5$ (correct to 1 decimal place)	1A

Solution (alternative)	Marks
$\beta^{-x}=e^{-\lambda x}$ for all $x\ge 0$	
$\lambda=\ln\beta$	
$\lambda=0.125$	1A accept $\lambda\approx 0.1249$ a−1 for r.t. 0.125
$\frac{dy}{dx}=-\alpha\beta^{-x}$	
$\frac{dy}{dx}=-\alpha e^{-\lambda x}$	1M can be absorbed
$[y]_0^2=-\alpha\int_0^2 e^{-\lambda x}dx$	1M for integrating from $x=0$ to $x=2$ on both sides
$y(2)-y(0)=\frac{\alpha}{\lambda}\left[e^{-\lambda x}\right]_0^2$	1A for correct integration
So, we have $\frac{\alpha}{\lambda}(1-e^{-2\lambda})=16.8$. Hence, we have	1M
[illegible]	
$\alpha\approx 9.5$ (correct to 1 decimal place)	1A
	----------(8)

Solution	Marks
(b) (i) By (a)(ii),	
[illegible]	
$C\approx 0.0504$	accept $C\in[-0.08,\ 0.06]$
So, $y\approx 75.94963597e^{-0.125x}+0.050364028$	accept $y\approx Be^{-0.125x}+C$ where $B\in[75.94,\ 76.08]$
When $y=25.2$, we have	
$25.2\approx 75.94963597e^{-0.125x}+0.050364028$	1M for leaving x only
[illegible]	
$x\approx 8.8$ (correct to 1 decimal place)	1A provided B and C both acceptable
Thus, the altitude of the mountain is 8.8 km above sea-level (correct to the nearest 0.1 km).	
(ii) $\frac{\alpha}{\lambda}e^{-\lambda h}-\frac{\alpha}{\lambda}e^{-2\lambda h}=13$	1M for using $y(h)-y(2h)=13$
$\frac{\alpha}{\lambda}(e^{-\lambda h})^2-\frac{\alpha}{\lambda}e^{-\lambda h}+13=0$	1M for transforming into a quadratic equation
$75.94963597(e^{-0.125h})^2-75.94963597e^{-0.125h}+13\approx 0$	
$e^{-0.125h}\approx 0.780773822$ or $e^{-0.125h}\approx 0.219226177$	
$h\approx 1.979758169$ or $h\approx 12.1412048$	1M for taking ln to find h
Note that $h\approx 12.1412048$ is rejected since $h>8.8$ is impossible.	
Thus, we have $h\approx 2.0$ (correct to 1 decimal place).	2A provided (b)(i) is correct
	----------(7)

(a)		Fair. Some candidates still forgot the integration constant.
(b)		Not satisfactory. Difficulties mainly arose from misunderstanding the question. Some candidates thought that $y(2h)-y(h)=13$.

21. (2001 ASL-M&S Q9)

(a) (i) $\ln P'(t) = -kt + \ln\dfrac{0.04ak}{1-a}$ — 1A

From the graph,

$-k \approx \dfrac{-8-(-3.5)}{18-0}$, $k \approx 0.25$ — 1A a–1 for more than 2 d.p.

$\ln\dfrac{0.04ak}{1-a} \approx -3.5$, $a \approx 0.7512 \approx 0.75$ — 1A a–1 for more than 2 d.p.

$P'(t) \approx 0.03e^{-0.25t}$

$P(t) \approx -0.12e^{-0.25t} + c$ for some constant c — 1M

Since $P(0) = 0.09$, $\therefore$ $c \approx 0.21$ — 1A

Hence $P(t) \approx -0.12e^{-0.25t} + 0.21$

(ii) $\mu = P(3) \approx 0.1533$ — 1A $\mu \in [0.1530, 0.1533]$

(iii) Stabilized PPI in town $A = \lim\limits_{t\to\infty} P(t) = 0.21$ — 1M+1A

---------(8)

(b) (i) Suppose $b = 0.09$.

(I) $Q'(t) = 0.24(3t+4)^{-\frac{3}{2}}$

$Q(t) = \dfrac{1}{3}(0.24)(-2)(3t+4)^{-\frac{1}{2}} + c$ for some constant c — 1A

$= -0.16(3t+4)^{-\frac{1}{2}} + c$

Since $Q(0) = 0.09$, $\therefore$ $c = 0.17$ — 1A

If $Q(t) = \mu \approx 0.1533$

$-0.16(3t+4)^{-\frac{1}{2}} + 0.17 \approx 0.1533$ — 1M

$(3t+4)^{\frac{1}{2}} \approx \dfrac{0.16}{0.0167}$

Since $3t+4 > 0$

$\therefore$ $t \approx 29.3$ — $t \in [28.2, 29.3]$

i.e. the PPI will reach the value of μ. — 1

Since $Q(0) = 0.09$, $\lim\limits_{t\to\infty} Q(t) = 0.17$ and

Q is continuous and strictly increasing ($Q'(t) > 0$) , — [1M]

$\therefore$ Q can reach any value between 0.09 and 0.17 including $\mu \approx 0.1533$. — [1]

(II) Stabilized PPI in town $B = \lim\limits_{t\to\infty} Q(t) = 0.17$

$\therefore$ The stabilized PPI will be reduced by 0.04 . — 1A

(ii) $0.05 < b$ [≤ 1]. — 1A

Otherwise, $Q'(t) \le 0$ and the PPI will not increase.

It follows that the epidemic will not break out. — 1

---------(7)

22. (2000 ASL-M&S Q10)

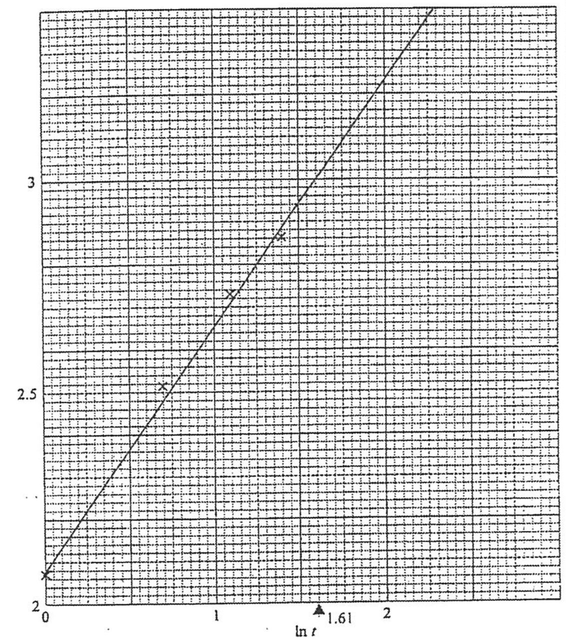
(a) (i) $r(t) = at^b$

$\ln r(t) = \ln a + b\ln t$ — 1A

(ii)

$\ln t$	0	0.69	1.10	1.39
$\ln r(t)$	2.07	2.51	2.73	2.86

Accept using common logarithm



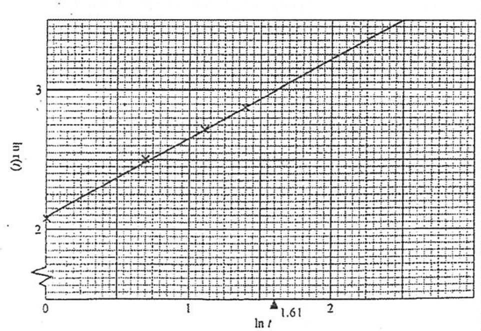
1M scale and labelling
1A points and line

When $t = 5$, $\ln t \approx 1.61$

$\therefore$ $\ln r(5) \approx 3$; from the graph. — 1M for either

$r(5) \approx 20.1$ — 1A $r(5) \in [19.1, 21.1]$; $\ln r(5) \in [2.95, 3.05]$

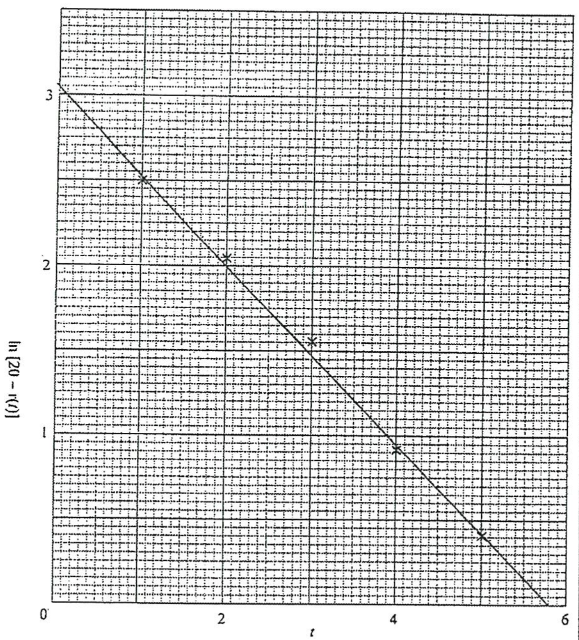
Alternative graph for 10(a)(ii)



(b) (i) $r(t) = 20 - pe^{-qt}$
$\ln[20 - r(t)] = \ln p - qt$ — 1A

(ii)

t	1	2	3	4	5
$\ln[20 - r(t)]$	2.49	2.04	1.55	0.92	0.41



1M scale and labelling
1A ignoring the point at $t = 5$

From the graph, $\ln p \approx 3.05$
$p \approx 21.1$

1A $p \in [20.1, 23.3]$
$\ln p \in [3.00, 3.15]$

$-q \approx \dfrac{0.41 - 3.05}{5} \approx -0.528$
$q \approx 0.528$

1A $q \in [0.518, 0.550]$

No marks for p or q if the graph is not correct.

(iii) The total number, in thousands, of bacteria after 15 days of cultivation

$= \int_0^{15} [20 - pe^{-qt}]dt + 100$ — {1M definite integral; 1M adding 100}

$= \left[20t + \dfrac{p}{q}e^{-qt}\right]_0^{15} + 100$ — 1M for integration

$= 300 + \dfrac{p}{q}e^{-15q} - \dfrac{p}{q} + 100$

$\approx 260 + 100$ — 1A $\int_0^{15}[20 - pe^{-qt}]dt \in [255, 263]$

≈ 360 — 1A Ans. $\in [355, 363]$
pp−1 for wrong/missing unit

Alternatively,
Let N(t) thousand be the total number of bacteria after t days of cultivation. Then

$N(t) = \int [20 - pe^{-qt}]dt$ — 1M

$= 20t + \dfrac{p}{q}e^{-qt} + c$ — 1M for integration

$\because$ N(0) = 100

$\therefore$ $100 = \dfrac{p}{q} + c$

$c = 100 - \dfrac{p}{q} \approx 60.04$ — 1A $c \in [55.02, 63.46]$

Hence the total number, in thousands, of bacteria after 15 days of cultivation is

$N(15) = 20 \times 15 + \dfrac{p}{q}e^{-15q} + c \approx 360$ — 1M+1A N(15) $\in [355, 363]$

23. (1999 ASL-M&S Q9)

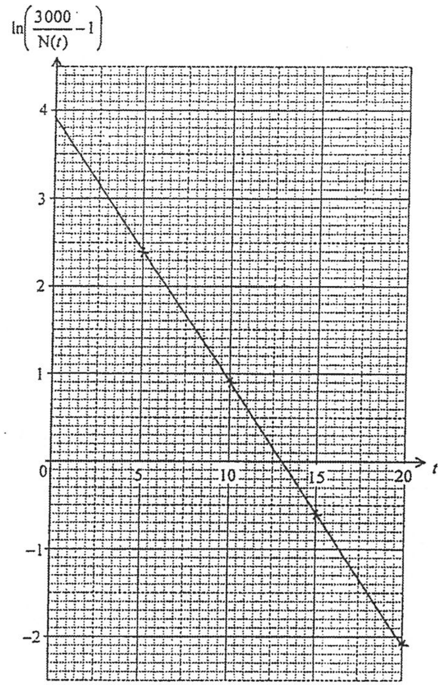
(a) (i) $N(t)=\dfrac{3000}{1+ae^{-bt}} \Leftrightarrow \dfrac{3000}{N(t)}-1=ae^{-bt}$

$\Leftrightarrow \ln\left(\dfrac{3000}{N(t)}-1\right)=-bt+\ln a$ — 1A — *pp*−1 for $-bt\ln e+\ln a$

(ii)

t	5	10	15	20
$\ln\left(\dfrac{3000}{N(t)}-1\right)$	2.40 (2.4)	0.90 (0.9)	−0.60 (−0.6)	−2.09 (−2.1)

1A — Correct to 1 d.p.



1A — the line must pass through all the 4 points

From the graph, $\ln a \approx 3.9$, — accept 3.85 – 3.95

$a \approx 49.4$ — 1A — accept 47.0 – 51.9

$b \approx -\dfrac{-2.09-2.40}{20-5} \approx 0.3$ — 1A

(b) (i) $N(t)=\dfrac{3000}{1+ae^{-bt}}$ (or $\dfrac{3000}{1+49.4e^{-0.3t}}$)

$N'(t)=\dfrac{3000abe^{-bt}}{(1+ae^{-bt})^2}$ — 1M

$=\dfrac{3000(49.4)(0.3)e^{-0.3t}}{(1+49.4e^{-0.3t})^2}$ (or $\dfrac{44460e^{-0.3t}}{(1+49.4e^{-0.3t})^2}$) — 1A — accept $a\in[47.0, 51.9]$ and $3000ab\in[42300, 46710]$

$\therefore$ $N'(t)>0$ for all t

$N(t)$ is increasing — 1

(ii) If $N'(t)=\dfrac{1}{100}N(t)$

$\dfrac{3000abe^{-bt}}{(1+ae^{-bt})^2}=\dfrac{1}{100}\cdot\dfrac{3000}{1+ae^{-bt}}$ — 1M — $a\in[47.0, 51.9]$, $b=0.3$

$e^{-bt}=\dfrac{1}{a(100b-1)}$ — $a\in[47.0, 51.9]$, $b=0.3$

$t=\dfrac{1}{0.3}\ln[a(100b-1)]$

OR

$\dfrac{44460e^{-0.3t}}{(1+49.4e^{-0.3t})^2}=\dfrac{1}{100}\cdot\dfrac{3000}{1+49.4e^{-0.3t}}$ — 1M

$1482e^{-0.3t}=1+49.4e^{-0.3t}$

$t\approx 24.2242$ — 1A — $t\in[24.0581, 24.3887]$

$\because$ $N(\dfrac{1}{0.3}\ln[a(100b-1)])=\dfrac{3000}{1+ae^{-b(\frac{1}{0.3}\ln[a(100b-1)])}}=2900$ — 1M

OR

$N(24.2242)=\dfrac{3000}{1+49.4e^{-0.3(24.2242)}}\approx 2900$ — 1M

$\therefore$ The greatest number of migrants found at Mai Po is 2900. — 1A

(iii) Suppose all the migrants leave Mai Po in x days.

Then $\int_0^x 60\sqrt{s}\,ds=2900$ — 1M

$\left[40s^{\frac{3}{2}}\right]_0^x=2900$ — 1A — for integration (including limits)

$x\approx 17.3870$

$\therefore$ The number of days in which we can see the migrants is $24.2242+17.3870\approx 42$ — 1A — r.t. 42

24. (1998 ASL-M&S Q10)

(a) (i) (I): $\ln r(t) = \ln\alpha + \beta\ln t$ — 1A

(II): $\ln r(t) = \ln\gamma + \lambda t$ — 1A

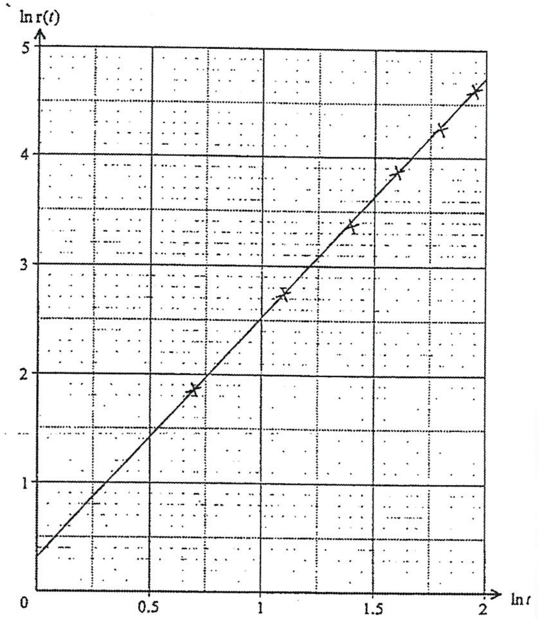
(ii)

t	2	3	4	5	6	7
$r(t)$	6.4	15.7	29.5	48.3	72.2	101.2
$\ln t$	0.69	1.10	1.39	1.61	1.79	1.95
$\ln r(t)$	1.86	2.75	3.38	3.88	4.28	4.62

1A — Correct to 1 d.p.

1A — Correct to 1 d.p.

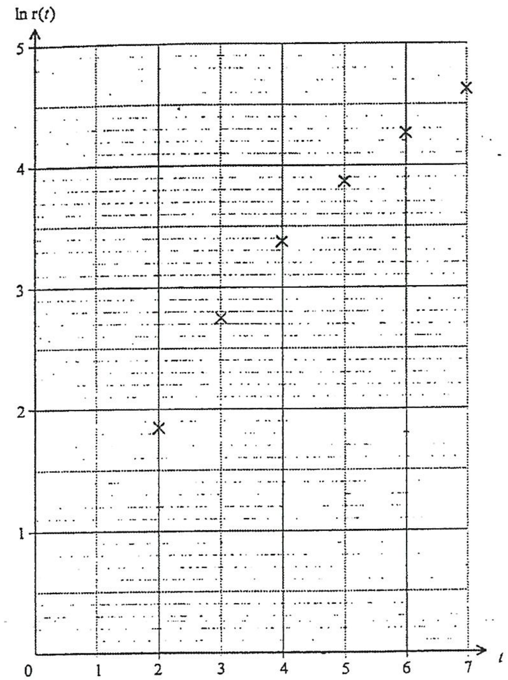
(I)



1A — for any 2 points being correct

1A — for all the 6 points being correct

(II)



1A — for any 2 points being correct

1A — for all the 6 points being correct

From the graphs, equation (I) would be a better model and

$\ln\alpha \approx 0.3$ — Accept 0.3 - 0.4

$\alpha \approx e^{0.3} \approx 1.3$ — 1A — Accept 1.3 - 1.5

$\beta \approx \dfrac{4.62-1.86}{1.95-0.69} \approx 2.2$ — 1A — Accept 2.0 – 2.4

(b) $\int_0^{14} \alpha t^{\beta}\,dt$ where $\alpha \approx 1.3$, $\beta \approx 2.2$ — 1M+1A — Accept $\alpha \in [1.3, 1.5]$, $\beta \in [2.0, 2.4]$

$= \dfrac{\alpha}{\beta+1}\left[t^{\beta+1}\right]_0^{14}$ $\quad\left(\approx \dfrac{1.3}{3.2}\left[t^{3.2}\right]_0^{14}\right)$

≈ 1889

$\therefore$ 1889 hundred of trees would be destroyed in the first 14 days. — 1A — Accept 1498 - 3015

Consider $\int_0^{k} \alpha t^{\beta}\,dt = 1889\times 2$

$\dfrac{1.3}{3.2}\left[t^{3.2}\right]_0^{k} = 3778$ — 1M

$k^{3.2} \approx 9299.69$

$k \approx e^{\frac{\ln 9299.69}{3.2}} \approx 17.3839$

$\therefore$ The total number of trees destroyed will be doubled in 4 days more. — 1A

25. (1997 ASL-M&S Q9)

(a) $b \approx \dfrac{7.49-7.95}{8-3.4}$		
≈ -0.1	1A	
Sub. (8, 7.49) into $\ln N(x) = -0.1x + \ln a$.		
$7.49 \approx \ln a - 0.8$	1M	
$a \approx 4000$	1A	
(b) (i) $N(x) = ae^{bx} = 4000e^{-0.1x}$	1M	
Daily profit (in dollars) of selling N(x) clams:		
$P(x) = N(x)\cdot x - (2N(x)+5000)$	1A	for 2N(x)+5000
$= (x-2)N(x) - 5000$		
$= 4000(x-2)e^{-0.1x} - 5000$	1A	
(ii) $P'(x) = 4000[(x-2)(-0.1e^{-0.1x}) + e^{-0.1x}]$	1A	
$= 400e^{-0.1x}(12-x)$		
$P'(x) \begin{cases} >0 & \text{if } 0<x<12 \\ =0 & \text{if } x=12 \\ <0 & \text{if } x>12 \end{cases}$	1	
$\therefore$ P(x) attains its maximum when $x = 12$.		
Hence the selling price of each clam = \$12	1A	
the number of clams sold per day $= N(12)$		
$= 4000e^{-0.1(12)}$		
≈ 1205	1A	
(c) The difference between the numbers of clams sold on the n-th and $(n-1)$-th days after the launch of the promotion programme		
$= M(n) - M(n-1)$		
$= \left[1500+1000(1-e^{-0.1n})\right] - \left[1500+1000(1-e^{-0.1(n-1)})\right]$	1M	
$= 1000(-e^{-0.1n} + e^{-0.1n}\cdot e^{0.1})$		
$= 1000e^{-0.1n}(e^{0.1}-1)$	1A	
If $M(n) - M(n-1) < 15$	1M	
then $e^{-0.1n} < \dfrac{15}{1000(e^{0.1}-1)}$		
$n > 19.475$	1M	
$\therefore$ The promotion programme should run for 20 days.	1A	

26. (1994 ASL-M&S Q9)

(a) (i) The machine will cease producing cloth when $x=0$,	1M	
$100e^{-0.01t} - 65e^{-0.02t} - 35 = 0$		
Put $y = e^{-0.01t}$,		
$100y - 65y^2 - 35 = 0$		
$13y^2 - 20y + 7 = 0$		
$(y-1)(13y-7) = 0$		
$y = 1$ or $\dfrac{7}{13}$		
$\therefore$ $e^{-0.01t} = 1$ or $\dfrac{7}{13}$		
$t = 0$ (rej.) or $t = \dfrac{\ln\frac{7}{13}}{-0.01}$		
$= 61.9039$	1A	r.t. 62
It will cease producing cloth in February, 2000		
(ii) The total amount of cloth produced during the lifespan of the machine		
$= \int_0^{61.904} x\,dt$	1M	Accept $\int_0^{62} x\,dt$
$= \int_0^{61.904} (100e^{-0.01t} - 65e^{-0.02t} - 35)\,dt$		
$= -10000e^{-0.01t} + \dfrac{65}{0.02}e^{-0.02t} - 35t \Big\|_0^{61.904}$	1M	
≈ 141 (km)	1A	
(b) Let P be the monthly profit, then		
$P = 800x - 300x - 300$	1A	
$= 500x - 300$		
$= 500(100e^{-0.01t} - 65e^{-0.02t} - 35) - 300$		
$= 50000e^{-0.01t} - 32500e^{-0.02t} - 17800$		
$\dfrac{dP}{dt} = -500e^{-0.01t} + 650e^{-0.02t}$	1M	
$\dfrac{dP}{dt} = 0$ when $650e^{-0.02t} = 500e^{-0.01t}$		
or $t = t_0$ where $t_0 = \dfrac{1}{0.01}\ln\left(\dfrac{650}{500}\right) \approx 26.2364$	1A	r.t. 26
$\left.\dfrac{d^2P}{dt^2}\right\|_{t=t_0} = (5e^{-0.01t} - 13e^{-0.02t})\|_{t=t_0} \approx -3.85 < 0$	1M	For proving max.
Hence P is maximum when $t = t_0$		

Alternatively		
$x = 100e^{-0.01t} - 65e^{-0.02t} - 35$		
$\dfrac{dx}{dt} = -e^{-0.01t} + 1.3e^{-0.02t}$	1M	
$\dfrac{dx}{dt} = 0$ when $e^{-0.01t} = 1.3e^{-0.02t}$		
or $t = t_0$ where $t_0 = \dfrac{\ln 1.3}{0.01} \approx 26.2364$	1A	
$\because$ $P = 800x - 300x - 300 = 500x - 300$	1A	
$\therefore$ P is maximum when x is maximum		
$\left.\dfrac{d^2x}{dt^2}\right\|_{t=t_0} = (0.01e^{-0.01t} - 0.026e^{-0.02t})\|_{t=t_0} \approx -0.0077 < 0$	1M	
Hence P is maximum when $t = t_0$		

$\because$ $P\|_{t=26} \approx 1431$ $P\|_{t=27} \approx 1430$ $\therefore$ The greatest monthly profit will be obtained when $t=26$, i.e. in February, 1997.	1	For checking P when $t=26, 27$
The greatest monthly profit is US$1431.	1A	Accept $P\|_{t=t_0} = 1431$, r.t. 1431
(c) If $P = 500$, then $500 = 50000e^{-0.01t} + 32500e^{-0.02t} - 17800$ $5 = 500e^{-0.01t} - 325e^{-0.02t} - 178$	1A	
<u>Alternatively</u> $500x - 300 = 500$ $x = 1.6$ $100e^{-0.01t} - 65e^{-0.02t} - 35 = 1.6$ $500e^{-0.01t} - 325e^{-0.02t} - 183 = 0$	1A	
Put $y = e^{-0.01t}$, $325y^2 - 500y + 183 = 0$ $(65y - 61)(5y - 3) = 0$ $y = \frac{61}{65}$ or $\frac{3}{5}$ $e^{-0.01t} = \frac{61}{65}$ or $\frac{3}{5}$ $t = \frac{1}{-0.01}\ln\frac{61}{65}$ or $\frac{1}{-0.01}\ln\frac{3}{5}$ ≈ 6.35 or 51.08	1A	
$\because$ P is increasing when $t \approx 6.35$ (OR The machine has not yet reached its production climax when $t \approx 6.35$)	1	
$\therefore$ The machine should be replaced when $t \approx 51.08$, i.e. in April, 1999.	1A	

3. Derivatives and Differentiation of Functions

Learning Unit	Learning Objective
Calculus Area	
Differentiation with Its Applications	
3. Derivative of a function	3.1 recognise the intuitive concept of the limit of a function 3.2 find the limits of algebraic functions, exponential functions and logarithmic functions 3.3 recognise the concept of the derivative of a function from first principles 3.4 recognise the slope of the tangent of the curve $y = f(x)$ at a point $x = x_0$
4. Differentiation of a function	4.1 understand the addition rule, product rule, quotient rule and chain rule of differentiation 4.2 find the derivatives of algebraic functions, exponential functions and logarithmic functions
5. Second derivative	5.1 recognise the concept of the second derivative of a function 5.2 find the second derivative of an explicit function

Implicit differentiation is **not** required.

~~Logarithmic differentiation is required.~~

Logarithmic differentiation is **not** required.

~~• $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$~~

Section A

1. Consider the curve C: $y=\frac{x}{\sqrt{x-2}}$, where $x>2$.

(a) Find $\frac{dy}{dx}$.

(b) A tangent to C passes through the point $(9,0)$. Find the slope of this tangent.

(7 marks) (2017 DSE-MATH-M1 Q7)

2. Consider the curve C: $y=(2x+8)^{\frac{3}{2}}+3x^2$, where $x>-4$.

(a) Find $\frac{dy}{dx}$.

(b) Someone claims that two of the tangents to C are parallel to the straight line $6x+y+4=0$. Do you agree? Explain your answer.

(7 marks) (2016 DSE-MATH-M1 Q7)

3. Consider the curve C: $y=x\sqrt{2x^2+1}$.

(a) Find $\frac{dy}{dx}$.

(b) Two of the tangents to C are perpendicular to the straight line $3x+17y=0$. Find the equations of the two tangents.

(7 marks) (2015 DSE-MATH-M1 Q7)

4. Consider the curve C: $y=x(x-2)^{\frac{1}{3}}$ and the straight line L that passes through the origin and is parallel to the tangent to C at $x=3$.

(a) Find the equation of L.

(b) Find the x-coordinates of the two intersecting points of C and L.

(4 marks) (2013 DSE-MATH-M1 Q3a, b)

5. It is given that $t=y^3+2y^{\frac{-1}{2}}+1$ and $e^t=x^{x^2+1}$.

(a) Find $\frac{dt}{dy}$.

(b) By expressing t in terms of x, find $\frac{dt}{dx}$.

(c) Find $\frac{dy}{dx}$ in terms of x and y.

(5 marks) (PP DSE-MATH-M1 Q2)

6. Consider the curve C: $y=x(2x-1)^{\frac{1}{2}}$, where $x>\frac{1}{2}$.

(a) Find $\frac{dy}{dx}$.

(b) Using (a), find the equations of the two tangents to the curve C which are parallel to the straight line $2x-y=0$.

(6 marks) (PP DSE-MATH-M1 Q4)

7. Let $x=\ln\dfrac{1+t}{1-t}$, where $-1<t<1$.

(a) Find $\dfrac{dx}{dt}$.

(b) Let $y=1+e^{-x}-e^{-2x}$.

(i) Find $\dfrac{dy}{dx}$.

(ii) Find the value of $\dfrac{dy}{dt}$ when $t=\dfrac{1}{2}$.

(6 marks) (2013 ASL-M&S Q2)

8. Let $y=\dfrac{1-e^{4x}}{1+e^{8x}}$.

(a) Find the value of $\dfrac{dy}{dx}$ when $x=0$.

(b) Let $(z^2+1)e^{3z}=e^{\alpha+\beta x}$, where α and β are constants.

(i) Express $\ln(z^2+1)+3z$ as a liner function of x.

(ii) It is given that the graph of the linear function obtained in (b)(i) passes through the origin and the slope of the graph is 2. Find the values of α and β.

(iii) Using the values of α and β obtained in (b)(ii), find the value of $\dfrac{dy}{dz}$ when $z=0$.

(7 marks) (2007 ASL-M&S Q3)

9. A chemical X is continuously added to a solution to form a substance Y. The total amount of Y formed is given by

$$y=3+\frac{4x-9}{\sqrt{4x^2+3x+9}},$$

where x grams and y grams are the total amount of X added and the total amount of Y formed respectively.

(a) Find $\dfrac{dy}{dx}$ when 10 grams of X is added to the solution.

(b) Estimate the total amount of Y formed if X is indefinitely added to the solution.

(6 marks) (2003 ASL-M&S Q3)

10. Let $u=e^{2x}$, and $\dfrac{dy}{du}=\dfrac{1}{u}-2u$.

(a) Express $\dfrac{du}{dx}$ and $\dfrac{dy}{dx}$ in terms of x.

(b) It is known that $y=1$ when $x=0$. Express y in terms of x.

(5 marks) (2001 ASL-M&S Q2)

11. Let $y=xe^{\frac{1}{x}}$ where $x>0$. Show that $x^4\dfrac{d^2y}{dx^2}-y=0$.

(5 marks) (1998 ASL-M&S Q1)

12. The population size x of an endangered species of animals is modeled by the equation

$$\frac{d^2x}{dt^2}-2\frac{dx}{dt}-3x=0,$$

where t denotes the time.

It is known that $x=100e^{kt}$ where k is a negative constant. Determine the value of k.

(5 marks) (1994 ASL-M&S Q2)

13. Let $x=-\dfrac{5}{t^2}+2e^{-3t}$ and $y=\dfrac{10}{t^2}+e^{2t}$ $(t\neq 0)$. It is given that $\dfrac{dy}{dx}=-2$. By considering $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$, find the value of t.

(4 marks) (*modified from* 2002 ASL-M&S Q1)

14. It is given that $\begin{cases}x=\ln(2t+4)\\ y=e^{t^2+4t+4}\end{cases}$, where $t>-2$.

(a) By considering $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$, express $\dfrac{dy}{dx}$ in terms of y only.

(b) Find the value of $\dfrac{d^2y}{dx^2}$ when $x=0$.

(6 marks) (*modified from* 2012 ASL-M&S Q2)

Limit at infinity (Section A)

15. Define $f(x)=\dfrac{6-x}{x+3}$ for all $x>-3$.
 (a) Prove that $f(x)$ is decreasing.
 (b) Find $\lim\limits_{x\to\infty} f(x)$.
 (c) Find the exact value of the area of the region bounded by the graph of $y=f(x)$, the x-axis and the y-axis.

(6 marks) (2019 DSE-MATH-M1 Q5)

16. Let $f(x)$ be a continuous function such that $f'(x)=\dfrac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x.
 (a) If $f(x)$ attains its minimum value at $x=\alpha$, find α.
 (b) It is given that the extreme value of $f(x)$ is 5 . Find
  (i) $f(x)$,
  (ii) $\lim\limits_{x\to\infty} f(x)$.

(6 marks) (2018 DSE-MATH-M1 Q5)

17. The value $R(t)$, in thousand dollars, of a machine can be modelled by

$$R(t)=Ae^{-0.5t}+B\ ,$$

where t (≥ 0) is the time, in years, since the machine has been purchased. At $t=0$, its value is 500 thousand dollars and in the long run, its value is 10 thousand dollars.
 (a) Find the values of A and B .
 (b) The machine can generate revenue at a rate of $P'(t)=600e^{-0.3t}$ thousand dollars per year, where t is the number since the machine has been purchased. Richard purchased the machine for his factory and used it for 5 years before he sold it. How much did he gain in this process? Correct your answer to the nearest thousand dollars.

(6 marks) (2013 ASL-M&S Q3)

18. An advertising company starts a media advertisement to recruit new members for a club. Past experience shows that the rate of change of the number of members N (in thousand) is given by

$$\frac{dN}{dt}=\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}\ ,$$

where t (≥ 0) is the number of weeks elapsed after the launch of the advertisement. The club has 500 members before the launch of the advertisement.
 (a) Using the substitution $u=1+e^{-0.2t}$, express N in terms of t .
 (b) Find the increase in the number of members of the club 4 weeks after the launch of the advertisement. Correct your answer to the nearest integer.

 (c) Will the number of members of the club ever reach 1300 after the launch of the advertisement? Explain your answer.

(7 marks) (2012 ASL-M&S Q3)

19. A company launches a promotion plan to raise revenue. The total amount of money X (in million dollars) invested in the plan can be modelled by

$$\frac{dX}{dt}=6\left(\frac{t}{0.2t^3+1}\right)^2\ ,\quad t\geq 0\ ,$$

where t is the number of months elapsed since the launch of the plan.
Initially, 4 million dollars are invested in the plan.
 (a) Using the substitution $u=0.2t^3+1$, or otherwise, express X in terms of t .
 (b) Find the number of months elapsed since the launch of the plan if a total amount of 13 million dollars are invested in the plan.
 (c) If the company has a budget of 14.5 million dollars only, can the plan be run for a long time? Explain your answer.

(7 marks) (2011 ASL-M&S Q2)

20. The rate of change of concentration of a drug in the blood of a patient can be modelled by

$$\frac{dx}{dt}=5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t}\ ,$$

where x is the concentration measured in mg/L and t is the time measured in hours after the patient has taken the drug. It is given that $x=0$ when $t=0$.
 (a) Find x in terms of t .
 (b) Find the concentration of the drug after a long time.

(6 marks) (2008 ASL-M&S Q3)

21. A researcher models the rate of change of the number of fish in a lake by

$$\frac{dN}{dt}=\frac{6}{(e^{\frac{t}{4}}+e^{-\frac{t}{4}})^2}\ ,$$

where N is the number in thousands of fish in the lake recorded yearly and $t(\geq 0)$ is the time measured in years from the start of the research. It is known that $N=8$ when $t=0$.
 (a) Prove that $\dfrac{dN}{dt}=\dfrac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$. Using the substitution $u=e^{\frac{t}{2}}+1$, or otherwise, express N in terms of t .
 (b) Estimate the number of fish in the lake after a very long time.

(6 marks) (2004 ASL-M&S Q2)

22. An engineer conducts a test for a certain brand of air-purifier in a smoke-filled room. The percentage of smoke in the room being removed by the air-purifier is given by S %. The engineer models the rate of change of S by

$$\frac{dS}{dt}=\frac{8100t}{(3t+10)^3},$$

where $t\ (\geq 0)$ is measured in hours from the start of the test.

(a) Using the substitution $u = 3t + 10$, or otherwise, find the percentage of smoke removed from the room in the first 10 hours.

(b) If the air-purifier operates indefinitely, what will the percentage of smoke removed from the room be?

(5 marks) (2002 ASL-M&S Q4)

23. An adventure estimates the volume of his hot air balloon by $V(r)=\frac{4}{3}\pi r^3+5\pi$, where r is measured in metres and V is measured in cubic metres. When the balloon is being inflated, r will increase with time $t(\geq 0)$ in such a way that,

$$r(t)=\frac{18}{3+2e^{-t}}$$

where t is measured in hours.

(a) Find the rate of change of volume of the balloon at t = 2. Give your answer correct to 2 decimal places.

(b) If the balloon is being inflated over a long period of time, what will the volume of the balloon be? Give your answer correct to 2 decimal places.

(5 marks) (2002 ASL-M&S Q2)

Limit at infinity (Section B)

24. Let y be the amount (in suitable units) of suspended particulate in a laboratory. It is given that

$$(E):\quad y=\frac{340}{2+e^{-t}-2e^{-2t}}\quad (t\geq 0),$$

where t is the time (in hours) which has elapsed since an experiment started.

(a) Will the value of y exceed 171 in the long run? Justify your answer.

(2 marks)

(b) Find the greatest value and least value of y.

(6 marks)

(c) (i) Rewrite (E) as a quadratic equation in e^{-t}.

(ii) It is known that the amounts of suspended particulate are the same at the time $t=\alpha$

and $t=3-\alpha$. Given that $0\leq\alpha<3-\alpha$, find α.

(4 marks)

(2014 DSE-MATH-M1 Q11)

25. A researcher models the rate of change of the population size of a kind of insects in a forest by

$$P'(t)=kte^{\frac{a}{20}t},$$

where $P(t)$, in thousands, is the population size, t $(\geq 0)$is the time measured in weeks since the start of the research, and a, k are integers.

The following table shows some values of t and $P'(t)$.

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60

(a) Express $\ln\frac{P'(t)}{t}$ as a linear function of t.

(1 mark)

(b) By plotting a suitable straight line on the graph paper on next page, estimate the integers a and k.

(5 marks)

(c) Suppose that $P(0)=30$. Using the estimates in (b),

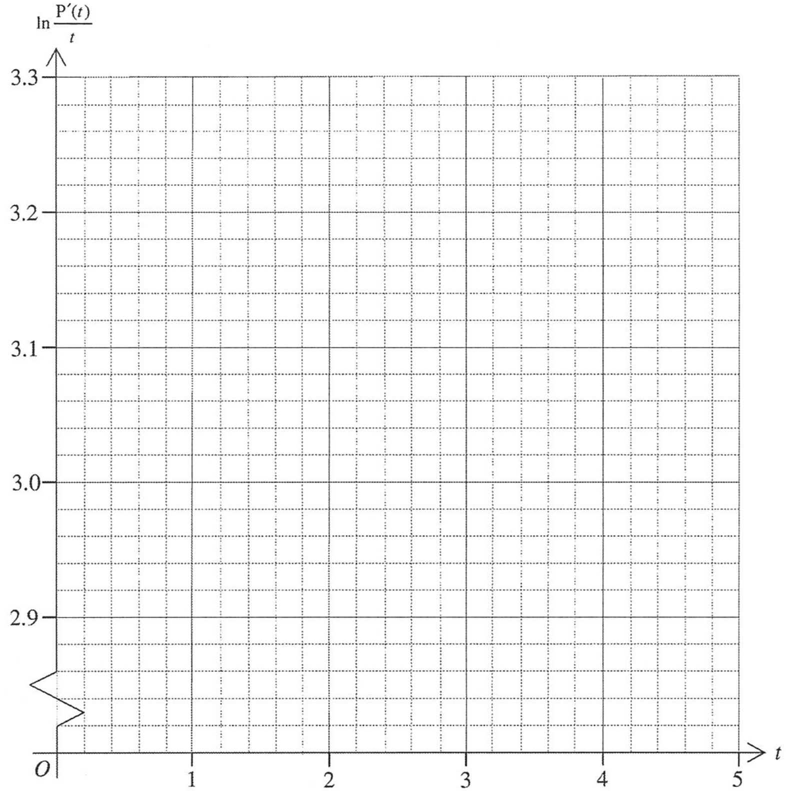
(i) find the value of t such that the rate of change of the population size of the insect is the greatest;

(ii) find $\frac{d}{dt}\left(te^{\frac{a}{20}t}\right)$ and hence, or otherwise, find $P(t)$;

(iii) estimate the population size after a very long time.

[Hint: You may use the fact that $\lim_{t\to\infty}\frac{t}{e^{mt}}=0$ for any positive constant m.]

(9 marks)



(PP DSE-MATH-M1 Q11)

26. The manager, Mary, of a theme park starts a promotion plan to increase **the daily number of visits** to the park. The rate of change of **the daily number of visits** to the park can be modelled by

$$\frac{dN}{dt}=\frac{k(25-t)}{e^{0.04t}+4t} \quad (t \geq 0),$$

where N is **the daily number of visits** (in hundreds) recorded at the end of a day, t is the number of days elapsed since the start of the plan and k is a positive constant.

Mary finds that at the start of the plan, $N=10$ and $\frac{dN}{dt}=50$.

(a) (i) Let $v=1+4te^{-0.04t}$, find $\frac{dv}{dt}$.

(ii) Find the value of k, and hence express N in terms of t.

(7 marks)

(b) (i) When will **the daily number of visits** attain the greatest value?

(ii) Mary claims that there will be more than 50 hundred visits on a certain day after the start of the plan. Do you agree? Explain your answer.

(3 marks)

(c) Mary's supervisor believes that **the daily number of visits** to the park will return to the original one at the start of the plan after a long period of time. Do you agree? Explain your answer.

(Hint: $\lim_{t\to\infty} te^{-0.04t}=0$.)

(2 marks)

(SAMPLE DSE-MATH-M1 Q11)

27. The population of a kind of bacterium $p(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$p(t)=\frac{a}{b+e^{-t}}+c \quad , \quad -\infty<t<\infty$$

where a , b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a , b and c ,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population*.

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[p(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a , b and c .

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the primordial population and ultimate population. Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a , b , c and $p(t)$, express $p'(t)$ in the form of $\frac{-b}{a}[p(t)-\alpha][p(t)-\beta]$, where $\alpha<\beta$. Hence express α and β in terms of a , b and c . Sketch $p'(t)$ against $p(t)$ for $\alpha<p(t)<\beta$ and hence verify your answer in (c).

(5 marks)

(2010 ASL-M&S Q9)

28. A shop owner wants to launch two promotion plans A and B to raise the revenue. Let R and Q (in million dollars) be the respective cumulative weekly revenues of the shop after the launching of the promotion plans A and B . It is known that R and Q can be modelled by

$$\frac{dR}{dt}=\begin{cases}\ln(2t+1) & \text{when } 0\le t\le 6\\ 0 & \text{when } t>6\end{cases},$$

and

$$\frac{dQ}{dt}=\begin{cases}45t(1-t)+\dfrac{1.58}{t+1} & \text{when } 0\le t\le 1\\ \dfrac{30e^{-t}}{(3+2e^{-t})^2} & \text{when } t>1\end{cases}$$

respectively, where t is the number of weeks elapsed since the launching of a promotion plan.

(a) Suppose plan A is adopted.

(i) Using the trapezoidal rule with 6 sub-intervals, estimate the total amount of revenue in the first 6 weeks since the start of the plan.

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Explain your answer briefly.

(4 marks)

(b) Suppose plan B is adopted.

(i) Find the total amount of revenue in the first week since the start of the plan.

(ii) Using the substitution $u=3+2e^{-t}$, or otherwise, find the total amount of revenue in the first n weeks, where $n>1$, since the start of the plan. Express your answer in terms of n .

(6 marks)

(c) Which of the plans will produce more revenue in the long run? Explain your answer briefly.

(5 marks)

(2009 ASL-M&S Q9)

29. A researcher studied the soot reduction effect of a petrol additive on soot emission of a car. Let t be the number of hours elapsed after the petrol additive has been used and $r(t)$, measured in ppm per hour, be the rate of change of the amount of soot reduced. The researcher suggested that $r(t)$ can be modeled by $r(t)=\alpha te^{-\beta t}$, where α and β are positive constants.

(a) Express $\ln\dfrac{r(t)}{t}$ as a linear function of t.

(1 mark)

(b) It is given that the slope and the intercept on the vertical axis of the graph of the linear function obtained in (a) are –0.50 and 2.3 respectively. Find the values of α and β correct to 1 significant figure.

Hence find the greatest rate of change of the amount of soot reduced after the petrol additive has been used. Give your answer correct to 1 significant figure.

(6 marks)

(c) Using the values of α and β obtained in (b) correct to 1 significant figure,

(i) find $\dfrac{d}{dt}\left((t+\dfrac{1}{\beta})e^{-\beta t}\right)$ and hence find, in terms of T, the total amount of soot reduced when the petrol additive has been used for T hours;

(ii) estimate the total amount of soot reduced when the petrol additive has been used for a very long time.

[Note: Candidates may use $\lim\limits_{T\to\infty}(Te^{-\beta T})=0$ without proof.]

(8 marks)

(2005 ASL-M&S Q8)

30. A researcher monitors the process of using micro-organisms to decompose food waste to fertilizer. He records daily the pH value of the waste and models its pH value by

$$P(t)=a+\frac{1}{5}\left(t^2-8t-8\right)e^{-kt},$$

where $t(\geq 0)$ is the time measured in days, a and k are positive constants.

When the decomposition process starts (i.e. $t=0$), the pH value of the waste is 5.9. Also, the researcher finds that $P(8)-P(4)=1.83$.

(a) Find the values of a and k correct to 1 decimal place.

(5 marks)

(b) Using the value of k obtained in (a),

(i) determine on which days the maximum pH value and the minimum pH value occurred respectively;

(ii) prove that $\dfrac{d^2P}{dt^2}>0$ for all $t\geq 23$.

(8 marks)

(c) Estimate the pH value of the waste after a very long time.

[Note: Candidates may use $\lim\limits_{t\to\infty}\left(t^2e^{-kt}\right)=0$ without proof.]

(2 marks)

(2003 ASL-M&S Q9)

31. The spread of an epidemic in a town can be measured by the value of PPI (the proportion of population infected). The value of PPI will increase when the epidemic breaks out and will stabilize when it dies out.

The spread of the epidemic in town A last year could be modelled by the equation

$P'(t)=\dfrac{0.04ake^{-kt}}{1-a}$, where a, $k>0$ and $P(t)$ was the PPI t days after the outbreak of the epidemic. The figure shows the graph of $\ln P'(t)$ against t, which was plotted based on some observed data obtained last year. The initial value of PPI is 0.09 (i.e. $P(0)=0.09$).

(a) (i) Express $\ln P'(t)$ as a linear function of t and use the figure to estimate the values of a and k correct to 2 decimal places.

Hence find $P(t)$.

(ii) Let μ be the PPI 3 days after the outbreak of the epidemic. Find μ.

(iii) Find the stabilized PPI.

(8 marks)

(b) In another town B, the health department took precautions so as to reduce the PPI of the epidemic. It is predicted that the rate of spread of the epidemic will follow the equation

$Q'(t)=6(b-0.05)(3t+4)^{\frac{-3}{2}}$, where $Q(t)$ is the PPI t days after the outbreak of the epidemic in town B and b is the initial value of PPI.

(i) Suppose $b=0.09$.

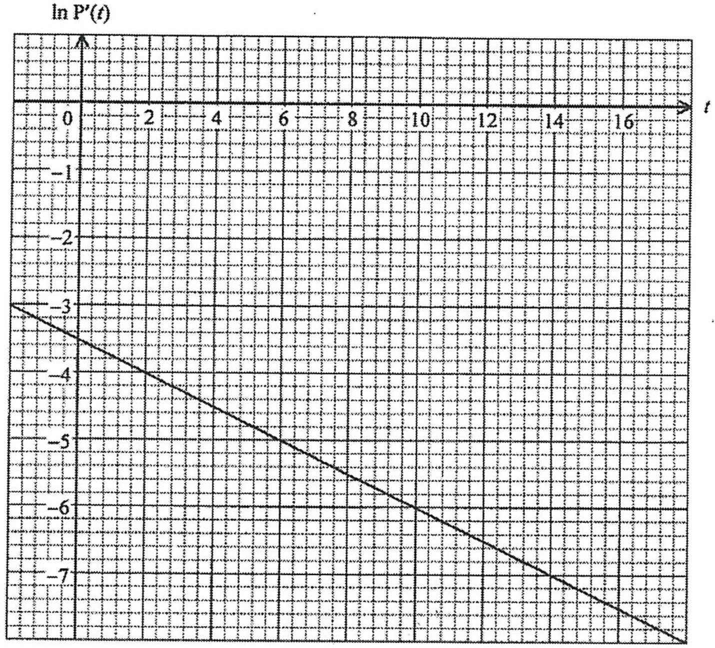
(I) Determine whether the PPI in town B will reach the value of μ in (a)(ii).

(II) How much is the stabilized PPI reduced in town B as compared with that in town A?

(ii) Find the range of possible values of b if the epidemic breaks out in town B. Explain your answer briefly.

(7 marks)

The graph of $\ln P'(t)$ against t



(2001 ASL-M&S Q9)

32. A department store has two promotion plans, F and G, designed to increase its profit, from which only one will be chosen. A marketing agent forecasts that if x hundred thousand dollars is spent on a promotion plan, the respective rates of change of its profit with respect to x can be modelled by

$$f(x)=16+4xe^{-0.25x} \quad \text{and} \quad g(x)=16+\frac{6x}{\sqrt{1+8x}} .$$

(a) Suppose that promotion plan F is adopted.

(i) Show that $f(x)\le f(4)$ for $x>0$.

(ii) If six hundred thousand dollars is spent on the plan, use the trapezoidal rule with 6 sub-intervals to estimate the expected increase in profit to the nearest hundred thousand dollars.

(6 marks)

(b) Suppose that promotion plan G is adopted.

(i) Show that $g(x)$ is strictly increasing for $x>0$.
As x tends to infinity, what value would $g(x)$ tend to?

(ii) If six hundred thousand dollars is spent on the plan, use the substitution $u=\sqrt{1+8x}$, or otherwise, to find the expected increase in profit to the nearest hundred thousand dollars.

(7 marks)

(c) The manager of the department store notices that if six hundred thousand dollars is spent on promotion, plan F will result in a bigger profit than G . Determine which plan will eventually result in a bigger profit if the amount spent on promotion increases indefinitely. Explain your answer briefly.

(2 marks)

(2000 ASL-M&S Q9)

33. A researcher studied the commercial fishing situation in a certain fishing zone. Denoting the total catch of coral fish in that zone in t years time from January 1, 1992 by $N(t)$ (in thousand tonnes), he obtained the following data:

t	2	4
$N(t)$	55	98

The researcher modelled $N(t)$ by $\ln N(t) = a - e^{1-kt}$ where a and k are constants.

(a) Show that $e^{-4k} - e^{-2k} + \frac{1}{e}\ln\frac{98}{55} = 0$.

Hence find, to 2 decimal places, two sets of values of a and k .

(4 marks)

(b) The researcher later found out that $N(7) = 170$. Determine which set of values of a and k obtained in (a) will make the model fit for the known data.

Hence estimate, to the nearest thousand tonnes, the total possible catch of coral fish in that zone since January 1, 1992.

(4 marks)

(c) The rate of change of the total catch of coral fish in that zone since January 1, 1992 by at time t is given by $\frac{dN(t)}{dt}$.

(i) Show that $\frac{dN(t)}{dt} = kN(t)e^{1-kt}$.

(ii) Using the values of a and k chosen in (b), determine in which year the maximum rate of change occurred.

Hence find, to the nearest integer, the volume of fish caught in that year.

(7 marks)

(Part c is out of Syllabus) (2000 ASL-M&S Q11)

34. A vehicle tunnel company wants to raise the tunnel fees. An expert predicts that after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day will drop drastically in the first week and on the t-th day after the first week, the number $N(t)$ (in thousands) of vehicles passing through the tunnel can be modelled by

$$N(t) = \frac{40}{1 + be^{-rt}} \qquad (t \geq 0)$$

where b and r are positive constants.

(a) Suppose that by the end of the first week after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day drops to 16 thousand and by the end of the second week, the number increases to 17.4 thousand, find b and r correct to 2 decimal places.

(5 marks)

(b) Show that $N(t)$ is increasing.

(3 marks)

(c) As time passes, $N(t)$ will approach the average number N_a of vehicles passing through the tunnel each day before the increase in the tunnel fees. Find N_a .

(2 marks)

(d) The expert suggests that the company should start to advertise on the day when the rate of increase of the number of cars passing through the tunnel per day is the greatest. Using the values of b and r obtained in (a),

(i) find $N''(t)$, and

(ii) hence determine when the company should start to advertise.

(5 marks)

(1997 ASL-M&S Q8)

Summary of Limit at Infinity

Limits	Question
$\lim_{x\to\infty} \frac{6-x}{x+3} = -1$	(2019 DSE-MATH-M1 Q5)
$\lim_{t\to\infty}\left(3+\frac{4x-9}{\sqrt{4x^2+3x+9}}\right)=5$	(2003 ASL-M&S Q3)
$\lim_{t\to\infty} \frac{-2}{3x^2-24x+49}+7=7$	(2018 DSE-MATH-M1 Q5)
$\lim_{t\to\infty} Ae^{-0.5t}+B=B$	(2013 ASL-M&S Q3)
$\lim_{t\to\infty}\left[\frac{3}{2(1+2e^{-0.2t})}-\frac{1}{4}\right]=1.25$	(2012 ASL-M&S Q3)
$\lim_{t\to\infty}\left(\frac{-10}{0.2t^3+1}+14\right)=14$	(2011 ASL-M&S Q2)
$\lim_{t\to\infty}\{5.3[\ln(t+2)-\ln(t+5)]-12^{-0.1t}+16.8563\}=16.8563$	(2008 ASL-M&S Q3)
$\lim_{t\to\infty}\left(14-\frac{12}{e^{\frac{t}{2}}+1}\right)=14$	(2004 ASL-M&S Q2)
$\lim_{T\to\infty}\left(\frac{-1}{3T+10}+\frac{5}{(3T+10)^2}+0.05\right)=0.05$	(2002 ASL-M&S Q4)
$\lim_{t\to\infty}\left(\frac{18}{3+2e^{-t}}\right)=6$	(2002 ASL-M&S Q2)
$\lim_{t\to\infty}\left(\frac{340}{2+e^{-t}-2e^{-2t}}\right)=170$	(2014 DSE-MATH-M1 Q11)
$\lim_{t\to\infty}\left(9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}\right)=9630$	(PP DSE-MATH-M1 Q11)
$\lim_{t\to\infty}[12.5\ln(1+4te^{-0.04t})+10]=10$	(SAMPLE DSE-MATH-M1 Q11)
$\lim_{t\to-\infty}\left(\frac{a}{b+e^{-t}}+c\right)=c$	(2010 ASL-M&S Q9)
$\lim_{t\to\infty}\left(\frac{a}{b+e^{-t}}+c\right)=\frac{a}{b}+c$	
$\lim_{n\to\infty}\left(\frac{15}{2}+1.58\ln 2-\frac{15}{3+2e^{-1}}+\frac{15}{3+2e^{-n}}\right)\approx 9.5799$	(2009 ASL-M&S Q9)
$\lim_{T\to\infty}[40-20(T+2)e^{-0.5T}]=40$	(2005 ASL-M&S Q8)

$\lim_{t\to\infty}\left(7.5+\frac{1}{5}(t^2-8t-8)e^{-0.2t}\right)=7.5$	(2003 ASL-M&S Q9)

$\lim_{t\to\infty}\left(-0.12e^{-0.25t}+0.21\right)=0.21$	(2001 ASL-M&S Q9)
$\lim_{t\to\infty}\left(-0.16(3t+4)^{\frac{-1}{2}}+0.17\right)=0.17$	
$\lim_{x\to\infty}\left(16+\frac{6x}{\sqrt{1+8x}}\right)=\infty$	(2000 ASL-M&S Q9)
$\ln N(t)=5.89-e^{1-0.18t}$ $\lim_{t\to\infty}N(t)=\lim_{t\to\infty}\left(e^{5.89-e^{1-0.18t}}\right)=e^{5.89}\approx 361$	(2000 ASL-M&S Q11)
$\lim_{t\to\infty}\left(\frac{40}{1+be^{-rt}}\right)=40$	(1997 ASL-M&S Q8)

Out of syllabus

35. Let $y=\sqrt[3]{\frac{3x-1}{x-2}}$, where $x>2$.

(a) Use logarithmic differentiation to express $\frac{1}{y}\cdot\frac{dy}{dx}$ in terms of x .

(b) Using the result of (a), find $\frac{d^2y}{dx^2}$ when $x=3$.

(Part a is out of Syllabus) (6 marks) (2012 DSE-MATH-M1 Q4)

36. Let $u=\sqrt{\frac{2x+3}{(x+1)(x+2)}}$, where $x>-1$.

(a) Use logarithmic differentiation to express $\frac{du}{dx}$ in terms of u and x.

(b) Suppose $u=3^y$, express $\frac{dy}{dx}$ in terms of x.

(Part a is out of Syllabus) (5 marks) (SAMPLE DSE-MATH-M1 Q6)

37. It is given that $\begin{cases}x=\ln(2t+4)\\ y=e^{t^2+4t+4}\end{cases}$, where $t>-2$.

(a) Express $\frac{dy}{dx}$ in terms of y only.

(b) Find the value of $\frac{d^2y}{dx^2}$ when $x=0$.

(Out of Syllabus) (6 marks) (2012 ASL-M&S Q2)

38. Let C be the curve $x=y^4-y$.

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent to C if the slope of the tangent is $\frac{1}{3}$.

(Out of Syllabus) (7 marks) (2009 ASL-M&S Q3)

39. Suppose $y^3-uy=1$ and $u=2^{x^2}$.

(a) Find $\frac{dy}{du}$ in terms of u and y .

(b) Find $\frac{du}{dx}$ in terms of x .

(c) Find $\frac{dy}{dx}$ in terms of x and y .

(Out of Syllabus) (7 marks) (2008 ASL-M&S Q2)

40. Let $w=\sqrt{\dfrac{(x-1)^3}{(x+2)(2x+1)}}$, where $x>1$.

(a) Express $\ln w$ in the form $a\ln(x-1)+b\ln(x+2)+c\ln(2x+1)$ where a , b and c are constants.

Hence find $\dfrac{dw}{dx}$.

(b) Suppose $w=2^y$.

Express $\dfrac{dy}{dw}$ in terms of w .

Hence express $\dfrac{dy}{dx}$ in terms of x .

(Out of Syllabus) (7 marks) (2005 ASL-M&S Q3)

41. Let $x=-\dfrac{5}{t^2}+2e^{-3t}$ and $y=\dfrac{10}{t^2}+e^{2t}$ $(t\neq 0)$. If $\dfrac{dy}{dx}=-2$, find the value of t .

(Out of Syllabus) (4 marks) (2002 ASL-M&S Q1)

42. Let $\ln(xy)=\dfrac{x}{y}$ where $x , y>0$. Show that $\dfrac{dy}{dx}=\dfrac{xy-y^2}{xy+x^2}$.

(Out of Syllabus) (4 marks) (2000 ASL-M&S Q1)

43. It is given that $e^{xy}=\dfrac{x(x+1)^3}{x^2+1}$, where $x>0$.

(a) Find the value of y when $x=1$.

(b) Find the value of $\dfrac{dy}{dx}$ when $x=1$.

(Out of Syllabus) (5 marks) (1999 ASL-M&S Q1)

44. (a) If $e^x+e^y=xy$, find $\dfrac{dy}{dx}$.

(b) If $y=\dfrac{1}{x+1}\sqrt{\dfrac{(x-2)(x+3)}{x+1}}$ where $x>2$, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

(Out of Syllabus) (6 marks) (1995 ASL-M&S Q2)

2021 DSE Q5

Let $f(x)=e^{-x^{\frac{1}{3}}}$.

(a) Let $g(u)=e^{-u}(u^2+2u+2)$, where $u=x^{\frac{1}{3}}$. Find the constant β such that $\dfrac{dg(u)}{dx}=\beta f(x)$.

(b) Express, in terms of e , the area of the region bounded by the curve $y=f(x)$, the x-axis, the y-axis and the straight line $x=8$.

(6 marks)

2021 DSE Q7

Let $y=\dfrac{e^x}{x^3-x+2}$, where $0\leq x\leq 5$. Find

(a) $\dfrac{dy}{dx}$,

(b) the greatest value and the least value of y .

3. Derivative and Differentiation of Functions

Section A

1. (2017 DSE-MATH-M1 Q7)

(a) $y=\dfrac{x}{\sqrt{x-2}}$		
$\dfrac{dy}{dx}=\dfrac{\sqrt{x-2}-x\left(\dfrac{1}{2}\right)(x-2)^{\frac{-1}{2}}}{x-2}$	1M	for quotient rule
$\dfrac{dy}{dx}=\dfrac{x-4}{2(x-2)^{\frac{3}{2}}}$	1A	
(b) Let (h,k) be the coordinates of the point of contact. So, the slope of this tangent is $\dfrac{h-4}{2(h-2)^{\frac{3}{2}}}$.		
$\dfrac{k-0}{h-9}=\dfrac{h-4}{2(h-2)^{\frac{3}{2}}}$	1M+1A	1M for using (a)
$\dfrac{h}{\sqrt{h-2}}2(h-2)^{\frac{3}{2}}=(h-4)(h-9)$		
$h^2+9h-36=0$ $h=3$ or $h=-12$ (rejected)	1M	
The slope of this tangent $=\dfrac{3-4}{2(3-2)^{\frac{3}{2}}}$	1M	
$=\dfrac{-1}{2}$	1A	
	----------(7)	

(a)	Good. Many candidates were able to find $\dfrac{dy}{dx}$ but some candidates did not simplify the answer.
(b)	Fair. Many candidates wrongly thought that $(9,0)$ was the point of contact.

2. (2016 DSE-MATH-M1 Q7)

(a) $\dfrac{dy}{dx}$		
$=\left(\dfrac{3}{2}\right)(2x+8)^{\frac{1}{2}}(2)+6x$	1M	for chain rule
$=3\sqrt{2x+8}+6x$	1A	
(b) Note that the slope of the straight line $6x+y+4=0$ is -6 . So, the slope of the tangent is -6 .		
$3\sqrt{2x+8}+6x=-6$	1M+1A	1M for using (a)
$\sqrt{2x+8}=-2(x+1)$		
$2x+8=4(x+1)^2$		
$2x^2+3x-2=0$	1M	for $ax^2+bx+c=0$
$x=-2$ or $x=\dfrac{1}{2}$ (rejected)	1A	for '$x=-2$ or $x=\dfrac{1}{2}$'
Hence, there is only one tangent to C parallel to the straight line $6x+y+4=0$. Thus, the claim is disagreed.	1A	f.t.
	----------(7)	

(a)	Very good. Nearly all of the candidates were able to apply chain rule to find $\dfrac{dy}{dx}=3\sqrt{2x+8}+6x$.
(b)	Good. Some candidates were unable to solve the equation involving radical $3\sqrt{2x+8}+6x=-6$, and many candidates were unable to reject the inappropriate root $x=\dfrac{1}{2}$.

3. (2015 DSE-MATH-M1 Q7)

(a) $y = x\sqrt{2x^2+1}$		
$\frac{dy}{dx} = \sqrt{2x^2+1} + x\left(\frac{1}{2}\right)(2x^2+1)^{\frac{-1}{2}}(4x)$	1M	for chain rule
$\frac{dy}{dx} = \frac{4x^2+1}{\sqrt{2x^2+1}}$	1A	
(b) Note that the slope of the straight line is $\frac{-3}{17}$.		
So, the slope of each tangent is $\frac{17}{3}$.		
$\frac{4x^2+1}{\sqrt{2x^2+1}} = \frac{17}{3}$	1M+1A	1M for using (a)
$3(4x^2+1) = 17\sqrt{2x^2+1}$		
$9(4x^2+1)^2 = 289(2x^2+1)$		
$72x^4 - 253x^2 - 140 = 0$	1M	for $ax^4+bx^2+c=0$
$x=2$ or $x=-2$		
For $x=2$, we have $y=6$. The equation of the tangent to C at the point $(2, 6)$ is		
$y-6 = \frac{17}{3}(x-2)$	1M	either one
$17x - 3y - 16 = 0$		
For $x=-2$, we have $y=-6$. The equation of the tangent to C at the point $(-2,-6)$ is		
$y+6 = \frac{17}{3}(x+2)$		
$17x - 3y + 16 = 0$	1A	for both
	-----(7)	

(a)	Very good. Most candidates were able to apply chain rule to find $\frac{dy}{dx}$.
(b)	Good. Some candidates made careless mistakes in simplifying the equation involving radical, and some candidates failed to write a quadratic equation in x^2.

4. (2013 DSE-MATH-M1 Q3a,b)

(a) $y = x(x-2)^{\frac{1}{3}}$		
$\frac{dy}{dx} = (x-2)^{\frac{1}{3}} + \frac{1}{3}(x-2)^{\frac{-2}{3}}x$	1M	For product rule
When $x=3$, $\frac{dy}{dx}=2$.		
Hence the equation of L is $y=2x$.	1A	
(b) Solving C and L:		
$x(x-2)^{\frac{1}{3}} = 2x$	1M	
$x\left[(x-2)^{\frac{1}{3}} - 2\right] = 0$		
$x=0$ or 10	1A	

(a)	Good. Some candidates found the equation of the tangent to C at $x=3$ instead of the equation of L.
(b)	Good. Some candidates did not know how to solve equations with fraction exponents or missed out the root $x=0$ by dividing both sides of an equation by x.

5. (PP DSE-MATH-M1 Q2)

(a) $t = y^3 + 2y^{\frac{-1}{2}} + 1$		
$\frac{dt}{dy} = 3y^2 - y^{\frac{-3}{2}}$	1A	
(b) $e^t = x^{x^2+1}$		
$t = (x^2+1)\ln x$	1A	
$\frac{dt}{dx} = \frac{x^2+1}{x} + 2x\ln x$	1A	
(c) $\frac{dy}{dx} = \frac{dt}{dx} \div \frac{dt}{dy}$	1M	
$= \frac{(x^2+1+2x^2\ln x)y^{\frac{3}{2}}}{x\left(3y^{\frac{7}{2}}-1\right)}$	1A	OR $\frac{\frac{x^2+1}{x}+2x\ln x}{3y^2-y^{\frac{-3}{2}}}$
	(5)	

(a)	甚佳。少數學生誤以為 $\frac{dt}{dy} = \frac{dy}{dt}$。
(b)	平平。部分學生未能正確運算 $\frac{d}{dx}[(x^2+1)\ln x]$。
(c)	甚佳。少數學生未能正確應用鏈式法則。

6. (PP DSE-MATH-M1 Q4)

Solution	Marks	Remarks
(a) $y = x(2x-1)^{\frac{1}{2}}$		
$\frac{dy}{dx} = (2x-1)^{\frac{1}{2}} + x \cdot \frac{1}{2}(2x-1)^{\frac{-1}{2}}(2)$	1M	For product rule
$= \frac{3x-1}{(2x-1)^{\frac{1}{2}}}$	1A	
(b) For tangents parallel to $2x - y = 0$, we need $\frac{dy}{dx} = 2$.		
$\frac{3x-1}{(2x-1)^{\frac{1}{2}}} = 2$	1M	
$9x^2 - 6x + 1 = 4(2x-1)$		
$9x^2 - 14x + 5 = 0$		
$x = 1$ or $\frac{5}{9}$	1A	
For $x = 1$, $y = 1$ and hence the equation of the tangent is		
$y - 1 = 2(x-1)$		
$2x - y - 1 = 0$	1A	
For $x = \frac{5}{9}$, $y = \frac{5}{27}$ and hence the equation of the tangent is		
$y - \frac{5}{27} = 2\left(x - \frac{5}{9}\right)$		
$54x - 27y - 25 = 0$	1A	
	(6)	

(a)	甚佳。大數學生明白函數的微分法。
(b)	平平。部分學生未能求出切線的方程。

7. (2013 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $x = \ln\frac{1+t}{1-t}$		
$\frac{dx}{dt} = \frac{1-t}{1+t} \cdot \frac{(1-t)(1)-(1+t)(-1)}{(1-t)^2}$	1M	
Alternative Solution $x = \ln(1+t) - \ln(1-t)$; $\frac{dx}{dt} = \frac{1}{1+t} + \frac{1}{1-t}$	1M	
$= \frac{2}{1-t^2}$	1A	
(b) (i) $y = 1 + e^{-x} - e^{-2x}$		
$\frac{dy}{dx} = -e^{-x} + 2e^{-2x}$	1A	
(ii) $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$		
$= (-e^{-x} + 2e^{-2x})\left(\frac{2}{1-t^2}\right)$	1M	
When $t = \frac{1}{2}$, $x = \ln 3$.	1A	
$\frac{dy}{dt} = (-e^{-\ln 3} + 2e^{-2\ln 3})\left[\frac{2}{1-\left(\frac{1}{2}\right)^2}\right]$		
$= \frac{-8}{27}$	1A	OR -0.2963
	(6)	

Good.
Some candidates were not able to apply the quotient rule.

8. (2007 ASL-M&S Q3)

Solution	Marks
(a) $y = \frac{1-e^{4x}}{1+e^{8x}}$	
$\frac{dy}{dx} = \frac{(1+e^{8x})(-4e^{4x}) - (1-e^{4x})(8e^{8x})}{(1+e^{8x})^2}$	1M for quotient rule or product rule
When $x = 0$, we have $\frac{dy}{dx} = -2$.	1A
(b) (i) Since $(z^2+1)e^{3z} = e^{\alpha+\beta x}$, we have $\ln(z^2+1) + 3z = \alpha + \beta x$.	1A
(ii) Since the graph of the linear function passes through the origin and the slope of the graph is 2, we have $\alpha = 0$ and $\beta = 2$.	1A for both correct
(iii) $\ln(z^2+1) + 3z = 2x$	
$\frac{2z}{z^2+1} + 3 = 2\frac{dx}{dz}$	
Therefore, we have $\left.\frac{dx}{dz}\right\vert_{z=0} = \frac{3}{2}$.	1A
Note that $x = 0$ when $z = 0$.	
Also note that $\left.\frac{dy}{dx}\right\vert_{x=0} = -2$.	
$\left.\frac{dy}{dz}\right\vert_{z=0}$	
$= \left(\left.\frac{dy}{dx}\right\vert_{x=0}\right)\left(\left.\frac{dx}{dz}\right\vert_{z=0}\right)$	
$= (-2)\left(\frac{3}{2}\right)$	1M for chain rule
$= -3$	1A
$y = \frac{1-e^{6z+2\ln(z^2+1)}}{1+e^{12z+4\ln(z^2+1)}}$	1A
$y = \frac{1-(z^2+1)^2e^{6z}}{1+(z^2+1)^4e^{12z}}$	
$\frac{dy}{dz} = \frac{\left(1+(z^2+1)^4e^{12z}\right)\left(-6(z^2+1)^2e^{6z} - 2(z^2+1)(2z)e^{6z}\right) - \left(1-(z^2+1)^2e^{6z}\right)\left(12(z^2+1)^4e^{12z} + 4(z^2+1)^3(2z)e^{12z}\right)}{\left(1+(z^2+1)^4e^{12z}\right)^2}$	1M for quotient rule or product rule
$\left.\frac{dy}{dz}\right\vert_{z=0} = -3$	1A
	(7)

Good. Most candidates could handle quotient rule and product rule. It is more efficient to apply $\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz}$ but some candidates went through the tedious way by expressing the function of y in terms of z.

9. (2003 ASL-M&S Q3)

(a)
$$\frac{dy}{dx}=\frac{4(4x^2+3x+9)^{\frac{1}{2}}-\frac{1}{2}(4x^2+3x+9)^{\frac{-1}{2}}(8x+3)(4x-9)}{4x^2+3x+9}$$

1M for quotient rule + 1M for chain rule + 1A

$$=\frac{8(4x^2+3x+9)-(8x+3)(4x-9)}{2(4x^2+3x+9)^{\frac{3}{2}}}$$
$$=\frac{84x+99}{2(4x^2+3x+9)^{\frac{3}{2}}}$$

When $x=10$, $\frac{dy}{dx}=\frac{84(10)+99}{2(4(10)^2+3(10)+9)^{\frac{3}{2}}}$

≈ 0.051043308

≈ 0.0510

1A a−1 for r.t. 0.051

(b) The required amount

$$=\lim_{x\to\infty}\left(3+\frac{4x-9}{\sqrt{4x^2+3x+9}}\right)$$
$$=3+\lim_{x\to\infty}\frac{4-\frac{9}{x}}{\sqrt{4+\frac{3}{x}+\frac{9}{x^2}}}$$

1M can be absorbed

$=3+2$

$=5$ grams

1A

----------(6)

Fair. Handling quotient rule and chain rule together needs more practice. Some candidates were not familiar with the techniques of taking limits.

DSE Mathematics Module 1 — 3. Derivative and Differentiation of Functions

10. (2001 ASL-M&S Q2)

(a) Since $u=e^{2x}$, $\therefore \frac{du}{dx}=2e^{2x}$. 1A

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\left(\frac{1}{u}-2u\right)\cdot 2u=2-4u^2=2-4e^{4x}$ 1M+1A 1M for $\left(\frac{1}{u}-2u\right)\cdot 2u$

Alternative solution	Marks
$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\left(\frac{1}{u}-2u\right)\cdot 2e^{2x}=\left(\frac{1}{e^{2x}}-2e^{2x}\right)\cdot 2e^{2x}=2-4e^{4x}$	1M+1A
$y=\ln u-u^2+c$ $y=\ln e^{2x}-(e^{2x})^2+c$ $y=2x-e^{4x}+c$ $\frac{dy}{dx}=2-4e^{4x}$	1M 1A f.t.

(b) Using (a), $y=\int(2-4e^{4x})dx$ 1M

$=2x-e^{4x}+c$ for some constant c.

Putting $x=0$ and $y=1$, we have $c=2$.

$\therefore\ y=2x-e^{4x}+2$ 1A

----------(5)

11. (1998 ASL-M&S Q1)

$y=xe^{\frac{1}{x}}$

$\frac{dy}{dx}=e^{\frac{1}{x}}+xe^{\frac{1}{x}}(-\frac{1}{x^2})$ 1M+1M 1M for product rule, 1M for chain rule

$=e^{\frac{1}{x}}-\frac{1}{x}e^{\frac{1}{x}}$ 1A

$\frac{d^2y}{dx^2}=-\frac{1}{x^2}e^{\frac{1}{x}}+\frac{1}{x^2}e^{\frac{1}{x}}+\frac{1}{x^3}e^{\frac{1}{x}}$ 1A

$=\frac{1}{x^3}e^{\frac{1}{x}}$

$\therefore\ x^4\frac{d^2y}{dx^2}-y=x^4\left(\frac{1}{x^3}e^{\frac{1}{x}}\right)-xe^{\frac{1}{x}}$

$=0$ 1 accept $\frac{d^2y}{dx^2}=\frac{y}{x^4}$

(5)

12. (1994 ASL-M&S Q2)

$\frac{dx}{dt}=100ke^{kt}$ 1A

$\frac{d^2x}{dt^2}=100k^2e^{kt}$ 1A

Hence $100k^2e^{kt}-200ke^{kt}-300e^{kt}=0$ 1M

$k^2-2k-3=0$ 1A

$(k-3)(k+1)=0$

$k=-1$ or 3

$\because k$ is negative

$\therefore k=-1$ 1A

5

13. (2002 ASL-M&S Q1)

$\frac{dx}{dt}=\frac{10}{t^3}-6e^{-3t}$

$\frac{dy}{dt}=-\frac{20}{t^3}+2e^{2t}$ — 1M+1A (1M for $(e^{at})'=ae^{at}$)

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{1}{\frac{dx}{dt}}\right)=\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}$ — 1M for Chain Rule and Inverse Function Rule

For $\frac{dy}{dx}=-2$

$\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}=-2$ — 1M

$e^{5t}=6$

$t=\frac{1}{5}\ln 6(\approx 0.3584)$ — 1A a−1 for r.t. 0.358

---------(5)

14. (2012 ASL-M&S Q2)

(a) $y=e^{t^2+4t+4}$ and $x=\ln(2t+4)$		
$\frac{dy}{dt}=e^{t^2+4t+4}(2t+4)$ and $\frac{dx}{dt}=\frac{1}{t+2}$	1A	For both
$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$		
$=2e^{t^2+4t+4}(t+2)\cdot(t+2)$	1M	
Alternative Solution		
$\ln y=(t+2)^2$ and $x=\ln 2+\ln(t+2)$		OR $\cdots$ and $t+2=\frac{1}{2}e^x$
$\therefore\ x=\ln 2+\frac{1}{2}\ln(\ln y)$	1A	OR $\ln y=\frac{1}{4}e^{2x}$
$\frac{dx}{dy}=\frac{1}{2}\cdot\frac{1}{\ln y}\cdot\frac{1}{y}$	1M	OR $\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{4}e^{2x}\cdot 2$
$\frac{dy}{dx}=2y\ln y$	1A	
(b) $\frac{d^2y}{dx^2}=\left(2y\cdot\frac{1}{y}+2\ln y\right)\frac{dy}{dx}$	1M	For chain rule
$=4y\ln y(1+\ln y)$		
When $x=0$, $t=\frac{-3}{2}$ and so $y=e^{\frac{1}{4}}$.	1A	
$\therefore\ \frac{d^2y}{dx^2}=4e^{\frac{1}{4}}\left(\frac{1}{4}\right)\left(1+\frac{1}{4}\right)$		
$=\frac{5}{4}e^{\frac{1}{4}}$	1A	OR 1.6050
	(6)	

Limit at infinity (Section A)

15. (2019 DSE-MATH-M1 Q5)

(a) For all $x>-3$,		
$f'(x)$		
$=\frac{(x+3)(-1)-(6-x)(1)}{(x+3)^2}$		
$=\frac{-9}{(x+3)^2}$		
<0		
Thus, $f(x)$ is decreasing.	1	
Note that $f(x)=\frac{9}{x+3}-1$ for all $x>-3$. Thus, $f(x)$ is decreasing.	1	
(b) $\lim_{x\to\infty} f(x)$		
$=\lim_{x\to\infty}\frac{\frac{6}{x}-1}{1+\frac{3}{x}}$		
$=-1$	1A	
$\lim_{x\to\infty} f(x)$ $=\lim_{x\to\infty}\left(\frac{9}{x+3}-1\right)$ $=-1$	1A	
(c) For $y=0$, we have $x=6$.		
The required area		
$=\int_0^6 f(x)\,dx$	1M	
$=\int_0^6\frac{6-x}{x+3}\,dx$		
$=\int_0^6\left(\frac{9}{x+3}-1\right)dx$	1M	
$=[9\ln(x+3)-x]_0^6$	1M	
$=9\ln 3-6$	1A	
For $y=0$, we have $x=6$. The required area		
$=\int_0^6 f(x)\,dx$	1M	
$=\int_0^6\frac{6-x}{x+3}\,dx$		
$=\int_3^9\frac{6-(u-3)}{u}\,du$ (by letting $u=x+3$)	1M	
$=\int_3^9\left(\frac{9}{u}-1\right)du$		
$=[9\ln u-u]_3^9$	1M	
$=9\ln 3-6$	1A	
	---------(6)	

(a)	Good. Some candidates were unable to show that $f'(x) < 0$ to complete the proof.
(b)	Good. Some candidates were unable to consider $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\frac{6}{x}-1}{1+\frac{3}{x}}$ to obtain the required limit.
(c)	Good. Many candidates were able to use integration to obtain the required area, but some candidates were unable to give the answer in exact value.

16. (2018 DSE-MATH-M1 Q5)

Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$

$\dfrac{12x-48}{(3x^2-24x+49)^2}=0$ — 1M

$x=4$

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	$+$

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. — 1A

$f'(x)=0$

$\dfrac{12x-48}{(3x^2-24x+49)^2}=0$ — 1M

$x=4$

$f''(x)$

$=\dfrac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$

$f''(4)$

$=12$

>0

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. — 1A

(b) (i) Let $v=3x^2-24x+49$. Then, we have $\dfrac{dv}{dx}=6x-24$.

$f(x)$

$=\displaystyle\int\frac{12x-48}{(3x^2-24x+49)^2}dx$

$=\displaystyle\int\frac{2}{v^2}dv$ — 1M

$=\dfrac{-2}{v}+C$

$=\dfrac{-2}{3x^2-24x+49}+C$

Since $f(x)$ has only one extreme value, we have $f(4)=5$. — 1M

$\dfrac{-2}{3(4)^2-24(4)+49}+C=5$

$C=7$

Thus, we have $f(x)=\dfrac{-2}{3x^2-24x+49}+7$. — 1A

(ii) $\lim_{x\to\infty} f(x)$

$=7$ — 1A

----------(6)

(a)	Very good. Over 85% of the candidates were able to find the value of α .
(b) (i)	Good. Many candidates were able to find $f(x)$ by indefinite integral but some candidates were unable to use a suitable substitution.
(ii)	Fair. Only some candidates were able to find the constant of integration in (b)(i), and thus the required limit.

17. (2013 ASL-M&S Q3)

(a) $R(t) = Ae^{-0.5t} + B$

$R(t) \to 10$ when $t \to \infty$ 1M

$\therefore\ B = 10$ 1A

$R(0) = 500$

$500 = A + B$

$\therefore\ A = 490$ 1A

(b) $\int_0^5 P'(t)\,dt + R(5) - R(0)$ 1M

$= \int_0^5 600e^{-0.3t}dt + \left[490e^{-0.5(5)} + 10\right] - 500$

$= [-2000e^{-0.3t}]_0^5 + 490e^{-2.5} - 490$ 1A For $[-2000e^{-0.3t}]_0^5$

$= -2000e^{-1.5} + 490e^{-2.5} + 1510$

≈ 1104 1A

Hence Richard gains 1104 thousand dollars in the process.

(6)

Good.
In (b), some candidates did not consider the depreciation of the value of the machine in five years.

18. (2012 ASL-M&S Q3)

(a) Let $u = 1 + e^{-0.2t}$.

$du = -0.2e^{-0.2t}dt$ 1A

$N = \int \frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}dt$

$N = \int \frac{0.3}{u^2} \cdot \frac{du}{-0.2}$

$= \frac{3}{2u} + C$ 1A

$= \frac{3}{2(1+e^{-0.2t})} + C$

When $t = 0$, $N = 0.5$.

$\therefore\ C = \frac{-1}{4}$

i.e. $N = \frac{3}{2(1+e^{-0.2t})} - \frac{1}{4}$ 1A

(b) $N(4) - N(0)$

$= \frac{3}{2(1+e^{-0.2\times 4})} - \frac{1}{4} - 0.5$ 1M

≈ 0.284961721

Hence the increase in the number of people is 285. 1A

(c) $\frac{dN}{dt} = \frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2} > 0$ for all $t \ge 0$

Hence N is always increasing.

} Withhold the last mark if this argument is missing

$\lim_{t\to\infty} N = \lim_{t\to\infty}\left[\frac{3}{2(1+e^{-0.2t})} - \frac{1}{4}\right]$

$= 1.25$ 1A

Hence the number of members will never reach 1300. 1

OR by arguing that $e^{-0.2t} > 0 \Rightarrow N < \frac{3}{2} - \frac{1}{4}$

OR by arguing that $\frac{3}{2(1+e^{-0.2t})} - \frac{1}{4} = 1.3$ has no real solution

(7)

Satisfactory.
Many candidates overlooked the units and did not use 0.5 to represent 500 since N was given in thousand. A number of candidates could not well explain their answer in (c) because they did not state clearly that N was an increasing function.

19. (2011 ASL-M&S Q2)

(a) $\dfrac{dX}{dt}=6\left(\dfrac{t}{0.2t^3+1}\right)^2$

$X=6\displaystyle\int\frac{t^2}{(0.2t^3+1)^2}\,dt$

Let $u=0.2t^3+1$, and therefore $du=0.6t^2\ dt$. — 1M — For $u=0.2t^3+1$ or $u=(0.2t^3+1)^2$

$\therefore\ X=6\displaystyle\int\frac{1}{0.6u^2}\,du$ — 1M

$=\dfrac{-10}{u}+C$ — 1A

$=\dfrac{-10}{0.2t^3+1}+C$

When $t=0$, $X=4$ and hence $C=14$.

i.e. $X=\dfrac{-10}{0.2t^3+1}+14$ — 1A

(b) $13=\dfrac{-10}{0.2t^3+1}+14$

$t=\sqrt[3]{45}$ months — 1A — OR 3.5569 months

(c) $X=14-\dfrac{10}{0.2t^3+1}<14$ for any value of t . — 1M

Hence the plan can be run for a long time. — 1A

(7)

Good.
In part (c), although most candidates found the limit of X when $t\to\infty$, the proof was incomplete without showing that the function was increasing.

20. (2008 ASL-M&S Q3)

(a) $\dfrac{dx}{dt}=5.3\left(\dfrac{1}{t+2}-\dfrac{1}{t+5}\right)+1.2e^{-0.1t}$

$x=\displaystyle\int\left[5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t}\right]dt$ — 1M

$=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+C$ (since $t\ge 0$) — 1A — OR $5.3[\ln|t+2|-\ln|t+5|]-12e^{-0.1t}+C$

When $t=0$, $x=0$.

$\therefore 0=5.3(\ln 2-\ln 5)-12+C$ — 1M

$C=5.3(\ln 5-\ln 2)+12$

≈ 16.8563

i.e. $x=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563$ — 1A — OR $x=\cdots+5.3\ln 2.5+12$

(b) $\displaystyle\lim_{t\to\infty}\left\{5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563\right\}$ — 1M

$=5.3\displaystyle\lim_{t\to\infty}\ln\frac{t+2}{t+5}-12\lim_{t\to\infty}e^{-0.1t}+16.8563$

$=16.8563$ — 1A

i.e. the concentration of the drug after a long time = 16.8563 mg/L

(6)

Good. Some candidates could not present the mathematical notation of the limit of x properly.

21. (2004 ASL-M&S Q2)

(a) $\dfrac{dN}{dt}$

$=\dfrac{6}{(e^{\frac{t}{4}}+e^{\frac{-t}{4}})^2}$

$=\dfrac{6}{(e^{\frac{-t}{4}}(e^{\frac{t}{2}}+1))^2}$

$=\dfrac{6}{e^{\frac{-t}{2}}(e^{\frac{t}{2}}+1)^2}$

$=\dfrac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$ — 1 must show steps

Let $u=e^{\frac{t}{2}}+1$.

Then, we have $\dfrac{du}{dt}=\dfrac{1}{2}e^{\frac{t}{2}}$.

Also, $dt=\dfrac{2du}{u-1}$. Now,

N

$=\displaystyle\int\frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}\,dt$

$=\displaystyle\int\frac{12(u-1)}{u^2(u-1)}\,du$

$=\displaystyle\int\frac{12}{u^2}\,du$ — 1A accept $\dfrac{dN}{du}=\dfrac{12}{u^2}$.

So, we have $N=\dfrac{-12}{u}+C$ [redacted] — 1A

Now, $N=\dfrac{-12}{e^{\frac{t}{2}}+1}+C$.

Using the condition that $N=8$ when $t=0$, we have $8=-6+C$.

Hence, $C=14$. — 1M for finding C

Thus, $N=14-\dfrac{12}{e^{\frac{t}{2}}+1}$. — 1A

(b) The required number of fish

$=\displaystyle\lim_{t\to\infty}\left(14-\frac{12}{e^{\frac{t}{2}}+1}\right)$

$=14-\displaystyle\lim_{t\to\infty}\frac{12}{e^{\frac{t}{2}}+1}$

$=14$ thousands — 1A

---------(6)

Fair. Many candidates were not able to relate mathematical presentations to integrations and taking limits.

22. (2002 ASL-M&S Q4)

(a) $S=\int_0^{10}\frac{8100t}{(3t+10)^3}dt$. — 1A

Let $u=3t+10$.
$du=3dt$.
When $t=0$, $u=10$.
When $t=10$, $u=40$.

$$S=\int_{10}^{40}\frac{8100\left(\frac{u-10}{3}\right)\cdot\frac{1}{3}du}{u^3}$$

$$=900\int_{10}^{40}\left(\frac{1}{u^2}-\frac{10}{u^3}\right)du$$ — 1M change of variable

$$=900\left[-\frac{1}{u}+\frac{5}{u^2}\right]_{10}^{40}$$

$$=\frac{405}{16}=25.3125$$ — 1A

The percentage of smoke removed is 25.3125%.

$$S=\int_0^{10}\frac{8100t}{(3t+10)^3}dt$$ — 1A

$$=900\int_0^{10}\left[\frac{1}{(3t+10)^2}-\frac{10}{(3t+10)^3}\right]d(3t+10)$$ — 1M change of variable

$$=900\left[-\frac{1}{3t+10}+\frac{5}{(3t+10)^2}\right]_0^{10}$$

$=25.3125$ — 1A

$$S=\int\frac{8100t}{(3t+10)^3}dt=900\left[-\frac{1}{3t+10}+\frac{5}{(3t+10)^2}\right]+C$$ — 1M+1M for change of variable

When $t=0$, $S=0$. Hence, we have $C=45$.

So, $S=900\left[-\frac{1}{3T+10}+\frac{5}{(3T+10)^2}\right]+45$.

When $t=10$, $S=25.3125$. — 1A

(b) $$S=\int_0^T\frac{8100t}{(3t+10)^3}dt$$

$$=900\left[-\frac{1}{u}+\frac{5}{u^2}\right]_{10}^{3T+10}$$

$$=900\left[-\frac{1}{3T+10}+\frac{5}{(3T+10)^2}+0.05\right]$$

$$\lim_{T\to\infty}S=\lim_{T\to\infty}900\left[-\frac{1}{3T+10}+\frac{5}{(3T+10)^2}+0.05\right]$$ — 1M taking limit and in terms of T

$=45$ — 1A

$\therefore$ 45% of smoke will be removed.

---------(5)

23. (2002 ASL-M&S Q2)

(a) At $t=2$, $r(2)=5.5035$(m)

$$\frac{dV}{dr}=4\pi r^2$$

$$\frac{dr}{dt}=\frac{18\times 2e^{-t}}{(3+2e^{-t})^2}=\frac{36e^{-t}}{(3+2e^{-t})^2}$$ — 1A

At $t=2$, $\frac{dV}{dr}=380.6109$

$\frac{dr}{dt}=0.45545$

$\therefore$ $\frac{dV}{dt}=\frac{dV}{dr}\frac{dr}{dt}$

At $t=2$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ — 1M

$=380.6109\times 0.45545$

$=173.35$ (m³/h) — 1A (Accept : 173.31–173.39) a–1 for more than 2 d.p.

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$ — 1M

$$\frac{dr}{dt}=\frac{36e^{-t}}{(3+2e^{-t})^2}$$ — 1A

$$\frac{dV}{dt}=4\pi r^2\cdot\frac{36e^{-t}}{(3+2e^{-t})^2}$$

$$=\frac{144\pi r^2e^{-t}}{(3+2e^{-t})^2}$$ — (Accept : $\frac{dV}{dt}=\frac{46656\pi e^{-t}}{(3+2e^{-t})^4}$)

At $t=2$,
$r\approx 5.50346$

$\therefore$ $\frac{dV}{dt}\approx 173.35$ (m^3/h) — 1A (Accept : 173.31–173.39) a–1 for more than 2 d.p.

(b) $$\lim_{t\to\infty}r(t)=\lim_{t\to\infty}\frac{18}{3+2e^{-t}}=6\text{ (m)}$$ — 1M

$\therefore$ the volume of the balloon will be

$$V=\frac{4}{3}\pi(6)^3+5\pi$$

$=293\pi$

$=920.49$ (m³) — 1A a–1 for more than 2 d.p.

---------(5)

Limit at infinity (Section B)

24. (2014 DSE-MATH-M1 Q11)

(a) $\lim_{t\to\infty}\frac{340}{2+e^{-t}-2e^{-2t}}=\frac{340}{2+0-2\cdot 0}$ — 1M

$=170$

Hence the value of y will not exceed 171 in the long run. — 1A

(2)

(b) $\frac{dy}{dt}=340[-(2+e^{-t}-2e^{-2t})^{-2}](-e^{-t}+4e^{-2t})$ — 1A

$=\frac{340(e^{-t}-4e^{-2t})}{(2+e^{-t}-2e^{-2t})^2}$

$\therefore \frac{dy}{dt}=0$ when $e^{-t}-4e^{-2t}=0$ — 1M (For $\frac{dy}{dt}=0$)

i.e. $t=\ln 4$ — 1A

t	$0\le t<\ln 4$	$t=\ln 4$	$t>\ln 4$
$\frac{dy}{dt}$	-ve	0	+ve

1M

Hence y is minimum when $t=\ln 4$.

The minimum value of $y=\frac{340}{2+e^{-\ln 4}-2e^{-2\ln 4}}$

$=160$ — 1A

When $t=0$, $y=\frac{340}{2+e^0-2e^0}$

$=340$

As the graph of y is continuous, and by (a), the greatest value of y is 340 and the least value of y is 160. — 1A

(6)

(c) (i) $y=\frac{340}{2+e^{-t}-2e^{-2t}}$

$2y+ye^{-t}-2ye^{-2t}=340$

$2y(e^{-t})^2-ye^{-t}+340-2y=0$ — 1A

(ii) Since $e^{-\alpha}$ and $e^{\alpha-3}$ are roots of the equation in (i),

$\frac{340-2y}{2y}=e^{-\alpha}e^{\alpha-3}$ — 1M

$340-2y=2ye^{-3}$

Hence the equation becomes $2y(e^{-t})^2-ye^{-t}+2ye^{-3}=0$

i.e. $2(e^{-t})^2-e^{-t}+2e^{-3}=0$ — 1A

$\therefore e^{-\alpha}=\frac{1+\sqrt{1-16e^{-3}}}{4}$ or $\frac{1-\sqrt{1-16e^{-3}}}{4}$ (rejected as $e^{-\alpha}$ is the greater root)

i.e. $\alpha=-\ln\frac{1+\sqrt{1-16e^{-3}}}{4}$ — 1A (OR $\ln\frac{1-\sqrt{1-16e^{-3}}}{4}+3$ OR 1.0140)

(4)

(a)	Good. Some candidates thought that $\lim_{t\to+\infty} e^{-t}=1$.
(b)	Fair. Quite a lot of candidates failed to consider both the value of y at $t=0$ and the limit found in (a).
(c)	Very poor. Most candidates wrote wrongly the equation required in (i).

25. (PP DSE-MATH-M1 Q11)

(a) $P'(t)=kte^{\frac{a}{20}t}$

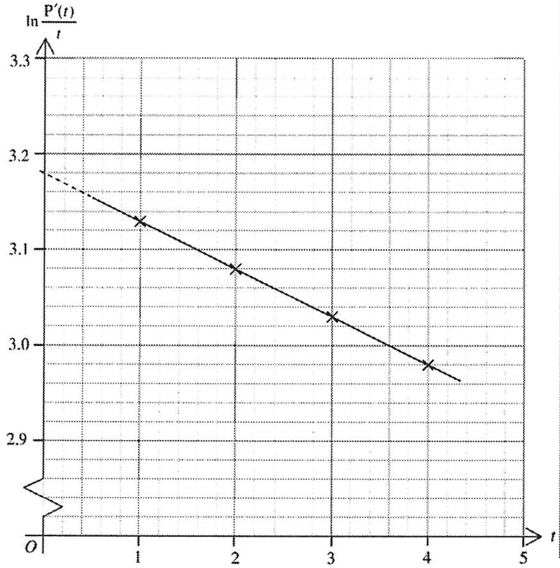
$\ln\frac{P'(t)}{t}=\frac{a}{20}t+\ln k$ — 1A

(1)

(b)

t	1	2	3	4
$P'(t)$	22.83	43.43	61.97	78.60
$\ln\frac{P'(t)}{t}$	3.13	3.08	3.03	2.98

1A



1A

From the graph, $\frac{a}{20}\approx\frac{2.98-3.13}{4-1}$ — 1M

$a\approx -1$ — 1A

From the graph, $\ln k\approx 3.18$

$k\approx 24$ — 1A

Either one

(5)

(c) (i) $\frac{d}{dt}P'(t)=\frac{d}{dt}\left(24te^{\frac{-t}{20}}\right)$

$=24e^{\frac{-t}{20}}\left(1-\frac{t}{20}\right)$ — 1A

$\therefore \frac{d}{dt}P'(t)=0$ when $t=20$

t	<20	20	>20
$\frac{d}{dt}P'(t)$	+ve	0	-ve

1M

Alternative Solution

$\frac{d^2}{dt^2}P'(t)=24e^{\frac{-t}{20}}\left[\frac{-1}{20}\left(1-\frac{t}{20}\right)+\frac{-1}{20}\right]$

$=\frac{6}{5}e^{\frac{-t}{20}}\left(\frac{t}{20}-2\right)$

$\therefore \frac{d^2}{dt^2}P'(t)<0$ when $t=20$ — 1M

Hence the rate of change of the population size is greatest when $t=20$. — 1A

(ii) $\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)=e^{\frac{-t}{20}}-\frac{1}{20}te^{\frac{-t}{20}}$ — 1A

$24te^{\frac{-t}{20}}=480e^{\frac{-t}{20}}-480\frac{d}{dt}\left(te^{\frac{-t}{20}}\right)$

$\int 24te^{\frac{-t}{20}}\,dt=-9600e^{\frac{-t}{20}}-480te^{\frac{-t}{20}}+C$ — 1M

$P(t)=C-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ — 1A

Since $P(0)=30$, we have

$C-480(0)e^0-9600e^0=30$ — 1M

$C=9630$

$\therefore P(t)=9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}$ — 1A

(iii) $\lim_{t\to\infty}P(t)=\lim_{t\to\infty}\left(9630-480te^{\frac{-t}{20}}-9600e^{\frac{-t}{20}}\right)$

$=9630$ — 1A

$\therefore$ the population size after a very long time is estimated to be 9630 thousands.

(9)

(a)	良好。 大部分學生能正確表達線性函數。
(b)	良好。 部分學生看漏了精確度的要求。
(c) (i)	平平。 部分學生在取得第一導數並使該導數相等於零之後，沒有作出極大測試。
(ii)	平平。 大部分學生不懂如何利用 $\frac{d}{dt}\left(te^{\frac{a}{20}t}\right)$ 的結果求 P(t)。
(iii)	甚差。 大部分學生因爲在前部分出錯而得出錯誤的結論。

26. (SAMPLE DSE-MATH-M1 Q11)

(a) (i) Let $v=1+4te^{-0.04t}$. Then we have

$\frac{dv}{dt}=4e^{-0.04t}-0.16te^{-0.04t}$ — 1M+1A (1M for product rule)

$=0.16e^{-0.04t}(25-t)$

(ii) When $t=0$, $\frac{dN}{dt}=50$. So we have $25k=50$.

Thus, we have $k=2$. — 1A

$N=\int\frac{2(25-t)}{e^{0.04t}+4t}\,dt$

$=2\int\frac{e^{-0.04t}(25-t)}{1+4te^{-0.04t}}\,dt$ — 1M

$=\frac{2}{0.16}\int\frac{dv}{v}$ — 1M (For using (a)(i))

$=12.5\ln|v|+C$

$=12.5\ln(1+4te^{-0.04t})+C$

When $t=0$, $N=10$. So, we have $C=10$. — 1M (For finding C)

i.e. $N=12.5\ln(1+4te^{-0.04t})+10$ — 1A

(7)

(b) (i) $\frac{dN}{dt}=\frac{2(25-t)}{e^{0.04t}+4t}$

$\begin{cases} >0 & \text{when } 0\le t<25 \\ =0 & \text{when } t=25 \\ <0 & \text{when } t>25 \end{cases}$ — 1M

So, N attains its greatest value when $t=25$. — 1A

Alternative Solution

$\frac{dN}{dt}=\frac{2(25-t)}{e^{0.04t}+4t}$

$\frac{dN}{dt}=0$ when $t=25$

$\frac{d^2N}{dt^2}=2\left[\frac{(e^{0.04t}+4t)(-1)-(0.04e^{0.04t}+4)(25-t)}{(e^{0.04t}+4t)^2}\right]$

$=-4\left[\frac{(1-0.02t)e^{0.04t}+50}{(e^{0.04t}+4t)^2}\right]$ — 1M

$\left.\frac{d^2N}{dt^2}\right|_{t=25}=-4\left[\frac{0.5e+50}{(e+100)^2}\right]<0$

So, N attains its greatest value when $t=25$. — 1A

(ii) $N(25)=12.5\ln(1+4te^{-0.04t})+10\approx 55.4>50$

Thus, the claim is agreed. — 1

(3)

(c) $\lim_{t\to\infty}N=\lim_{t\to\infty}[12.5\ln(1+4te^{-0.04t})+10]$

$=12.5\ln(1+0)+10$ — 1M — For using $\lim_{t\to\infty}te^{-0.04t}=0$

$=10$

Thus, the belief of Mary's supervisor is agreed. — 1

(2)

27. (2010 ASL-M&S Q9)

(a) (i) $p(t)=\frac{a}{b+e^{-t}}+c$

$p'(t)=\frac{ae^{-t}}{(b+e^{-t})^2}$ — 1A

$p''(t)=\frac{(b+e^{-t})^2(-ae^{-t})-(ae^{-t})2(b+e^{-t})(-e^{-t})}{(b+e^{-t})^4}$

$=\frac{ae^{-t}(e^{-t}-b)}{(b+e^{-t})^3}$ — 1A

Hence $p''(t)=0$ when $e^{-t}-b=0$.

i.e. $t=-\ln b$

t	$t<-\ln b$	$t=-\ln b$	$t>-\ln b$
$p''(t)$	+	0	−

Hence the growth rate attains the maximum value when $t=-\ln b$ — 1A — Follow through

(ii) *primordial population* $=\lim_{t\to-\infty}\left(\frac{a}{b+e^{-t}}+c\right)=c$ — 1A

(iii) *ultimate population* $=\lim_{t\to\infty}\left(\frac{a}{b+e^{-t}}+c\right)=\frac{a}{b}+c$ — 1A

(5)

(b) $\ln[p(t)-c]=-\ln(b+e^{-t})+\ln a$

$\therefore \ln a=\ln 8000$

$a=8000$ — 1A

$\therefore p'(0)=\frac{8000}{(b+1)^2}=2000$

$b=1$ or -3 (rejected) — 1A

$\therefore p(0)=\frac{8000}{1+1}+c=6000$

$c=2000$ — 1A

(3)

(c) The population at the time of maximum growth rate is

$p(-\ln b)=\frac{a}{2b}+c$ — 1A

The mean of the *primordial population* and *ultimate population* is

$\frac{1}{2}\left[c+\left(\frac{a}{b}+c\right)\right]=\frac{a}{2b}+c$

Hence the scientist's claim is agreed. — 1

(2)

(d) $p(t)=\dfrac{a}{b+e^{-t}}+c$

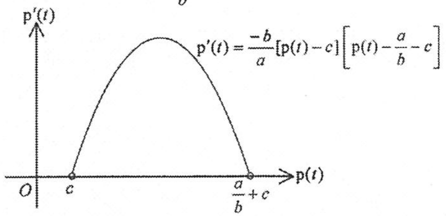
$e^{-t}=\dfrac{a}{p(t)-c}-b$ — 1A

$\therefore\ p'(t)=\dfrac{a\left[\dfrac{a}{p(t)-c}-b\right]}{\left[b+\left(\dfrac{a}{p(t)-c}-b\right)\right]^2}$ — 1M

$=\dfrac{a[p(t)-c]\{a-b[p(t)-c]\}}{a^2}$

$=\dfrac{-b}{a}[p(t)-c]\left[p(t)-\dfrac{a}{b}-c\right]$

Hence $\alpha=c$ and $\beta=\dfrac{a}{b}+c$. — 1A



1A

From the graph, we can see that $p'(t)$ is maximum when $p(t)$ is the mean of c and $\dfrac{a}{b}+c$, i.e. the mean of the *primordial population* and *ultimate population*. — 1 — Follow through

(5)

(a) (i)		Fair. Many candidates confused the maximum growth rate and the maximum population and hence could not determine the time required.
(ii) (iii)		Fair. Some candidates mistook $p(0)$ to be the primordial population.
(b)		Fair.
(c)		Poor. Most candidates did not understand the question.
(d)		Very poor. Most candidates could not go beyond expressing e^{-t} in terms of a, b, c and $p(t)$.

28. (2009 ASL-M&S Q9)

(a) (i) $R_6=\int_0^6\ln(2t+1)\,dt$

$\approx\frac{1}{2}\{\ln(2\cdot0+1)+2[\ln(2\cdot1+1)+\ln(2\cdot2+1)+\ln(2\cdot3+1)+\ln(2\cdot4+1)$
$+\ln(2\cdot5+1)]+\ln(2\cdot6+1)\}$ — 1M

$=10.53155488$

The total amount of revenue in the first 6 weeks is 10.5316 million dollars. — 1A

(ii) Let $f(t)=\ln(2t+1)$

$f'(t)=\dfrac{2}{2t+1}$

$f''(t)=\dfrac{-4}{(2t+1)^2}$ — 1A

<0 for $0\le t\le 6$

$\therefore$ $f(t)$ is concave downward for $0\le t\le 6$.

Hence the estimate in (a)(i) is an under-estimate. — 1A — Follow through

(4)

(b) (i) $Q_1=\int_0^1\left[45t(1-t)+\dfrac{1.58}{t+1}\right]dt$ — 1A

$=\left[45\left(\dfrac{t^2}{2}-\dfrac{t^3}{3}\right)+1.58\ln|t+1|\right]_0^1$

$=\dfrac{15}{2}+1.58\ln2$

≈8.595172545

The total amount of revenue in the first week is 8.5952 million dollars. — 1A

(ii) $Q_n=Q_1+\int_1^n\dfrac{30e^{-t}}{(3+2e^{-t})^2}dt$ — 1M

Let $u=3+2e^{-t}$

$du=-2e^{-t}dt$

$\therefore\ Q_n=Q_1+\int_{3+2e^{-1}}^{3+2e^{-n}}\dfrac{-15}{u^2}du$ — 1M — For $\dfrac{-15}{u^2}$

$=Q_1+\left[\dfrac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$ — 1A — For $\left[\dfrac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$

$=\dfrac{15}{2}+1.58\ln2+\dfrac{15}{3+2e^{-n}}-\dfrac{15}{3+2e^{-1}}$

Hence the total amount of revenue in the first n weeks is

$\left(\dfrac{15}{2}+1.58\ln2+\dfrac{15}{3+2e^{-n}}-\dfrac{15}{3+2e^{-1}}\right)$ million dollars, where $n>1$. — 1A — Accept $4.5799+\dfrac{15}{3+2e^{-n}}$

(6)

(c) For $n>6$, $R_n=R_6+\int_6^n0\,dt\approx10.5316$ (by (a)(i)) — 1M — For $\int_6^n0\,dt=0$

When $n\to\infty$, $e^{-n}\to0$ and so $Q_n\to4.5799+\dfrac{15}{3+0}=9.5799$ — 1M — For $e^{-n}\to0$

Therefore, over a long period of time, plan A produces approximately 10.5316 million dollars and plan B produces 9.5799 million dollars of revenue. — 1A

Moreover, the revenue of plan A is even an under-estimate. — 1M

Hence, plan A will produce more revenue over a long period of time. — 1A — Follow through

(5)

(a)	(i)		Good.
	(ii)		Poor. The poor performance was rather unexpected since applying the concept of concave and convex curves should be quite standard.
(b)	(i)		Good.
	(ii)		Poor. The problem might look unfamiliar. Many candidates did not realize that the lower and upper limits of the integral should be 1 and n, and Q_1 should be added to the integral to get Q_n.
(c)			Very poor. Most candidates got the wrong conclusion due to mistakes made in the previous parts.

29. (2005 ASL-M&S Q8)

(a) $r(t)=\alpha t e^{-\beta t}$

$\frac{r(t)}{t}=\alpha e^{-\beta t}$

$\ln\frac{r(t)}{t}=\ln\alpha-\beta t$ 1A

----------(1)

(b) $\because \ln\alpha=2.3$

$\therefore \alpha\approx 10$ (correct to 1 significant figure) 1A

Also, we have $\beta\approx 0.5$ (correct to 1 significant figure). 1A

$r(t)=10te^{-0.5t}$

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$

$=10e^{-0.5t}-5te^{-0.5t}$ 1A

$=(10-5t)e^{-0.5t}$

$\frac{dr(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<2\\ =0 & \text{if } t=2\\ <0 & \text{if } t>2\end{cases}$ 1M for testing+1A

So, $r(t)$ attains its greatest value when $t=2$.

Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx 7.357588823$.

Thus, the greatest rate of change is 7 ppm per hour. 1A

$\frac{dr(t)}{dt}=10t(-0.5e^{-0.5t})+10e^{-0.5t}$ $=10e^{-0.5t}-5te^{-0.5t}$ $=(10-5t)e^{-0.5t}$	1A
$\frac{d^2r(t)}{dt^2}=-5e^{-0.5t}+(-5+2.5t)e^{-0.5t}=(2.5t-10)e^{-0.5t}$	
$\frac{dr(t)}{dt}=0$ when $t=2$ only and $\left.\frac{d^2r(t)}{dt^2}\right\|_{t=2}=-5e^{-1}<0$	1M for testing+1A
So, $r(t)$ attains its greatest value when $t=2$. Hence, greatest value of $r(t)$ is $(10)(2)e^{-0.5(2)}\approx 7.357588823$. Thus, the greatest rate of change is 7 ppm per hour.	1A

----------(6)

(c) (i) $\frac{d}{dt}\left((t+\frac{1}{\beta})e^{-\beta t}\right)$

$=\frac{d}{dt}\left((t+2)e^{-0.5t}\right)$

$=e^{-0.5t}-0.5(t+2)e^{-0.5t}$ 1M for product rule or chain rule

$=-0.5te^{-0.5t}$ 1M accept $-\beta te^{-\beta t}$

The required amount

$=\int_0^T r(t)dt$

$=\int_0^T 10te^{-0.5t}dt$ 1M

$=\left[-20(t+2)e^{-0.5t}\right]_0^T$ 1M+1A

$=\left(40-20(T+2)e^{-0.5T}\right)$ ppm 1A

Note that $\int r(t)dt$ $=\int 10te^{-0.5t}dt$ $=-20(t+2)e^{-0.5t}+C$	1M+1A
Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours. Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$.	1M
Since $A(0)=0$, we have $C=40$. So, we have $A(t)=\left(40-20(t+2)e^{-0.5t}\right)$. Note that $A(0)=0$. Thus, the required amount $=A(T)=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

Note that $\int r(t)dt$ $=\int 10te^{-0.5t}dt$ $=-20(t+2)e^{-0.5t}+C$	1M+1A
Let $A(t)$ ppm be the amount of soot reduced when the petrol additive has been used for t hours. Then, we have $A(t)=-20(t+2)e^{-0.5t}+C$.	1M
The required amount $=A(T)-A(0)$ $=\left(-20(T+2)e^{-0.5T}+C\right)-\left(-40+C\right)$ $=\left(40-20(T+2)e^{-0.5T}\right)$ ppm	1A

(ii) The required amount

$=\lim_{T\to\infty}\left(40-20(T+2)e^{-0.5T}\right)$

$=40-20\lim_{T\to\infty}Te^{-0.5T}-40\lim_{T\to\infty}e^{-0.5T}$

$=40-20(0)-40(0)$ 1M for $\lim_{T\to\infty}e^{-0.5T}=0$ and can be absorbed

$=40$ ppm 1A

----------(8)

(a)		Very Good.
(b)		Good. Some candidates were not able to show that the stationary point is a maximum point.
(c) (i)		Fair. The first part was done well but the later part was less satisfactory. Only some candidates were able to work out the total amount of soot reduced.
(ii)		Poor. Many candidates were not able to complete this part because they failed to solve (c)(i).

30. (2003 ASL-M&S Q9)

Solution	Marks
(a) $\because\ P(0)=5.9$	
$\therefore\ a+\frac{1}{5}(0-0-8)=5.9$	
So, $a=7.5$	1A
$P(t)=7.5+\frac{1}{5}(t^2-8t-8)e^{-kt}$	
$\because\ P(8)-P(4)=1.83$	
$\therefore\ -1.6e^{-8k}+4.8e^{-4k}=1.83$	1M+1A
$160(e^{-4k})^2-480e^{-4k}+183=0$	1M can be absorbed
$e^{-4k}\approx 2.551784198$ or $e^{-4k}\approx 0.448215801$	
$k\approx -0.2341982$ or $k\approx 0.200620116$	
$\because\ k>0$	
$\therefore\ k\approx 0.2$ (correct to 1 decimal place)	1A
	----------(5)
(b) $P(t)=\frac{15}{2}+\frac{1}{5}(t^2-8t-8)e^{-0.2t}$	
(i) $\frac{dP(t)}{dt}=\frac{-1}{25}\left[(t^2-8t-8)-5(2t-8)\right]e^{-0.2t}$	1M for Product Rule or Chain Rule
$=\frac{-1}{25}(t^2-18t+32)e^{-0.2t}$	1A independent of the obtained value of a
$=\frac{-1}{25}(t-2)(t-16)e^{-0.2t}$	
For $\frac{dP(t)}{dt}=0$, we have $t=2$ or $t=16$.	
$\frac{dP(t)}{dt}\begin{cases}<0 & \text{if } 0\le t<2\\ =0 & \text{if } t=2\\ >0 & \text{if } 2<t<16\end{cases}$	
So, the minimum pH value occurred at $t=2$.	1M+1A
$\frac{dP(t)}{dt}\begin{cases}>0 & \text{if } 2<t<16\\ =0 & \text{if } t=16\\ <0 & \text{if } t>16\end{cases}$	
So, the maximum pH value occurred at $t=16$.	1M+1A accept max at $t=0$ and at $t=16$

Solution	Marks
$\frac{dP(t)}{dt}=\frac{-1}{25}\left[(t^2-8t-8)-5(2t-8)\right]e^{-0.2t}$	1M for Product Rule or Chain Rule
$=\frac{-1}{25}(t^2-18t+32)e^{-0.2t}$	1A independent of the obtained value of a
$=\frac{-1}{25}(t-2)(t-16)e^{-0.2t}$	
For $\frac{dP(t)}{dt}=0$, we have $t=2$ or $t=16$.	
$\frac{d^2P(t)}{dt^2}=\frac{1}{125}[t^2-18t+32-5(2t-18)]e^{-0.2t}$	
$=\frac{1}{125}(t^2-28t+122)e^{-0.2t}$	
$\left.\frac{d^2P(t)}{dt^2}\right\rvert_{t=2}\approx 0.375379225>0$	
So, the minimum pH value occurred at $t=2$.	1M+1A
$\left.\frac{d^2P(t)}{dt^2}\right\rvert_{t=16}\approx -0.022826834<0$	
So, the maximum pH value occurred at $t=16$.	1M+1A accept max at $t=0$ and at $t=16$

Solution	Marks
(ii) $\frac{d^2P}{dt^2}=\frac{1}{125}[t^2-18t+32-5(2t-18)]e^{-0.2t}$	
$=\frac{1}{125}(t^2-28t+122)e^{-0.2t}$	1A
$\because\ \frac{d^2P}{dt^2}=\frac{1}{125}\left(t-(14-\sqrt{74})\right)\left(t-(14+\sqrt{74})\right)e^{-0.2t}$	
$5<14-\sqrt{74}<6$ and $22<14+\sqrt{74}<23$	
$\therefore\ \frac{d^2P}{dt^2}>0$ for all $t\ge 23$.	1
	----------(8)
(c) The required pH value	
$=\lim_{t\to\infty}\left(7.5+\frac{1}{5}(t^2-8t-8)e^{-0.2t}\right)$	
$=7.5+\frac{1}{5}\lim_{t\to\infty}\left(t^2e^{-0.2t}\right)-\frac{8}{5}\lim_{t\to\infty}\left(te^{-0.2t}\right)-\frac{8}{5}\lim_{t\to\infty}e^{-0.2t}$	
$=7.5+\frac{1}{5}(0)-\frac{8}{5}(0)-\frac{8}{5}(0)$ $\left(\because \lim_{t\to\infty}(te^{-0.2t})=\left(\lim_{t\to\infty}\frac{1}{t}\right)\left(\lim_{t\to\infty}t^2e^{-0.2t}\right)=(0)(0)=0\right)$	1A for $\lim_{t\to\infty}(te^{-0.2t})=0$ (can be absorbed)
$=7.5$	1M accept the required pH value $=a$
	----------(2)

(a)		Good. Some candidates were unable to transform the equation '$-1.6e^{-8k}+4.8e^{-4k}=1.83$' into a quadratic equation.
(b)		Good. Most candidates were able to differentiate functions involving 'exp' function.
(c)		Satisfactory. Some candidates had difficulty in finding the limit.

31. (2001 ASL-M&S Q9)

(a) (i) $\ln P'(t) = -kt + \ln \dfrac{0.04ak}{1-a}$ — 1A

From the graph,

$-k \approx \dfrac{-8-(-3.5)}{18-0}$, $k \approx 0.25$ — 1A a–1 for more than 2 d.p.

$\ln \dfrac{0.04ak}{1-a} \approx -3.5$, $a \approx 0.7512 \approx 0.75$ — 1A a–1 for more than 2 d.p.

$P'(t) \approx 0.03e^{-0.25t}$

$P(t) \approx -0.12e^{-0.25t} + c$ for some constant c — 1M

Since $P(0) = 0.09$, $\therefore$ $c \approx 0.21$ — 1A

Hence $P(t) \approx -0.12e^{-0.25t} + 0.21$

(ii) $\mu = P(3) \approx 0.1533$ — 1A $\mu \in [0.1530, 0.1533]$

(iii) Stabilized PPI in town $A = \lim_{t\to\infty} P(t) = 0.21$ — 1M+1A

----------(8)

(b) (i) Suppose $b = 0.09$.

(I) $Q'(t) = 0.24(3t+4)^{-\frac{3}{2}}$

$Q(t) = \dfrac{1}{3}(0.24)(-2)(3t+4)^{-\frac{1}{2}} + c$ for some constant c — 1A

$= -0.16(3t+4)^{-\frac{1}{2}} + c$

Since $Q(0) = 0.09$, $\therefore$ $c = 0.17$ — 1A

If $Q(t) = \mu \approx 0.1533$

$-0.16(3t+4)^{-\frac{1}{2}} + 0.17 \approx 0.1533$ — 1M

$(3t+4)^{\frac{1}{2}} \approx \dfrac{0.16}{0.0167}$

Since $3t+4 > 0$

$\therefore$ $t \approx 29.3$ — $t \in [28.2, 29.3]$

i.e. the PPI will reach the value of μ. — 1

Since $Q(0) = 0.09$, $\lim_{t\to\infty} Q(t) = 0.17$ and Q is continuous and strictly increasing ($Q'(t) > 0$), $\therefore$ Q can reach any value between 0.09 and 0.17 including $\mu \approx 0.1533$.	1M 1

(II) Stabilized PPI in town $B = \lim_{t\to\infty} Q(t) = 0.17$

$\therefore$ The stabilized PPI will be reduced by 0.04. — 1A

(ii) $0.05 < b \, (\le 1)$. — 1A

Otherwise, $Q'(t) \le 0$ and the PPI will not increase.

It follows that the epidemic will not break out. — 1

----------(7)

32. (2000 ASL-M&S Q9)

(a) (i) $f(x) = 16 + 4xe^{-0.25x}$

$f'(x) = 4e^{-0.25x}(1 - 0.25x)$ — {1M attempting to find f'; 1A}

accept considering $f'(x) = e^{-0.25x}(0.25x - 2)$

$\begin{cases} > 0 & \text{if } 0 < x < 4 \\ = 0 & \text{if } x = 4 \\ < 0 & \text{if } x > 4 \end{cases}$

$\therefore$ $f(x) \le f(4)$ for $x > 0$. — 1 follow through

(ii)

x	0	1	2	3	4	5	6
f(x)	16 (16)	19.1152 (19.1)	20.8522 (20.9)	21.6684 (21.7)	21.8861 (21.9)	21.7301 (21.7)	21.3551 (21.4)

1A correct to 1 d.p.

$\int_0^6 f(x)dx$

$\approx \dfrac{1}{2}[16 + 21.3551 + 2(19.1152 + 20.8523 + 21.6684 + 21.8861 + 21.7301)]$ — 1M

≈ 124

$\therefore$ The expected increase in profit is 124 hundred thousand dollars. — 1A a–1 for r.t. 124; pp–1 for wrong/missing unit

(b) (i) $g(x) = 16 + \dfrac{6x}{\sqrt{1+8x}}$

$g'(x) = \dfrac{6\sqrt{1+8x} - \dfrac{6x \cdot 8}{2\sqrt{1+8x}}}{1+8x}$ — 1A

$= \dfrac{6(1+4x)}{(1+8x)^{\frac{3}{2}}}$

> 0 for $x > 0$.

$\therefore$ $g(x)$ is strictly increasing for $x > 0$. — 1

$\because$ $\lim_{x\to\infty}\left(16 + \dfrac{6x}{\sqrt{1+8x}}\right) = \lim_{x\to\infty}\left(16 + \dfrac{6\sqrt{x}}{\sqrt{\dfrac{1}{x}+8}}\right)$

$\therefore$ $g(x) \to \infty$ as $x \to \infty$ — 1A

(ii) Let $u=\sqrt{1+8x}$, then $u^2=1+8x$, $2u\mathrm{d}u=8\mathrm{d}x$

$$\int_0^6 g(x)\mathrm{d}x=\int_0^6\left(16+\frac{6x}{\sqrt{1+8x}}\right)\mathrm{d}x \qquad \left(\text{or } \int_0^6 16\mathrm{d}x+\int_0^6\frac{6x}{\sqrt{1+8x}}\mathrm{d}x\right)$$

$$=\int_1^7\left(16+\frac{6(u^2-1)}{8u}\right)\frac{1}{4}u\mathrm{d}u \qquad \left(\text{or } [16x]_0^6+\int_1^7\frac{6(u^2-1)}{8u}\frac{1}{4}u\mathrm{d}u\right)$$ 1A integrand; 1A limits

$$=\int_1^7\left(\frac{3}{16}u^2+4u-\frac{3}{16}\right)\mathrm{d}u \qquad \left(\text{or } 96+\int_1^7\left(\frac{3}{16}u^2-\frac{3}{16}\right)\mathrm{d}u\right)$$

$$=\left[\frac{1}{16}u^3+2u^2-\frac{3}{16}u\right]_1^7 \qquad \left(\text{or } 96+\left[\frac{1}{16}u^3-\frac{3}{16}u\right]_1^7\right)$$ 1A ignore limits

$$=116\frac{1}{4}$$

$$\approx 116$$

∴ The expected increase in profit is 116 hundred thousand dollars. — 1A $a-1$ for r.t. 116; $pp-1$ for wrong/missing unit

(c) From (a)(i), $f(x)\le f(4)$ (≈ 21.8861) for $x>0$.
i.e. $f(x)$ is bounded above by $f(4)$.
From (b)(i), $g(x)$ increases to infinity as x increases to infinity. — 1M
∵ $f(x)>0$ and $g(x)>0$ for $x>0$,
the area under the graph of $g(x)$ will be greater than that of $f(x)$ as x increases indefinitely.
∴ Plan G will eventually result in a bigger profit. — 1A

33. (2000 ASL-M&S Q11)

(a) $$\begin{cases}\ln 55=a-e^{1-2k}\\ \ln 98=a-e^{1-4k}\end{cases}$$

Eliminating a , we have

$$e^{1-4k}-e^{1-2k}+\ln 98-\ln 55=0$$

$$e^{-4k}-e^{-2k}+\frac{1}{e}\ln\frac{98}{55}=0$$ 1

$$(e^{-2k})^2-e^{-2k}+\frac{1}{e}\ln\frac{98}{55}=0$$ 1M quadratic equation

$$e^{-2k}=\frac{1\pm\sqrt{1-\frac{4}{e}\ln\frac{98}{55}}}{2}$$

≈ 0.30635 or 0.69365

≈ 0.306 or 0.694 — 1A r.t. 0.306 , 0.694

$$\begin{cases}k\approx 0.5915\\ a\approx 4.8401\end{cases} \text{ or } \begin{cases}k\approx 0.1829\\ a\approx 5.8929\end{cases}$$

$$\begin{cases}k\approx 0.59\\ a\approx 4.84\end{cases} \text{ or } \begin{cases}k\approx 0.18\\ a\approx 5.89 \text{ (or } 5.90)\end{cases}$$ (2 d.p.) — 1A $a-1$ for more than 2 d.p.

(b) Using $\begin{cases}k\approx 0.59\\ a\approx 4.84\end{cases}$, $\ln N(7)\approx 4.80$, — 1M r.t. 4.80

$N(7)\approx 121$. — r.t. 121

Using $\begin{cases}k\approx 0.18\\ a\approx 5.89\end{cases}$, $\ln N(7)\approx 5.12$, — r.t. 5.12 – 5.14

$N(7)\approx 167$. (or comparing $\ln 170\approx 5.1358$) — r.t. 167 – 170

∴ $\begin{cases}k\approx 0.18\\ a\approx 5.89\end{cases}$ will make the model fit for the known data. — 1A follow through

∵ $N(t)=e^{\ln N(t)}\approx e^{5.89-e^{1-0.18t}}$ — 1M

∴ $N(t)\to e^{5.89}\approx 361$ as $t\to\infty$

The total possible catch of coral fish in that area since January 1, 1992 is 361 thousand tonnes. — 1A r.t. 361 – 365; $pp-1$ for wrong/missing unit

Solution	Marks	Remarks
(c) (i) $\because \ln N(t) = a - e^{1-kt}$		
$\therefore \frac{N'(t)}{N(t)} = ke^{1-kt}$		
$N'(t) = k\,N(t)e^{1-kt}$	1	
Alternatively, $N(t) = e^{a-e^{1-kt}}$		
$N'(t) = -e^{1-kt}(-k)e^{a-e^{1-kt}} = ke^{1-kt}N(t)$	1	
(ii) $N''(t) = k[N'(t)e^{1-kt} - k\,N(t)e^{1-kt}]$		
$= k^2 N(t)e^{1-kt}(e^{1-kt} - 1)$	1A	
$\begin{cases} >0 & \text{when } t < \frac{1}{k} \\ =0 & \text{when } t = \frac{1}{k} \\ <0 & \text{when } t > \frac{1}{k} \end{cases}$	1M	
$\therefore N'(t)$ is maximum at $t = \frac{1}{k}$	1A	
≈ 5.56		$t \in [5.47, 5.56]$
The maximum rate of change of the total catch of coral fish in that area since January 1, 1992 occurred in 1997.	1A	
$\ln N(6) \approx 4.97$, $N(6) \approx 143.6$		$\ln N(6) \in [4.97, 4.99]$ $N(6) \in [143.6, 146.3]$
$\ln N(5) \approx 4.78$, $N(5) \approx 119.7$		$\ln N(5) \in [4.78, 4.80]$ $N(5) \in [119.7, 122.0]$
$\therefore$ The volume of fish caught in 1997		
$= [N(6) - N(5)]$ thousand tonnes	1M	
≈ 24 thousand tonnes	1A	*pp*−1 for wrong/missing unit

34. (1997 ASL-M&S Q8)

Solution	Marks	Remarks
(a) $\because N(0) = 16$		
$\therefore \frac{40}{1+b} = 16$	1M	
$b = 1.5$	1A	
$\because N(7) = 17.4$		
$\therefore \frac{40}{1+1.5e^{-7r}} = 17.4$	1M	
$e^{-7r} = \frac{1}{1.5}\left(\frac{40}{17.4} - 1\right)$		
$r = \frac{1}{-7}\ln\left[\frac{1}{1.5}\left(\frac{40}{17.4} - 1\right)\right]$	1M	
≈ 0.02	1A	
(b) $N(t) = \frac{40}{1+be^{-rt}}$ (or $\frac{40}{1+1.5e^{-0.02t}}$)		
$N'(t) = \frac{-40(-bre^{-rt})}{(1+be^{-rt})^2}$ (or $\frac{-40(-1.5)(0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)	1M+1A	
$= \frac{40bre^{-0.02t}}{(1+be^{-rt})^2}$ (or $\frac{1.2e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)		
> 0		
$\therefore N(t)$ is increasing.	1	
(c) $\because \lim_{t\to\infty} e^{-rt} = 0$		
$\therefore N_a = \lim_{t\to\infty}\frac{40}{1+be^{-rt}}$ (or $\lim_{t\to\infty}\frac{40}{1+1.5e^{-0.02t}}$)	1M	
$= 40$	1A	
(d) (i) $N''(t)$		
$= \frac{[(1+1.5e^{-0.02t})(1.2) - 1.2e^{-0.02t}(2)(1.5)](1+1.5e^{-0.02t})(-0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^4}$	1M	
$= \frac{0.012e^{-0.02t}(3e^{-0.02t} - 2)}{(1+1.5e^{-0.02t})^3}$	1A	
(ii) From (i), $N''(t) \begin{cases} >0 & \text{when } t < t_0 \\ =0 & \text{when } t = t_0 \\ <0 & \text{when } t > t_0 \end{cases}$	1M	For Solving $N''(t) = 0$
where $t_0 = -\frac{1}{0.02}\ln\frac{2}{3} \approx 20.2733$		
$\therefore$ The rate of increase is the greatest when $t = t_0 \approx 20.2733$	1M	For checking maximum
$\because N'(20) \approx 0.199999$		
$N'(21) \approx 0.199989$		
$\therefore$ The company should start to advertise on the 20th day after the first week.	1A	

Out of syllabus

35. (2012 DSE-MATH-M1 Q4)

Solution	Marks	Remarks
(a) $y=\sqrt[3]{\dfrac{3x-1}{x-2}}$		
$\ln y=\dfrac{1}{3}\ln(3x-1)-\dfrac{1}{3}\ln(x-2)$	1A	
$\therefore\ \dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}$	1A	
(b) By (a), $\dfrac{dy}{dx}=\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}$	1A	
$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}+\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]\dfrac{d}{dx}\left(\sqrt[3]{\dfrac{3x-1}{x-2}}\right)$		
$=\left[\dfrac{-3}{(3x-1)^2}+\dfrac{1}{3(x-2)^2}\right]\sqrt[3]{\dfrac{3x-1}{x-2}}+\left[\dfrac{1}{3x-1}-\dfrac{1}{3(x-2)}\right]^2\sqrt[3]{\dfrac{3x-1}{x-2}}$ by (a)	1M	For using (a)
When $x=3$, $\dfrac{d^2y}{dx^2}=\left\{\dfrac{-3}{(3\cdot3-1)^2}+\dfrac{1}{3(3-2)^2}+\left[\dfrac{1}{3\cdot3-1}-\dfrac{1}{3(3-2)}\right]^2\right\}\sqrt[3]{\dfrac{3\cdot3-1}{3-2}}$	1M	
Alternative Solution		
When $x=3$, $y=2$ and so $\dfrac{dy}{dx}=\dfrac{-5}{12}$.	1A	For both y and $\dfrac{dy}{dx}$
By (a), $\dfrac{1}{y}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}\cdot\dfrac{dy}{dx}=\dfrac{-3}{(3x-1)^2}+\dfrac{1}{3(x-2)^2}$	1M	For chain rule
When $x=3$, $\dfrac{1}{2}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{2^2}\cdot\dfrac{-5}{12}\cdot\dfrac{-5}{12}=\dfrac{-3}{(3\cdot3-1)^2}+\dfrac{1}{3(3-2)^2}$	1M	
i.e. $\dfrac{d^2y}{dx^2}=\dfrac{95}{144}$	1A	OR 0.6597
	(6)	

(a)	Good. Some candidates wrote $\sqrt[3]{\dfrac{3x-1}{x-2}}=\left(\dfrac{3x-1}{x-2}\right)^{3/2}$. Some did not use logarithmic differentiation.
(b)	Fair. Some candidates did not use the result in (a). Some wrote $\dfrac{d^2y}{dx^2}=\left(\dfrac{dy}{dx}\right)^2$, $\dfrac{d}{dx}\left(\dfrac{1}{y}\cdot\dfrac{dy}{dx}\right)=-y^{-2}\dfrac{d^2y}{dx^2}$ or $\dfrac{d}{dx}\left(\dfrac{1}{y}\cdot\dfrac{dy}{dx}\right)=\dfrac{1}{y}\cdot\dfrac{d^2y}{dx^2}-\dfrac{1}{y^2}\dfrac{dy}{dx}$.

36. (SAMPLE DSE-MATH-M1 Q6)

Solution	Marks	Remarks
(a) $\ln u=\dfrac{1}{2}\ln(2x+3)-\dfrac{1}{2}\ln(x+1)-\dfrac{1}{2}\ln(x+2)$	1A	
Differentiate both sides with respect to x, we have		
$\dfrac{1}{u}\dfrac{du}{dx}=\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}$		
$\dfrac{du}{dx}=u\left[\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}\right]$	1A	
(b) $u=3^y$ gives $y=\dfrac{\ln u}{\ln 3}$		
$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$		
$=\dfrac{1}{u\ln 3}\cdot u\left[\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}\right]$	1M+1M	1M for chain rule
$=\dfrac{1}{\ln 3}\left[\dfrac{1}{2x+3}-\dfrac{1}{2(x+1)}-\dfrac{1}{2(x+2)}\right]$	1A	
	(5)	

37. (2012 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $y=e^{t^2+4t+4}$ and $x=\ln(2t+4)$		
$\dfrac{dy}{dt}=e^{t^2+4t+4}(2t+4)$ and $\dfrac{dx}{dt}=\dfrac{1}{t+2}$	1A	For both
$\dfrac{dy}{dx}=\dfrac{dy}{dt}\div\dfrac{dx}{dt}$		
$=2e^{t^2+4t+4}(t+2)\cdot(t+2)$	1M	
Alternative Solution		
$\ln y=(t+2)^2$ and $x=\ln 2+\ln(t+2)$		OR $\cdots$ and $t+2=\dfrac{1}{2}e^x$
$\therefore\ x=\ln 2+\dfrac{1}{2}\ln(\ln y)$	1A	OR $\ln y=\dfrac{1}{4}e^{2x}$
$\dfrac{dx}{dy}=\dfrac{1}{2}\cdot\dfrac{1}{\ln y}\cdot\dfrac{1}{y}$	1M	OR $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=\dfrac{1}{4}e^{2x}\cdot 2$
$\dfrac{dy}{dx}=2y\ln y$	1A	
(b) $\dfrac{d^2y}{dx^2}=\left(2y\cdot\dfrac{1}{y}+2\ln y\right)\dfrac{dy}{dx}$	1M	For chain rule
$=4y\ln y(1+\ln y)$		
When $x=0$, $t=\dfrac{-3}{2}$ and so $y=e^{\frac{1}{4}}$.	1A	
$\therefore\ \dfrac{d^2y}{dx^2}=4e^{\frac{1}{4}}\left(\dfrac{1}{4}\right)\left(1+\dfrac{1}{4}\right)$		
$=\dfrac{5}{4}e^{\frac{1}{4}}$	1A	OR 1.6050
	(6)	

Fair.
Candidates did not perform well in chain rule.

38. (2009 ASL-M&S Q3)

(a) $x = y^4 - y$		
$\frac{dx}{dy} = 4y^3 - 1$	1M	For finding $\frac{dx}{dy}$
$\therefore \frac{dy}{dx} = \frac{1}{4y^3 - 1}$	1M+1A	1M for $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
Alternative Solution		
$1 = 4y^3 \frac{dy}{dx} - \frac{dy}{dx}$	1M+1M	1M for finding $\frac{dy}{dx}$; 1M for chain rule
$\therefore \frac{dy}{dx} = \frac{1}{4y^3 - 1}$	1A	
(b) $\therefore \frac{1}{4y^3 - 1} = \frac{1}{3}$	1M	
$y = 1$		
$\therefore x = 1^4 - 1 = 0$	1A	
Hence the required equation of the tangent is		
$y - 1 = \frac{1}{3}(x - 0)$	1M	
i.e. $x - 3y + 3 = 0$	1A	
	(7)	

Good. Most candidates had good knowledge in differentiation of inverse function and were able to find the equation of tangent.

39. (2008 ASL-M&S Q2)

(a) $y^3 - uy = 1$		
$3y^2 \frac{dy}{du} - \left(u\frac{dy}{du} + y\right) = 0$	1M	
$\frac{dy}{du} = \frac{y}{3y^2 - u}$	1A	
Alternative Solution		
$u = y^2 - \frac{1}{y}$		
$\frac{du}{dy} = 2y + \frac{1}{y^2}$	1M	
$\frac{dy}{du} = \frac{y^2}{2y^3 + 1}$	1A	
(b) $u = 2^{x^2}$		
$\ln u = x^2 \ln 2$	1M	
$\frac{1}{u}\frac{du}{dx} = 2x \ln 2$	1M	
$\frac{du}{dx} = 2^{x^2} \cdot 2x \ln 2$	1A	
Alternative Solution		
$u = 2^{x^2} = e^{x^2 \ln 2}$	1M	
$\frac{du}{dx} = e^{x^2 \ln 2} \cdot 2x \ln 2$	1M	
$= 2^{x^2} \cdot 2x \ln 2$	1A	
(c) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$		
$= \frac{y}{3y^2 - u} \cdot 2^{x^2} \cdot 2x \ln 2$	1M	
$= \frac{2^{x^2} \cdot 2xy \ln 2}{3y^2 - 2^{x^2}}$	1A	OR $\frac{2^{x^2+1} xy \ln 2}{3y^2 - 2^{x^2}}$
	(7)	

Good. Some candidates were not familiar with the use of the logarithmic function in differentiation.

40. (2005 ASL-M&S Q3)

Solution	Marks
(a) $\ln w=\frac{3}{2}\ln(x-1)-\frac{1}{2}\ln(x+2)-\frac{1}{2}\ln(2x+1)$	1A
Differentiate both sides w.r.t. x, we have	
$\frac{1}{w}\frac{dw}{dx}=\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}$	1M
$\frac{dw}{dx}=w\left(\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}\right)$	1A
$\boxed{\frac{dw}{dx}=\sqrt{\frac{(x-1)^3}{(x+2)(2x+1)}}\left(\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}\right)}$	
$\boxed{\frac{dw}{dx}=\frac{w(2x^2+14x+11)}{2(x-1)(x+2)(2x+1)}}$	
(b) $w=2^y$	
$\ln w=y\ln 2$	1M for taking log on both sides and can be absorbed
$y=\frac{\ln w}{\ln 2}$	
$\frac{dy}{dw}=\frac{1}{w\ln 2}$	1A
$\frac{dy}{dx}=\frac{dy}{dw}\frac{dw}{dx}$	
$\frac{dy}{dx}=\left(\frac{1}{w\ln 2}\right)\left[w\left(\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}\right)\right]$	1M for Chain Rule
$\frac{dy}{dx}=\frac{1}{\ln 2}\left(\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}\right)$	1A accept $\frac{(2x^2+14x+11)}{2(x-1)(x+2)(2x+1)\ln 2}$
$w=2^y$	
$\ln w=y\ln 2$	1M for taking log on both sides and can be absorbed
Differentiate both sides w.r.t. y, we have	
$\frac{1}{w}\frac{dw}{dy}=\ln 2$	
$\frac{dw}{dy}=w\ln 2$	
$\frac{dy}{dw}=\frac{1}{w\ln 2}$	1A
$\frac{dy}{dx}=\frac{dy}{dw}\frac{dw}{dx}$	
$\frac{dy}{dx}=\left(\frac{1}{w\ln 2}\right)\left[w\left(\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}\right)\right]$	1M for Chain Rule
$\frac{dy}{dx}=\frac{1}{\ln 2}\left(\frac{3}{2(x-1)}-\frac{1}{2(x+2)}-\frac{1}{2x+1}\right)$	1A accept $\frac{(2x^2+14x+11)}{2(x-1)(x+2)(2x+1)\ln 2}$
	----------(7)

Good. Some candidates failed to apply the chain rule.

41. (2002 ASL-M&S Q1)

Solution	Marks
$\frac{dx}{dt}=\frac{10}{t^3}-6e^{-3t}$	1M+1A
$\frac{dy}{dt}=-\frac{20}{t^3}+2e^{2t}$	(1M for $(e^{at})'=ae^{at}$)
$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{1}{\frac{dx}{dt}}\right)=\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}$	1M for Chain Rule and Inverse Function Rule
For $\frac{dy}{dx}=-2$	
$\frac{-\frac{20}{t^3}+2e^{2t}}{\frac{10}{t^3}-6e^{-3t}}=-2$	1M
$e^{5t}=6$	
$t=\frac{1}{5}\ln 6(\approx 0.3584)$	1A $a-1$ for r.t. 0.358
	----------(5)

42. (2000 ASL-M&S Q1)

Solution	Marks
$\ln(xy)=\frac{x}{y}$	
$\ln x+\ln y=\frac{x}{y}$	
$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=\frac{y-x\frac{dy}{dx}}{y^2}$ (or $\frac{x\frac{dy}{dx}+y}{xy}=\frac{y-x\frac{dy}{dx}}{y^2}$)	1M differentaition of $\ln x$ 1M chain rule 1M quotient/product rule
$y^2+xy\frac{dy}{dx}=xy-x^2\frac{dy}{dx}$	
$\frac{dy}{dx}=\frac{xy-y^2}{xy+x^2}$	1 at least one step
Alternatively,	
$xy=e^{\frac{x}{y}}$	
$x\frac{dy}{dx}+y=e^{\frac{x}{y}}\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)$	1M differentaition of e^x 1M chain rule 1M quotient rule
$x\frac{dy}{dx}+y=xy\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)$	
$xy\frac{dy}{dx}+y^2=xy-x^2\frac{dy}{dx}$	
$\frac{dy}{dx}=\frac{xy-y^2}{xy+x^2}$	1
	----(4)

43. (1999 ASL-M&S Q1)

Solution	Marks	Remarks
(a) When $x=1$, $e^y=\frac{2^3}{2}=4$		
$y=\ln 4$ (or $y=2\ln 2$)	1A	
(or $y=1.3863$)		$a-1$ for r.t. 1.386
(b) $\because$ $e^{xy}=\frac{x(x+1)^3}{x^2+1}$		
$xy=\ln\left(\frac{x(x+1)^3}{x^2+1}\right)$	1A	taking log on both sides (one side must correct)
$xy=\ln x+3\ln(x+1)-\ln(x^2+1)$		
$x\frac{dy}{dx}+y=\frac{1}{x}+\frac{3}{x+1}-\frac{2x}{x^2+1}$	1M+1M	1M for product rule 1M for differentiating log
When $x=1$, $\frac{dy}{dx}+\ln 4=1+\frac{3}{2}-1$		
$\frac{dy}{dx}=\frac{3}{2}-\ln 4$ (or 0.1137)	1A	$a-1$ for r.t. 0.114
Alternatively,		
$e^{xy}(x\frac{dy}{dx}+y)=\frac{(x^2+1)[(x+1)^3+3(x+1)^2x]-x(x+1)^3(2x)}{(x^2+1)^2}$	1M+1M+1A	1M for differentiating e^{xy} 1M for product/quotient rule
When $x=1$, $e^{xy}(\frac{dy}{dx}+\ln 4)=6$		
$\frac{dy}{dx}=\frac{3}{2}-\ln 4$	1A	
	(5)	

44. (1995 ASL-M&S Q2)

Solution	Marks	Remarks
(a) $e^x+e^y=xy$		
$e^x+e^y\frac{dy}{dx}=y+x\frac{dy}{dx}$	1A + 1A	
$\frac{dy}{dx}=\frac{e^x-y}{x-e^y}$	1A	
(b) $y=\frac{(x-2)^{\frac{1}{2}}(x+3)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}}}$		
$\ln y=\frac{1}{2}\ln(x-2)+\frac{1}{2}\ln(x+3)-\frac{3}{2}\ln(x+1)$		
$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2(x-2)}+\frac{1}{2(x+3)}-\frac{3}{2(x+1)}$	1M + 1A	1M for taking log. on both sides and applying: $\ln ab=\ln a+\ln b$ $n\ln a=\ln a^n$ $\frac{d}{dx}\ln y=\frac{1}{y}\frac{dy}{dx}$
$\frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-2}+\frac{1}{x+3}-\frac{3}{x+1}\right)$	1A	
(or $\frac{1}{2(x+1)}\sqrt{\frac{(x-2)(x+3)}{x+1}}\left(\frac{1}{x-2}+\frac{1}{x+3}-\frac{3}{x+1}\right)$		
or $\frac{y(19-x^2)}{2(x-2)(x+3)(x+1)}$)		
	(6)	

4. Applications of Differentiation

Learning Unit	Learning Objective
Calculus Area	
Differentiation with Its Applications	
6. Applications of differentiation	6.1 use differentiation to solve problems involving tangents, rates of change, maxima and minima

Section A

1. Define $f(x)=\dfrac{6-x}{x+3}$ for all $x>-3$.

   (a) Prove that $f(x)$ is decreasing.

   (b) Find $\lim\limits_{x\to\infty} f(x)$.

(6 marks) (2019 DSE-MATH-M1 Q5a,b)

2. Let h be a constant. Consider the curve C: $y=x^2\sqrt{h-x}$, where $0<x<h$. It is given that $\dfrac{dy}{dx}=30$ when $x=4$.

   (a) Prove that $h=20$.

   (b) Find the maximum point(s) of C.

   (c) Write down the equation(s) of the horizontal tangent(s) to C.

(7 marks) (2018 DSE-MATH-M1 Q7)

3. Let $f(x)$ be a continuous function such that $f'(x)=\dfrac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x.

   If $f(x)$ attains its minimum value at $x=\alpha$, find α.

(3 marks) (2018 DSE-MATH-M1 Q5a)

4. Let $f(x)=4x^3+mx^2+nx+615$, where m and n are constants. It is given that $(-6,33)$ is a turning point of the graph of $y=f(x)$. Find

   (a) m and n ,

   (b) the minimum value(s) and the maximum value(s) of $f(x)$.

(6 marks) (2017 DSE-MATH-M1 Q6)

5. Air is leaking from a spherical balloon at a constant rate of 100 cm^3 per second. Find the rate of change of the radius of the balloon at the instant when the radius is 10 cm.

(3 marks) (2014 DSE-MATH-M1 Q1)

6. Let $f(x)=\dfrac{x^x}{(2x+13)^6}$, where $x>1$.

(a) By considering $\ln f(x)$, find $f'(x)$.

(b) Show that $f(x)$ is increasing for $x>1$.

(Part a is out of Syllabus) (6 marks) (2014 DSE-MATH-M1 Q2)

7. The population p (in million) of a city at time t (in years) can be modelled by

$$p=8-\frac{2.1}{\sqrt{t+4}} \quad \text{for } t\geq 0 .$$

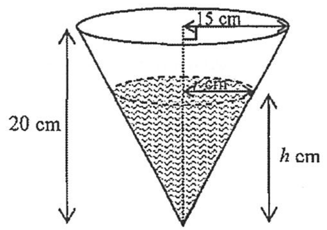
An environment study indicates that, when the population is p million, the concentration of carbon dioxide in the air is given by

$$C=2^p \text{ units} .$$

Find the rate of change of the concentration of carbon dioxide in the air at $t=5$.

(4 marks) (2013 DSE-MATH-M1 Q2)

8.



A glass container is in the shape of a vertically inverted right circular cone of base radius 15 cm and height 20 cm . Initially, the container is full of water. Suppose the water is running out from it at a constant rate of $2\pi\text{ cm}^3/\text{s}$. Let h cm be the depth of water remaining in the container, r cm

be the radius of the water surface (see the Figure), $V\text{ cm}^3$ be the volume of the water, and $A\text{ cm}^2$ be the area of the wet surface of the container. It is given that $V=\dfrac{1}{3}\pi r^2 h$ and $A=\pi r\sqrt{r^2+h^2}$.

(a) Express V and A in terms of r only.

(b) When $r=3$,

(i) find the rate of change of the radius of the water surface;

(ii) find the rate of change of the area of the wet surface of the container.

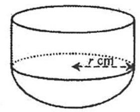
(6 marks) (PP DSE-MATH-M1 Q3)

9. When a hot air balloon is being blown up, its radius $r(t)$ (in m) will increase with time t (in hr). They are related by $r(t)=3-\dfrac{2}{2+t}$, where $t\geq 0$. It is known that the volume $V(r)$ (in m^3) of the balloon is given by $V(r)=\dfrac{4}{3}\pi r^3$.

Find the rate of change, in terms of π , of the volume of the balloon when the radius is 2.5 m .

(4 marks) (SAMPLE DSE-MATH-M1 Q2)

10. The figure shows a container (without a lid) consisting of a thin hollow hemisphere of radius r cm joined to the bottom of a right circular cylindrical thin pipe of base radius r cm. It is known that the area of the outer surface of the container is $162\pi\text{ cm}^2$.



(a) Prove that the capacity of the container is $\left(81\pi r-\dfrac{\pi r^3}{3}\right)\text{cm}^3$.

(b) As r varies, can the capacity of the container be greater than 1600 cm^3 ? Explain your answer.

(7 marks) (2004 ASL-M&S Q3)

11. At any time t (in hours), the relationship between the number N of tourists at a ski-resort and the air temperature $\theta°C$ can be modeled by

$$N=2930-(\theta+440)[\ln(\theta+49)]^2$$

where $-45\leq\theta\leq-40$.

(a) Express $\dfrac{dN}{dt}$ in terms of θ and $\dfrac{d\theta}{dt}$.

(b) At a certain moment, the air temperature is $-40°C$ and it is falling at a rate of $0.5°C$ per hour. Find, to the nearest integer, the rate of increase of the number of tourists at that moment.

(6 marks) (1996 ASL-M&S Q5)

12. An adventure estimates the volume of his hot air balloon by $V(r)=\frac{4}{3}\pi r^3+5\pi$, where r is measured in metres and V is measured in cubic metres. When the balloon is being inflated, r will increase with time $t(\geq 0)$ in such a way that,

$$r(t)=\frac{18}{3+2e^{-t}}$$

where t is measured in hours.

(a) Find the rate of change of volume of the balloon at $t=2$. Give your answer correct to 2 decimal places.

(b) If the balloon is being inflated over a long period of time, what will the volume of the balloon be? Give your answer correct to 2 decimal places.

(5 marks) (2002 ASL-M&S Q2)

13. Let $P(t)$ and $C(t)$ (in suitable units) be the electric energy produced and consumed respectively in a city during the time period $[0,t]$, where t is in years and $t\geq 0$. It is known that

$P'(t)=4\left(4-e^{\frac{-t}{5}}\right)$ and $C'(t)=9\left(2-e^{\frac{-t}{10}}\right)$. The redundant electric energy being generated during the time period $[0,t]$ is $R(t)$, where $R(t)=P(t)-C(t)$ and $t\geq 0$.

(a) Find t such that $R'(t)=0$.

(3 marks)

(b) Show that $R'(t)$ decreases with t.

(3 marks)

(2013 DSE-MATH-M1 Q11(a,b))

14. The current rate of selling of a certain kind of handbags is 30 thousand per day. The sales manager decides to raise the price of the handbags. After the price of the handbags has been raised for t days, the rate of selling of handbags $r(t)$ (in thousand per day) can be modelled by

$$r(t)=20-40e^{-at}+be^{-2at} \qquad (t\geq 0),$$

where a and b are positive constants. From past experience, it is known that after the increase in the price of the handbags, the rate of selling of handbags will decrease for 9 days.

(a) Find the value of b.

(1 mark)

(b) Find the value of a correct to 1 decimal place.

(3 marks)

(c) The sales manager will start to advertise when the rate of change of the rate of selling of handbags reaches a maximum. Use the results obtained in (a) and (b) to find the rate of selling of handbags when the sales manager starts to advertise.

(4 marks)

(2012 ASL-M&S Q9(a,b,c))

15. Mr. Lee has a fish farm in Sai Kung. Last week, the fish in his farm were affected by a certain disease. An expert told Mr. Lee that the number N of fish in his farm could be modelled by the function

$$N=\frac{5000e^{\lambda t}}{t} \qquad (0<t<120),$$

where λ is a constant and t is the number of days elapsed since the disease began to spread. Suppose that the numbers of fish will be the same when $t=15$ and $t=95$.

(a) Find the value of λ.

(b) How many days after the start of the spread of the disease will the number of fish decrease to the minimum?

(8 marks)

(1998 ASL-M&S Q8(a))

Section B

16. A researcher, Peter, models the number of crocodiles in a lake by

$$x = 4 + \frac{3k}{2^{\lambda t} - k} ,$$

where λ and k are positive constants, x is the number in thousands of crocodiles in the lake and t (≥ 0) is the number of years elapsed since the start of the research.

(a) (i) Express $(x-4)(x-1)$ in terms of λ, k and t.

(ii) Peter claims that the number of crocodiles in the lake does not lie between 1 thousand and 4 thousand. Is the claim correct? Explain your answer.

(3 marks)

(b) Peter finds that $\frac{dx}{dt} = \frac{-\ln 2}{24}(x-4)(x-1)$.

(i) Prove that $\lambda = \frac{1}{8}$.

(ii) For each of the following conditions (1) and (2), find k. Also determine whether the crocodiles in the lake will eventually become extinct or not. If your answer is 'yes', find the time it will take for the crocodiles to become extinct; if your answer is 'no', estimate the number of crocodiles in the lake after a very long time.

(1) When $t = 0$, $x = 0.8$.

(2) When $t = 0$, $x = 7$.

(9 marks)

(2017 DSE-MATH-M1 Q12)

17. The chickens in a farm are infected by a certain bird flu. The number of chickens (in thousand) in the farm is modelled by

$$N = \frac{27}{2 + \alpha t e^{\beta t}} ,$$

where t (≥ 0) is the number of days elapsed since the start of the spread of the bird flu and α and β are constants.

(a) Express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t.

(2 marks)

(b) It is given that the slope and the intercept on the horizontal axis of the graph of the linear function obtained in (a) are -0.1 and $10\ln 0.03$ respectively.

(i) Find α and β .

(ii) Will the number of chickens in the farm be less than 12 thousand on a certain day after the start of the spread of the bird flu? Explain your answer.

(iii) Describe how the rate of change of the number of chickens in the farm varies during the first 20 days after the start of the spread of the bird flu. Explain your answer.

(10 marks)

(2016 DSE-MATH-M1 Q12)

18. The population of a kind of bacterium $p(t)$ at time t (in days elapsed since 9 am on 16/4/2010, and can be positive or negative) is modelled by

$$p(t)=\frac{a}{b+e^{-t}}+c \quad , \quad -\infty<t<\infty$$

where a , b and c are positive constants. Define the *primordial population* be the population of the bacterium long time ago and the *ultimate population* be the population of the bacterium after a long time.

(a) Find, in terms of a , b and c ,

(i) the time when the growth rate attains the maximum value;

(ii) the *primordial population*;

(iii) the *ultimate population*.

(5 marks)

(b) A scientist studies the population of the bacterium by plotting a linear graph of $\ln[p(t)-c]$ against $\ln(b+e^{-t})$ and the graph shows the intercept on the vertical axis to be $\ln 8000$. If at 9 am on 16/4/2010 the population and the growth rate of the bacterium are 6000 and 2000 per day respectively, find the values of a , b and c .

(3 marks)

(c) Another scientist claims that the population of the bacterium at the time of maximum growth rate is the mean of the primordial population and ultimate population. Do you agree? Explain your answer.

(2 marks)

(d) By expressing e^{-t} in terms of a , b , c and $p(t)$, express $p'(t)$ in the form of $\frac{-b}{a}[p(t)-\alpha][p(t)-\beta]$, where $\alpha<\beta$.

Hence express α and β in terms of a , b and c .

Sketch $p'(t)$ against $p(t)$ for $\alpha<p(t)<\beta$ and hence verify your answer in (c).

(5 marks)

(2010 ASL-M&S Q9)

19. In a certain year, the amount of water (in million cubic metres) stored in a reservoir can be modeled by

$$A(t)=\left(-t^2+5t+a\right)e^{kt}+7 \qquad (0\le t\le 12) \ ,$$

where a and k are constants and t is the time measured in months from the start of the year. The amount of water stored in the reservoir is the greatest when $t=2$. It is found that $A(0)=3$.

(a) Find the value of a .

Hence find the amount of water stored in the reservoir when $t=1$.

(2 marks)

(b) Find the value of k .

(3 marks)

(c) In that year, the period during which the amount of water stored in the reservoir is 7 million cubic metres or more is terms *adequate*.

(i) How long does the *adequate* period last?

(ii) Find the least amount of water stored in the reservoir, within that year, after the *adequate* period has ended.

(iii) Find $\frac{d^2 A(t)}{dt^2}$.

(iv) Describe the behavior of $A(t)$ and $\frac{dA(t)}{dt}$, within that year, after the *adequate* period has ended for 6 months.

(10 marks)

(2007 ASL-M&S Q9)

20. A researcher monitors the process of using micro-organisms to decompose food waste to fertilizer. He records daily the pH value of the waste and models its pH value by

$$P(t)=a+\frac{1}{5}\left(t^2-8t-8\right)e^{-kt},$$

where $t(\ge 0)$ is the time measured in days, a and k are positive constants.

When the decomposition process starts (i.e. $t=0$) , the pH value of the waste is 5.9. Also, the researcher finds that $P(8)-P(4)=1.83$.

(a) Find the values of a and k correct to 1 decimal place.

(5 marks)

(b) Using the value of k obtained in (a) ,

(i) determine on which days the maximum pH value and the minimum pH value occurred respectively;

(ii) prove that $\frac{d^2P}{dt^2}>0$ for all $t\ge 23$.

(8 marks)

(c) Estimate the pH value of the waste after a very long time.

[Note: Candidates may use $\lim_{t\to\infty}\left(t^2e^{-kt}\right)=0$ without proof.]

(2 marks) (2003 ASL-M&S Q9)

21. A chemical factory continually discharges a constant amount of biochemical waste into a river. The microorganisms in the waste material flow down the river and remove dissolved oxygen from the water during biodegradation. The concentration of dissolved oxygen (CDO) of the river is given by

$$G(x)=2a-12e^{-kx}+(a+12)e^{-2kx} \ ,$$

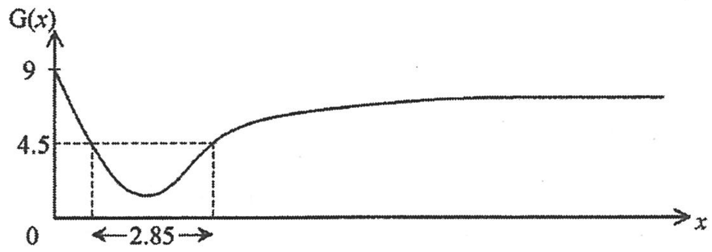
where $G(x)$ mg/L is the CDO of the river at position x km downstream from the location of discharge of the waste, and a , k are positive constants.

At the location of the discharge of waste (i.e. $x=0$) , the CDO of the river is 9 mg/L .

(a) (i) Show that $a=3$.

(ii) Find the minimum CDO of the river.

(b) The figure shows a sketch of the graph of $G(x)$ against x . It is found that downstream from the location of the discharge of waste, a stretch of 2.85 km of the river has a CDO of 4.5 mg/L or below.



(i) Find the value of k correct to 1 decimal place.

(ii) Find $G''(x)$.

Hence determine the position of the river, to the nearest 0.1 km, where the rate of change of the CDO is greatest.

(iii) A river is said to be *healthy* if the CDO of the river is 5.5 mg/L or above. Will the river in this case become *healthy*? If yes, find the position of the river, to the nearest 0.1 km, where it becomes *healthy* again.

(2001 ASL-M&S Q8)

22. A researcher studied the commercial fishing situation in a certain fishing zone. Denoting the total catch of coral fish in that zone in t years time from January 1, 1992 by $N(t)$ (in thousand tonnes), he obtained the following data:

t	2	4
$N(t)$	55	98

The researcher modelled $N(t)$ by $\ln N(t)=a-e^{1-kt}$ where a and k are constants.

(a) Show that $e^{-4k}-e^{-2k}+\frac{1}{e}\ln\frac{98}{55}=0$.

Hence find, to 2 decimal places, two sets of values of a and k .

(4 marks)

(b) The researcher later found out that $N(7)=170$. Determine which set of values of a and k obtained in (a) will make the model fit for the known data.

Hence estimate, to the nearest thousand tonnes, the total possible catch of coral fish in that zone since January 1, 1992.

(4 marks)

(c) The rate of change of the total catch of coral fish in that zone since January 1, 1992 by at time t is given by $\frac{dN(t)}{dt}$.

(i) Show that $\frac{dN(t)}{dt}=kN(t)e^{1-kt}$.

(ii) Using the values of a and k chosen in (b), determine in which year the maximum rate of change occurred.

Hence find, to the nearest integer, the volume of fish caught in that year.

(7 marks)

(Part c is out of Syllabus) (2000 ASL-M&S Q11)

23. A vehicle tunnel company wants to raise the tunnel fees. An expert predicts that after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day will drop drastically in the first week and on the t-th day after the first week, the number $N(t)$ (in thousands) of vehicles passing through the tunnel can be modelled by

$$N(t)=\frac{40}{1+be^{-rt}} \qquad (t\geq 0)$$

where b and r are positive constants.

(a) Suppose that by the end of the first week after the increase in the tunnel fees, the number of vehicles passing through the tunnel each day drops to 16 thousand and by the end of the second week, the number increases to 17.4 thousand, find b and r correct to 2 decimal places.

(5 marks)

(b) Show that $N(t)$ is increasing.

(3 marks)

(c) As time passes, $N(t)$ will approach the average number N_a of vehicles passing through the tunnel each day before the increase in the tunnel fees. Find N_a .

(2 marks)

(d) The expert suggests that the company should start to advertise on the day when the rate of increase of the number of cars passing through the tunnel per day is the greatest. Using the values of b and r obtained in (a),

(i) find $N''(t)$, and

(ii) hence determine when the company should start to advertise.

(5 marks)

(1997 ASL-M&S Q8)

24. A merchant sells compact discs (CDs). A market researcher suggests that if each CD is sold for \$$x$, the number $N(x)$ of CDs sold per week can be modeled by

$$N(x)=ae^{-bx}$$

where a and b are constants.

The merchant wants to determine the values of a and b based on the following results obtained from a survey:

x	20	30	40	50
$N(x)$	450	301	202	136

(a) (i) Express $\ln N(x)$ as a linear function of x .

(ii) Use a graph paper to estimate graphically the values of a and b correct to 2 decimal places.

(7 marks)

(b) Suppose the merchant wishes to sell 400 CDs in the next week. Use the values of a and b estimated in (a) to determine the price of each CD. Give your answer correct to 1 decimal place.

(2 marks)

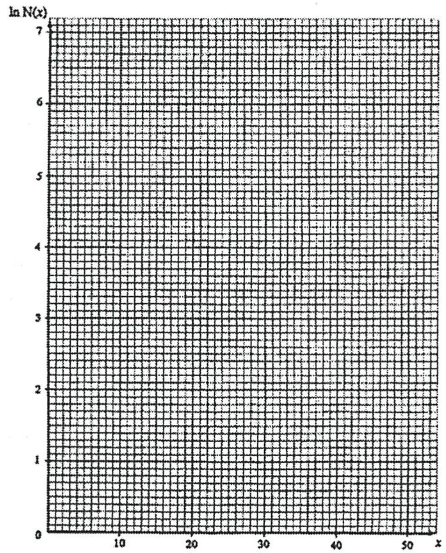
(c) It is known that the merchant obtains CDs at a cost of \$10 each. Let $G(x)$ dollars denote the weekly profit. Using the values of a and b estimated in (a),

(i) express $G(x)$ in terms of x.

(ii) find $G'(x)$ and hence determine the selling price for each CD in order to maximize the profit.

(6 marks)

(1995 ASL-M&S Q8)



2021 DSE Q8

Let $f(x)$ be a function such that $f'(x)=\frac{2^{kx}}{1+2^{kx}}$, where k is a constant. The straight line $8x-9y+10=0$ touches the curve $y=f(x)$ at the point A. It is given that the x-coordinate of A is 1. Find

(a) k,

(b) $f(x)$.

(7 marks)

2021 DSE Q12

A tank is used for collecting rain water. During a certain shower, rain water flows into the tank for 7 minutes. Let $V\text{ m}^3$ be the volume of rain water in the tank. It is given that

$$\frac{dV}{dt}=\sqrt{t+1}\sqrt{3-\sqrt{t+1}} \quad (0\le t\le 7),$$

where t is the number of minutes elapsed since rain water starts flowing into the tank. The tank is empty at $t=0$ and the rate of change of the volume of rain water in the tank attains its maximum value when $t=T$.

(a) Find T. (4 marks)

(b) Find the exact value of V when $t=T$. (5 marks)

(c) The tank is in the shape of an inverted right circular cone of height 1 m and base radius 6 m. The tank is held vertically. Let h m be the depth of rain water in the tank. Find

(i) the constant Q such that $\frac{dV}{dt}=Qh^2\frac{dh}{dt}$,

(ii) $\left.\frac{dh}{dt}\right|_{t=T}$.

(5 marks)

4. Application of Differentiation

Section A

1. (2019 DSE-MATH-M1 Q5a,b)

(a) For all $x>-3$, $f'(x)=\frac{(x+3)(-1)-(6-x)(1)}{(x+3)^2}=\frac{-9}{(x+3)^2}<0$. Thus, $f(x)$ is decreasing.	1	
Note that $f(x)=\frac{9}{x+3}-1$ for all $x>-3$. Thus, $f(x)$ is decreasing.	1	
(b) $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\frac{6}{x}-1}{1+\frac{3}{x}}=-1$	1A	
$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left(\frac{9}{x+3}-1\right)=-1$	1A	

(a)	Good. Some candidates were unable to show that $f'(x)<0$ to complete the proof.
(b)	Good. Some candidates were unable to consider $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\frac{6}{x}-1}{1+\frac{3}{x}}$ to obtain the required limit.

2. (2018 DSE-MATH-M1 Q7)

3. (2018 DSE-MATH-M1 Q5a)

Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	$+$

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

$f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

$f''(x)$

$=\frac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$

$f''(4)$

$=12$

>0

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

Very good. Over 85% of the candidates were able to find the value of α .

4. (2017 DSE-MATH-M1 Q6)

(a) $f(6)=-33$

$4(6^3)+m(6^2)+n(6)+615=-33$ 1M

$6m+n=-252$

$f'(x)=12x^2+2mx+n$

$f'(6)=0$

$12(6^2)+2m(6)+n=0$ 1M

$12m+n=-432$

Solving, we have $m=-30$ and $n=-72$. 1A for both correct

(b) $f'(x)=12x^2-60x-72$

$f'(x)=0$ when $x=-1$ or $x=6$. 1M

x	$(-\infty,-1)$	-1	$(-1,6)$	6	$(6,\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	653	↘	-33	↗

1M for testing

Thus, the minimum value is -33 and the maximum value is 653 . 1A for both correct

----------(6)

(a)	Very good. Most candidates were able to find the values of m and n .
(b)	Very good. Many candidates were able to find the maximum value and the minimum value.

5. (2014 DSE-MATH-M1 Q1)

Let V and r be the volume and radius of the spherical balloon respectively.

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ 1M

$\therefore\ -100=4\pi\cdot 10^2\frac{dr}{dt}$ 1M

$\frac{dr}{dt}=\frac{-1}{4\pi}$ 1A OR -0.0796

Hence the rate of change of the radius is $\frac{-1}{4\pi}$ cm s^{-1} .

(3)

Satisfactory.
Many candidates set $\frac{dV}{dt}$ equal to 100 rather than -100 .

6. (2014 DSE-MATH-M1 Q2)

(a) $f(x)=\frac{x^x}{(2x+13)^6}$

$\ln f(x)=x\ln x-6\ln(2x+13)$ 1A

$\frac{f'(x)}{f(x)}=\ln x+x\cdot\frac{1}{x}-6\cdot\frac{2}{2x+13}$ 1M+1A

$f'(x)=\left(\ln x+1-\frac{12}{2x+13}\right)f(x)$

$=\left(\ln x+\frac{2x+1}{2x+13}\right)\frac{x^x}{(2x+13)^6}$ 1A Accept $\left(\ln x+\frac{2x+1}{2x+13}\right)f(x)$

(b) For $x>1$, we have $\ln x>0$, $\frac{2x+1}{2x+13}>0$ and $\frac{x^x}{(2x+13)^6}>0$. 1M

$\therefore\ f'(x)>0$ OR $f'(x)\geq 0$

Hence $f(x)$ is an increasing function. 1

(6)

(a)	Good.
(b)	Very poor. Most candidates failed to show clearly why $f'(x)>0$.

7. (2013 DSE-MATH-M1 Q2)

$p = 8 - \dfrac{2.1}{\sqrt{t+4}}$

$\dfrac{dp}{dt} = \dfrac{2.1}{2(t+4)^{\frac{3}{2}}}$ 1A

$C = 2^p$

$\dfrac{dC}{dp} = 2^p \ln 2$ 1A

$\dfrac{dC}{dt} = \dfrac{dC}{dp} \cdot \dfrac{dp}{dt}$

$= 2^p \ln 2 \cdot \dfrac{2.1}{2(t+4)^{\frac{3}{2}}}$ 1M

When $t = 5$, $p = 7.3$ and hence

$\dfrac{dC}{dt} = 2^{7.3} \ln 2 \cdot \dfrac{2.1}{2(5+4)^{\frac{3}{2}}}$

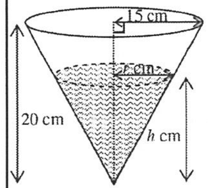
≈ 4.2479 1A

i.e. the rate of change of the concentration of carbon dioxide ≈ 4.2479 units/year.

(4)

Satisfactory. Some candidates found $\dfrac{dp}{dt}$ or $\dfrac{dC}{dp}$ wrongly (for example, writing $\dfrac{dC}{dp} = 2^p \ln p$ or $p2^{p-1}$), while some others obtained $\dfrac{dC}{dt}$ correctly but did not substitute 5 for t.

8. (PP DSE-MATH-M1 Q3)



(a) By similar triangles, we have $\dfrac{h}{r} = \dfrac{20}{15}$. 1M

$h = \dfrac{4r}{3}$

$\therefore\ V = \dfrac{1}{3}\pi r^2\left(\dfrac{4r}{3}\right)$

$= \dfrac{4}{9}\pi r^3$ 1A

$A = \pi r\sqrt{r^2 + \left(\dfrac{4r}{3}\right)^2}$

$= \dfrac{5}{3}\pi r^2$ 1A

(b) (i) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$ 1M (Either one)

$= \dfrac{4}{3}\pi r^2 \dfrac{dr}{dt}$

$-2\pi = \dfrac{4}{3}\pi(3)^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{-1}{6}$ 1A

Hence the rate of change of the radius of the water surface is $\dfrac{-1}{6}$ cm/s.

(ii) $\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dt}$

$= \dfrac{10}{3}\pi r \dfrac{dr}{dt}$

$= \dfrac{10}{3}\pi(3)\left(\dfrac{-1}{6}\right)$

$= \dfrac{-5}{3}\pi$ 1A

Hence the rate of change of the area of the wet surface is $\dfrac{-5}{3}\pi$ cm^2/s.

(6)

(a)	甚佳。部分學生未能利用相似三角形的特性。
(b)	平平。很多學生誤以爲 $\dfrac{dV}{dt} = +2\pi$ 。

9. (SAMPLE DSE-MATH-M1 Q2)

$\because \quad r = 3 - \dfrac{2}{2+t}$

$\therefore \quad \dfrac{dr}{dt} = \dfrac{2}{(2+t)^2}$ 1A

$V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ 1A

When $r = 2.5$, $2.5 = 3 - \dfrac{2}{2+t}$ 1M

i.e. $t = 2$

$\therefore \quad \left.\dfrac{dV}{dt}\right|_{r=2.5} = 4\pi(2.5)^2 \cdot \dfrac{2}{(2+2)^2}$

$= \dfrac{25}{8}\pi$ 1A

$\therefore$ the rate of change of volume of the balloon is $\dfrac{25}{8}\pi$ m^3/hr .

(4)

10. (2004 ASL-M&S Q3)

(a) Since $2\pi rh + 2\pi r^2 = 162\pi$, we have

$rh + r^2 = 81$. 1A

Therefore, either one

The required capacity

$= \pi r^2 h + \dfrac{2}{3}\pi r^3$

$= \pi r(81 - r^2) + \dfrac{2}{3}\pi r^3$

$= (81\pi r - \dfrac{1}{3}\pi r^3)$ cm^3 1

(b) Let $f(r) = 81\pi r - \dfrac{1}{3}\pi r^3$ for all $r \geq 0$. Then, we have

$\dfrac{df(r)}{dr} = 81\pi - \pi r^2$. 1A

$\dfrac{df(r)}{dr} = 0$ when $r = 9$ 1M

$\dfrac{df(r)}{dr} \begin{cases} > 0 & \text{if } 0 \leq r < 9 \\ = 0 & \text{if } r = 9 \\ < 0 & \text{if } r > 9 \end{cases}$ 1M for testing

So, $f(r)$ attains its greatest value when $r = 9$. 1A

Note that $f(9) = 486\pi$

≈ 1526.81403

≤ 1600

Thus, by (a), the capacity of the container cannot be greater than 1 600 cm^3 . 1A

Let $f(r) = 81\pi r - \dfrac{1}{3}\pi r^3$ for all $r \geq 0$. Then, we have

$\dfrac{df(r)}{dr} = 81\pi - \pi r^2$. 1A

$\dfrac{df(r)}{dr} = 0$ when $r = 9$ 1M

$\dfrac{d^2 f(r)}{dr^2} = -2\pi r < 0$ for any $r > 0$ 1M for testing

Note that there is only one local maximum

So, $f(r)$ attains its greatest value when $r = 9$. 1A

Note that $f(9) = 486\pi$

≈ 1526.81403

≤ 1600

Thus, by (a), the capacity of the container cannot be greater than 1 600 cm^3 . 1A

----------(7)

Good. Some candidates failed to prove that the extreme value is the greatest value.

11. (1996 ASL-M&S Q5)

(a) $\because \frac{dN}{d\theta} = -[\ln(\theta+49)]^2 - \frac{2(\theta+440)\ln(\theta+49)}{\theta+49}$ — 1M+1A+1A (1M for product rule; 1A for diff. of log.)

$= -\ln(\theta+49)\left[\ln(\theta+49)+\frac{2(\theta+440)}{\theta+49}\right]$

$\therefore \frac{dN}{dt} = \frac{dN}{d\theta}\cdot\frac{d\theta}{dt}$

$= -\ln(\theta+49)\left[\ln(\theta+49)+\frac{2(\theta+440)}{\theta+49}\right]\frac{d\theta}{dt}$ — 1A

(b) $\theta=-40$, $\frac{d\theta}{dt}=-0.5$

$\frac{dN}{dt} = -\ln(-40+49)\left[\ln(-40+49)+\frac{2(-40+440)}{-40+49}\right](-0.5)$ — 1M

≈ 100 — 1A

$\therefore$ The rate of increase of the number of tourists is 100 per hour.

(6)

12. (2002 ASL-M&S Q2)

(a) At $t=2$, $r(2)=5.5035$(m)

$\frac{dV}{dr}=4\pi r^2$

$\frac{dr}{dt}=\frac{18\times 2e^{-t}}{(3+2e^{-t})^2}=\frac{36e^{-t}}{(3+2e^{-t})^2}$ — 1A

At $t=2$, $\frac{dV}{dr}=380.6109$

$\frac{dr}{dt}=0.45545$

$\therefore \frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}$

At $t=2$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ — 1M

$=380.6109\times 0.45545$

$=173.35$ (m³/h) — 1A (Accept : 173.31–173.39) a–1 for more than 2 d.p.

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ — 1M

$\frac{dr}{dt}=\frac{36e^{-t}}{(3+2e^{-t})^2}$ — 1A

$\frac{dV}{dt}=4\pi r^2\cdot\frac{36e^{-t}}{(3+2e^{-t})^2}$

$=\frac{144\pi r^2e^{-t}}{(3+2e^{-t})^2}$ (Accept : $\frac{dV}{dt}=\frac{46656\pi e^{-t}}{(3+2e^{-t})^4}$)

At $t=2$,

$r\approx 5.50346$

$\therefore \frac{dV}{dt}\approx 173.35$ (m³/h) — 1A (Accept : 173.31–173.39) a–1 for more than 2 d.p.

(b) $\lim_{t\to\infty} r(t)=\lim_{t\to\infty}\frac{18}{3+2e^{-t}}=6$ (m) — 1M

$\therefore$ the volume of the balloon will be

$V=\frac{4}{3}\pi(6)^3+5\pi$

$=293\pi$

$=920.49$ (m³) — 1A a–1 for more than 2 d.p.

(5)

13. (2013 DSE-MATH-M1 Q11(a,b))

(a) $R'(t)=0$

$P'(t)-C'(t)=0$

$4(4-e^{\frac{-t}{5}})-9(2-e^{\frac{-t}{10}})=0$ — 1A

$-4\left(e^{\frac{-t}{10}}\right)^2+9e^{\frac{-t}{10}}-2=0$ — 1M — For $e^{\frac{-t}{5}}=\left(e^{\frac{-t}{10}}\right)^2$

$e^{\frac{-t}{10}}=0.25$ or 2

$t=20\ln 2$ or $-10\ln 2$ (rejected as $t\geq 0$) — 1A — OR $t\approx 13.8629$

(3)

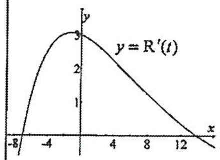
(b) $R'(t)=-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2$ — 1A

$R''(t)=\frac{4}{5}e^{\frac{-t}{5}}-\frac{9}{10}e^{\frac{-t}{10}}$

$=\frac{1}{10}e^{\frac{-t}{10}}\left(8e^{\frac{-t}{10}}-9\right)$ — 1M

<0 for $t\geq 0$ (since $e^{\frac{-t}{10}}\leq 1$ for $t\geq 0$)

Therefore $R'(t)$ decreases with t. — 1



(3)

(a)	Fair. Some candidates confused $R(t)$ with $R'(t)$, or found $R(t)=P(t)-C(t)$ by integration first and then obtained the expression for $R'(t)=P'(t)-C'(t)$ by differentiation. Many candidates failed to make use of knowledge about quadratic equations to solve for t. Some got wrong answers such as '$e^{\frac{-t}{5}}=0.25$ or 2' or did not reject $t=-10\ln 2$.
(b)	Very poor. Many candidates failed to find $R''(t)$ correctly. Among those who were able to find $R''(t)$, only few provided sufficient reasons to conclude that '$R'(t)$ decreases with t'.

14. (2012 ASL-M&S Q9)

(a) $r(t)=20-40e^{-at}+be^{-2at}$

$\because$ $r(0)=20-40e^{0}+be^{0}=30$

$\therefore$ $b=50$ — 1A

(1)

(b) $r'(t)<0$ for 9 days

$40ae^{-at}-100ae^{-2at}<0$ for $t<9$ — 1M

$20ae^{-2at}(2e^{at}-5)<0$

$e^{at}<2.5$

$t<\frac{\ln 2.5}{a}$ — 1A

$\therefore$ $\frac{\ln 2.5}{a}=9$

i.e. $a\approx 0.1$ (correct to 1 decimal place) — 1A

(3)

(c) The rate of change of the rate of selling of handbags is $r'(t)=4e^{-0.1t}-10e^{-0.2t}$.

$\frac{d}{dt}r'(t)=-0.4e^{-0.1t}+2e^{-0.2t}$

$\frac{d}{dt}r'(t)=0$ when $0.4e^{-0.1t}=2e^{-0.2t}$ — 1M

$e^{0.1t}=5$

$t=10\ln 5$ — 1A — OR 16.0944

$\frac{d^2}{dt^2}r'(t)=0.04e^{-0.1t}-0.4e^{-0.2t}$

When $t=10\ln 5$, $\frac{d^2}{dt^2}r'(t)=-0.008<0$ — 1M — OR by using sign test

Hence $r'(t)$ is maximum when $t=10\ln 5$

$r(10\ln 5)=20-40e^{-0.1(10\ln 5)}+50e^{-0.2(10\ln 5)}=14$

The rate of selling $=14$ thousand per day — 1A — OR 14000 per day

(4)

(a)		Very good.
(b)		Satisfactory. Many candidates used an equation rather than an inequality to solve for the value of a.
(c)		Fair. Some candidates overlooked that the given condition was for the rate of change of the rate of selling. When consider the maximum rate of change, candidates should set the second derivative $\frac{d^2r}{dt^2}$ zero.

15. (1998 ASL-M&S Q8)

Solution	Marks	Remarks
(i) If $\dfrac{5000e^{15\lambda}}{15}=\dfrac{5000e^{95\lambda}}{95}$	1A	
then $e^{80\lambda}=\dfrac{19}{3}$ $\lambda=\dfrac{1}{80}\ln(\dfrac{19}{3})$ ≈ 0.0231	1M+1A	
(ii) $N=\dfrac{5000e^{\lambda t}}{t}\approx\dfrac{5000e^{0.0231t}}{t}$ $\dfrac{dN}{dt}=5000\left(\dfrac{\lambda te^{\lambda t}-e^{\lambda t}}{t^2}\right)$	1M+1A	
$=\dfrac{5000e^{\lambda t}(\lambda t-1)}{t^2}$ $\begin{cases}<0 & \text{when } 0<t<\dfrac{1}{\lambda}\\ =0 & \text{when } t=\dfrac{1}{\lambda} \quad (\approx 43.3410)\\ >0 & \text{when } \dfrac{1}{\lambda}<t<120\end{cases}$	1M+1A	
$\therefore$ N attains its minimum when $t\approx 43.3410$ (The number of fish decreased to the minimum in about 43 days after the spread of the disease.)	1A	r.t. 43

Section B

16. (2017 DSE-MATH-M1 Q12)

Solution	Marks	Remarks
(a) (i) $x-4=\dfrac{3k}{2^{\lambda t}-k}$ $x-1=\dfrac{3(2^{\lambda t})}{2^{\lambda t}-k}$ $(x-4)(x-1)=\dfrac{9k2^{\lambda t}}{(2^{\lambda t}-k)^2}$	1A	
(ii) $\dfrac{9k2^{\lambda t}}{(2^{\lambda t}-k)^2}>0$ (as $k>0$) $(x-4)(x-1)>0$ (by (a)(i))	1M	
$x>4$ or $x<1$ Thus, the claim is correct.	1A ---------(3)	f.t.
(b) (i) $\dfrac{dx}{dt}=\dfrac{-3(\ln 2)k\lambda 2^{\lambda t}}{(2^{\lambda t}-k)^2}$	1A	
$\dfrac{-\ln 2}{24}(x-4)(x-1)=\dfrac{-3(\ln 2)k2^{\lambda t}}{8(2^{\lambda t}-k)^2}$ $\lambda=\dfrac{1}{8}$	1	
(ii) (1) When $t=0$, $x=0.8$. $-3.2=\dfrac{3k}{1-k}$ $k=16$	1A	
When $x=0$, we have $4+\dfrac{48}{2^{\frac{t}{8}}-16}=0$.	1M	either one
So, we have $2^{\frac{t}{8}}=4$.	1M	either one
Solving, we have $t=16$. Thus, the crocodiles in the lake will eventually become extinct in 16 years.	1A	
(2) When $t=0$, $x=7$. $3=\dfrac{3k}{1-k}$ $k=0.5$	1A	
When $x=0$, we have $4+\dfrac{1.5}{2^{\frac{t}{8}}-0.5}=0$.		
So, we have $2^{\frac{t}{8}}=0.125$.		
It is impossible as $2^{\frac{t}{8}}>1$ for $t>0$. Thus, the crocodiles in the lake will never become extinct.	1A	f.t.
Note that $\lim_{t\to\infty}x=\lim_{t\to\infty}\left(4+\dfrac{1.5}{2^{\frac{t}{8}}-0.5}\right)=4$. After a very long time, the estimated number of crocodiles in the lake is 4 000 .	1A ---------(9)	

(a) (i)	Poor. Only a few candidates were able to express $(x-4)(x-1)$ in terms of λ, k and t.
(ii)	Poor. Only a few candidates were able to use the result in (a)(i) to finish the argument.
(b) (i)	Fair. Many candidates were unable to find $\frac{dx}{dt}$.
(ii) (1)	Fair. Only some candidates were able to find the value of k.
(2)	Fair. Many candidates estimated the number of crocodiles in the lake after a very long time without first determining that the crocodiles in the lake will not become extinct eventually.

17. (2016 DSE-MATH-M1 Q12)

(a) $N=\frac{27}{2+\alpha te^{\beta t}}$		
$\frac{27-2N}{Nt}=\alpha e^{\beta t}$	1M	
$\ln\left(\frac{27-2N}{Nt}\right)=\ln\alpha+\beta t$	1A	
	----------(2)	
(b) (i) $\beta=-0.1$	1A	
$0=-0.1(10\ln 0.03)+\ln\alpha$ $\ln\alpha=\ln 0.03$ $\alpha=0.03$	1A	
(ii) $\frac{dN}{dt}$		
$=-27(2+0.03te^{-0.1t})^{-2}(0.03)(e^{-0.1t}-0.1te^{-0.1t})$	1M	for $\frac{d}{dt}e^{\beta t}=\beta e^{\beta t}$
$=\frac{0.081(t-10)e^{-0.1t}}{(2+0.03te^{-0.1t})^2}$	1A	
For $\frac{dN}{dt}=0$, we have $t=10$.		
(see table below)	1M	
So, N attains its least value when $t=10$.	1A	
The least value of $N=\frac{27}{2+0.3e^{-1}}\approx 12.79400243>12$. Thus, the number of chickens will not be less than 12 thousand on a certain day after the start of the spread of the bird flu.	1A	f.t.
(iii) $\frac{d^2N}{dt^2}$		
$=\frac{d}{dt}\left(\frac{dN}{dt}\right)$		
$=\frac{0.081(2+0.03te^{-0.1t})^2(e^{-0.1t}-0.1(t-10)e^{-0.1t})}{(2+0.03te^{-0.1t})^4}-\frac{0.081(t-10)e^{-0.1t}(2)(2+0.03te^{-0.1t})(0.03)(e^{-0.1t}-0.1te^{-0.1t})}{(2+0.03te^{-0.1t})^4}$	1M	for quotient rule
$=0.0081\left(\frac{(2+0.03te^{-0.1t})(20-t)e^{-0.1t}+0.06(t-10)^2e^{-0.2t}}{(2+0.03te^{-0.1t})^3}\right)$	1A	
Hence, we have $\frac{d^2N}{dt^2}>0$ for $0\le t\le 20$.		
So, $\frac{dN}{dt}$ increases for $0\le t\le 20$.	1A	f.t.
Thus, the rate of change of the number of chickens increases.	---------(10)	

t	$0\le t<10$	$t=10$	$t>10$
$\frac{dN}{dt}$	−	0	+

(a)	Very good. More than 70% of the candidates were able to express $\ln\left(\frac{27-2N}{Nt}\right)$ as a linear function of t.
(b) (i)	Good. Many candidates were able to use the slope of the linear function to find β , while a few candidates wrongly took the given horizontal intercept as the vertical intercept to find α .
(ii)	Fair. Many candidates wrongly gave the limiting value of N instead of the least value of N as the answer. Some candidates were unable to evaluate $\frac{d}{dt}te^{-0.1t}$ when finding $\frac{dN}{dt}$.
(iii)	Poor. Most candidates were unable to find the derivative of $\frac{dN}{dt}$ to describe how the rate of change of the number of chickens varies. Only a very small number of candidates were able to determine the sign of $\frac{d^2N}{dt^2}$ for $0\le t\le 20$.

18. (2010 ASL-M&S Q9)

(a) (i) $p(t)=\dfrac{a}{b+e^{-t}}+c$

$p'(t)=\dfrac{ae^{-t}}{(b+e^{-t})^2}$ 1A

$p''(t)=\dfrac{(b+e^{-t})^2(-ae^{-t})-(ae^{-t})2(b+e^{-t})(-e^{-t})}{(b+e^{-t})^4}$

$=\dfrac{ae^{-t}(e^{-t}-b)}{(b+e^{-t})^3}$ 1A

Hence $p''(t)=0$ when $e^{-t}-b=0$.

i.e. $t=-\ln b$

t	$t<-\ln b$	$t=-\ln b$	$t>-\ln b$
$p''(t)$	+	0	−

Hence the growth rate attains the maximum value when $t=-\ln b$ 1A Follow through

(ii) *primordial population* $=\lim\limits_{t\to-\infty}\left(\dfrac{a}{b+e^{-t}}+c\right)=c$ 1A

(iii) *ultimate population* $=\lim\limits_{t\to\infty}\left(\dfrac{a}{b+e^{-t}}+c\right)=\dfrac{a}{b}+c$ 1A

(5)

(b) $\ln[p(t)-c]=-\ln(b+e^{-t})+\ln a$

$\therefore$ $\ln a=\ln 8000$

$a=8000$ 1A

$\therefore$ $p'(0)=\dfrac{8000}{(b+1)^2}=2000$

$b=1$ or -3 (rejected) 1A

$\therefore$ $p(0)=\dfrac{8000}{1+1}+c=6000$

$c=2000$ 1A

(3)

(c) The population at the time of maximum growth rate is

$p(-\ln b)=\dfrac{a}{2b}+c$ 1A

The mean of the *primordial population* and *ultimate population* is

$\dfrac{1}{2}\left[c+\left(\dfrac{a}{b}+c\right)\right]=\dfrac{a}{2b}+c$

Hence the scientist's claim is agreed. 1

(2)

(d) $p(t)=\dfrac{a}{b+e^{-t}}+c$

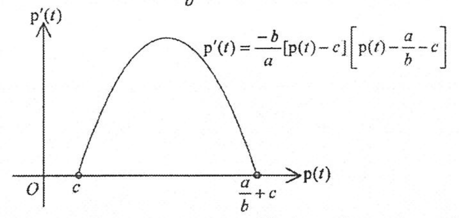
$e^{-t}=\dfrac{a}{p(t)-c}-b$ 1A

$\therefore$ $p'(t)=\dfrac{a\left[\dfrac{a}{p(t)-c}-b\right]}{\left[b+\left(\dfrac{a}{p(t)-c}-b\right)\right]^2}$ 1M

$=\dfrac{a[p(t)-c]\{a-b[p(t)-c]\}}{a^2}$

$=\dfrac{-b}{a}[p(t)-c]\left[p(t)-\dfrac{a}{b}-c\right]$

Hence $\alpha=c$ and $\beta=\dfrac{a}{b}+c$. 1A



1A

From the graph, we can see that $p'(t)$ is maximum when $p(t)$ is the mean of c and $\dfrac{a}{b}+c$, i.e. the mean of the *primordial population* and *ultimate population* 1 Follow through

(5)

(a) (i)		Fair. Many candidates confused the maximum growth rate and the maximum population and hence could not determine the time required.
(ii) (iii)		Fair. Some candidates mistook $p(0)$ to be the primordial population.
(b)		Fair.
(c)		Poor. Most candidates did not understand the question.
(d)		Very poor. Most candidates could not go beyond expressing e^{-t} in terms of a, b, c and $p(t)$.

19. (2007 ASL-M&S Q9)

(a) $A(t)=(-t^2+5t+a)e^{kt}+7$

Since $A(0)=3$, we have $a+7=3$.

Thus, we have $a=-4$. — 1A

The required amount of water stored

$=(-1^2+5-4)e^k+7$

$=7$ million cubic metres — 1A

---------(2)

(b) $A(t)=(-t^2+5t-4)e^{kt}+7$

$\frac{dA(t)}{dt}$

$=(-2t+5)e^{kt}+(-t^2+5t-4)(ke^{kt})$ — 1M for product rule

$=\left(-kt^2+(5k-2)t+5-4k\right)e^{kt}$

Note that when $t=2$, $\frac{dA(t)}{dt}=0$.

So, we have $2k+1=0$. — 1M

Thus, we have $k=\frac{-1}{2}$. — 1A

---------(3)

(c) (i) When $A(t)\ge 7$, we have

$-t^2+5t-4\ge 0$ — 1M accept setting quadratic equation

$t^2-5t+4\le 0$

$1\le t\le 4$ — 1A (accept $t=1 \to t=4$)

Thus, the *adequate* period lasts for 3 months.

(ii) Note that $A(t)=(-t^2+5t-4)e^{\frac{-t}{2}}+7$.

So, $\frac{dA(t)}{dt}=\left(\frac{t^2}{2}-\frac{9t}{2}+7\right)e^{\frac{-t}{2}}$

and $\frac{dA(t)}{dt}=0$ when $t=2$ (rejected since $t>4$) or $t=7$. — 1A

$$\frac{dA(t)}{dt}\begin{cases}<0 & \text{if } 4<t<7\\ =0 & \text{if } t=7\\ >0 & \text{if } 7<t\le 12\end{cases}$$ — 1M for testing + 1A

So, $A(t)$ attains its least value when $t=7$.

The least amount of water stored

$=A(7)$

≈ 6.456447098

≈ 6.4564 million cubic metres — 1A $a-1$ for r.t. 6.456 million cubic metres

Note that $A(t)=(-t^2+5t-4)e^{\frac{-t}{2}}+7$.

So, $\frac{dA(t)}{dt}=\left(\frac{t^2}{2}-\frac{9t}{2}+7\right)e^{\frac{-t}{2}}$

and $\frac{dA(t)}{dt}=0$ when $t=2$ (rejected since $t>4$) or $t=7$. — 1A

$\frac{d^2A(t)}{dt^2}$

$=\left(\frac{-t^2}{4}+\frac{9t}{4}-\frac{7}{2}\right)e^{\frac{-t}{2}}+\left(t-\frac{9}{2}\right)e^{\frac{-t}{2}}$

$=\left(\frac{-t^2}{4}+\frac{13t}{4}-8\right)e^{\frac{-t}{2}}$ — 1A

Therefore, we have $\left.\frac{d^2A(t)}{dt^2}\right|_{t=7}=\frac{5}{2}e^{\frac{-7}{2}}>0$. — 1M for testing + 1A

Note that there is only one local minimum after the *adequate* period.

So, $A(t)$ attains its least value when $t=7$.

The least amount of water stored

$=A(7)$

≈ 6.456447098

≈ 6.4564 million cubic metres — 1A $a-1$ for r.t. 6.456 million cubic metres

(either one)

(iii) $\frac{d^2A(t)}{dt^2}$

$=\left(\frac{-t^2}{4}+\frac{9t}{4}-\frac{7}{2}\right)e^{\frac{-t}{2}}+\left(t-\frac{9}{2}\right)e^{\frac{-t}{2}}$

$=\left(\frac{-t^2}{4}+\frac{13t}{4}-8\right)e^{\frac{-t}{2}}$

(iv) Since $\frac{dA(t)}{dt}=\frac{1}{2}\left((t-\frac{9}{2})^2-\frac{25}{4}\right)e^{\frac{-t}{2}}>0$ for $10<t<12$, — 1M for considering the sign

A(t) increases, within that year, after the *adequate* period has ended for 6 months. — 1A f.t.

(either one)

Since $\frac{d^2A(t)}{dt^2}=\frac{-1}{4}\left((t-\frac{13}{2})^2-\frac{41}{4}\right)e^{\frac{-t}{2}}<0$ for $10<t<12$,

$\frac{dA(t)}{dt}$ decreases, within that year, after the *adequate* period has ended for 6 months. — 1A f.t.

---------(10)

(a)		Very good.
(b)		Good.
(c) (i)		Good.
(ii)		Fair. Many candidates neglected the range of values of the variable and hence could not complete the solution.
(iii)		Good.
(iv)		Poor. Many candidates could not make use of the value of A''(t) to explain the behaviour of A'(t) and many did not specify the time period.

20. (2003 ASL-M&S Q9)

(a) $\because$ $P(0)=5.9$

$\therefore$ $a+\frac{1}{5}(0-0-8)=5.9$

So, $a=7.5$ — 1A

$P(t)=7.5+\frac{1}{5}(t^2-8t-8)e^{-kt}$

$\because$ $P(8)-P(4)=1.83$

$\therefore$ $-1.6e^{-8k}+4.8e^{-4k}=1.83$ — 1M+1A

$160(e^{-4k})^2-480e^{-4k}+183=0$ — 1M can be absorbed

$e^{-4k}\approx 2.551784198$ or $e^{-4k}\approx 0.448215801$

$k\approx -0.2341982$ or $k\approx 0.200620116$

$\because$ $k>0$

$\therefore$ $k\approx 0.2$ (correct to 1 decimal place) — 1A

---------(5)

(b) $P(t)=\frac{15}{2}+\frac{1}{5}(t^2-8t-8)e^{-0.2t}$

(i) $\frac{dP(t)}{dt}=\frac{-1}{25}\left[(t^2-8t-8)-5(2t-8)\right]e^{-0.2t}$ — 1M for Product Rule or Chain Rule

$=\frac{-1}{25}(t^2-18t+32)e^{-0.2t}$ — 1A independent of the obtained value of a

$=\frac{-1}{25}(t-2)(t-16)e^{-0.2t}$

For $\frac{dP(t)}{dt}=0$, we have $t=2$ or $t=16$.

$$\frac{dP(t)}{dt}\begin{cases}<0 & \text{if } 0\le t<2\\ =0 & \text{if } t=2\\ >0 & \text{if } 2<t<16\end{cases}$$

So, the minimum pH value occurred at $t=2$. — 1M+1A

$$\frac{dP(t)}{dt}\begin{cases}>0 & \text{if } 2<t<16\\ =0 & \text{if } t=16\\ <0 & \text{if } t>16\end{cases}$$

So, the maximum pH value occurred at $t=16$. — 1M+1A accept max at $t=0$ and at $t=16$

$\frac{dP(t)}{dt}=\frac{-1}{25}\left[(t^2-8t-8)-5(2t-8)\right]e^{-0.2t}$ — 1M for Product Rule or Chain Rule

$=\frac{-1}{25}(t^2-18t+32)e^{-0.2t}$ — 1A independent of the obtained value of a

$=\frac{-1}{25}(t-2)(t-16)e^{-0.2t}$

For $\frac{dP(t)}{dt}=0$, we have $t=2$ or $t=16$.

$\frac{d^2P(t)}{dt^2}=\frac{1}{125}[t^2-18t+32-5(2t-18)]e^{-0.2t}$

$=\frac{1}{125}(t^2-28t+122)e^{-0.2t}$

$\left.\frac{d^2P(t)}{dt^2}\right|_{t=2}\approx 0.375379225>0$

So, the minimum pH value occurred at $t=2$. — 1M+1A

$\left.\frac{d^2P(t)}{dt^2}\right|_{t=16}\approx -0.022826834<0$

So, the maximum pH value occurred at $t=16$. — 1M+1A accept max at $t=0$ and at $t=16$

(ii) $\frac{d^2P}{dt^2}=\frac{1}{125}[t^2-18t+32-5(2t-18)]e^{-0.2t}$

$=\frac{1}{125}(t^2-28t+122)e^{-0.2t}$ — 1A

$\because$ $\frac{d^2P}{dt^2}=\frac{1}{125}\left(t-(14-\sqrt{74})\right)\left(t-(14+\sqrt{74})\right)e^{-0.2t}$

$5<14-\sqrt{74}<6$ and $22<14+\sqrt{74}<23$

$\therefore$ $\frac{d^2P}{dt^2}>0$ for all $t\ge 23$. — 1

---------(8)

(c) The required pH value

$=\lim_{t\to\infty}\left(7.5+\frac{1}{5}(t^2-8t-8)e^{-0.2t}\right)$

$=7.5+\frac{1}{5}\lim_{t\to\infty}\left(t^2e^{-0.2t}\right)-\frac{8}{5}\lim_{t\to\infty}\left(te^{-0.2t}\right)-\frac{8}{5}\lim_{t\to\infty}e^{-0.2t}$

$=7.5+\frac{1}{5}(0)-\frac{8}{5}(0)-\frac{8}{5}(0)$ $\left(\because \lim_{t\to\infty}(te^{-0.2t})=\left(\lim_{t\to\infty}\frac{1}{t}\right)\left(\lim_{t\to\infty}t^2e^{-0.2t}\right)=(0)(0)=0\right)$ — 1A for $\lim_{t\to\infty}(te^{-0.2t})=0$ (can be absorbed)

$=7.5$ — 1M accept the required pH value $=a$

---------(2)

(a)		Good. Some candidates were unable to transform the equation '$-1.6e^{-8k}+4.8e^{-4k}=1.83$' into a quadratic equation.
(b)		Good. Most candidates were able to differentiate functions involving 'exp' function.
(c)		Satisfactory. Some candidates had difficulty in finding the limit.

21. (2001 ASL-M&S Q8)

(a) (i) Since $G(0)=9$,

$\therefore \quad 2a-12+(a+12)=9$

$a=3$ — 1

(ii) $G(x)=6-12e^{-kx}+15e^{-2kx}$

$G'(x)=12ke^{-kx}-30ke^{-2kx}$ — 1A

$=6ke^{-kx}(2-5e^{-kx})$

$G'(x)=0$ when $e^{-kx}=\frac{2}{5}$ or $x=\frac{1}{k}\ln\frac{5}{2}$ $\boxed{\frac{0.9163}{k}}$ — 1A

and $G'(x)\begin{cases}<0 & \text{when } 0\le x<\frac{1}{k}\ln\frac{5}{2}\\ >0 & \text{when } x>\frac{1}{k}\ln\frac{5}{2}\end{cases}$ — 1M

$\therefore$ $G(x)$ is minimum when $e^{-kx}=\frac{2}{5}$. — 1

$G''(x)=-12k^2e^{-kx}+60k^2e^{-2kx}$	
When $e^{-kx}=\frac{2}{5}$, $G''(x)=\frac{24}{5}k^2>0$	
Since $G(x)$ has only one stationary point for $x\ge 0$,	
$G(x)$ is minimum when $e^{-kx}=\frac{2}{5}$.	$\boxed{1M}$

(ii) $G(x)=6-12e^{-kx}+15e^{-2kx}$	
$=15(e^{-2kx}-\frac{4}{5}e^{-kx})+6$	
$=15(e^{-kx}-\frac{2}{5})^2+\frac{18}{5}$	$\boxed{1M}+\boxed{1A}$
$G(x)$ is minimum when $e^{-kx}=\frac{2}{5}$.	$\boxed{1M}$

The minimum CDO $=\left[6-12\left(\frac{2}{5}\right)+15\left(\frac{2}{5}\right)^2\right]$ mg/L

$=3.6$ mg/L — 1A f.t.

----------(5)

(b) (i) Solving $G(x)=4.5$, we have

$6-12e^{-kx}+15e^{-2kx}=4.5$ — 1M

$10(e^{-kx})^2-8e^{-kx}+1=0$

$e^{-kx}=\frac{4\pm\sqrt{6}}{10}$ — 1A

$x=-\frac{1}{k}\ln\frac{4\pm\sqrt{6}}{10}$

Hence $-\frac{1}{k}\ln\frac{4-\sqrt{6}}{10}+\frac{1}{k}\ln\frac{4+\sqrt{6}}{10}=2.85$ — 1M+1A

$\frac{1}{k}\ln\frac{4+\sqrt{6}}{4-\sqrt{6}}=2.85$

$k\approx 0.5$ (1 d.p.) — 1A

(ii) $G'(x)=6e^{-0.5x}-15e^{-x}$ $\boxed{12ke^{-kx}-30ke^{-2kx}}$

$G''(x)=-3e^{-0.5x}+15e^{-x}$ $\boxed{-12k^2e^{-kx}+60k^2e^{-2kx}}$ — 1M

$=3e^{-0.5x}(5e^{-0.5x}-1)$ $\boxed{12k^2e^{-kx}(5e^{-kx}-1)}$

$G''(x)=0$ when $x=-\frac{1}{0.5}\ln\frac{1}{5}$ (≈ 3.2) $\boxed{x=-\frac{1}{k}\ln\frac{1}{5}}$

and $G''(x)\begin{cases}<0 & \text{when } x>-\frac{1}{0.5}\ln\frac{1}{5} \quad \left[x>-\frac{1}{k}\ln\frac{1}{5}\right]\\ >0 & \text{when } 0\le x<-\frac{1}{0.5}\ln\frac{1}{5} \quad \left[0\le x<-\frac{1}{k}\ln\frac{1}{5}\right]\end{cases}$ — 1M

$G'''(x)=1.5e^{-0.5x}(1-10e^{-0.5x})$ $\boxed{12k^3e^{-kx}(1-10e^{-kx})}$	
When $e^{-kx}=\frac{1}{5}$, $G'''(x)=-0.3<0$ $\boxed{-2.4k^3<0}$	
Since $G'(x)$ has only one stationary point for $x\ge 0$,	
$G'(x)$ is greatest when $e^{-kx}=\frac{1}{5}$.	$\boxed{1M}$

$\therefore$ 3.2 km downstream from the location of discharge of the waste, the rate of change of the CDO is greatest. — 1A

(iii) Solving $G(x)=5.5$, we have

$30e^{-x}-24e^{-.5x}+1=0$

$e^{-0.5x}=\frac{12\pm\sqrt{114}}{30}$

$x=-\frac{1}{0.5}\ln\frac{12\pm\sqrt{114}}{30}$

$x\approx 0.6$ or 6.2 — 1A

$\therefore$ The river will return to be healthy 6.2 km downstream form the location of discharge of waste. — 1

Since $\lim\limits_{x\to\infty}G(x)=\lim\limits_{x\to\infty}(6-12e^{-0.5x}+15e^{-x})=6>5.5$	
$\therefore$ The river will return to be healthy.	$\boxed{1}$
Solving $G(x)=5.5$, we have $x\approx 0.6$ or 6.2	
$\therefore$ The river will return to be healthy 6.2 km downstream form the location of discharge of waste.	$\boxed{1A}$

----------(10)

22. (2000 ASL-M&S Q11)

(a) $\begin{cases} \ln 55 = a - e^{1-2k} \\ \ln 98 = a - e^{1-4k} \end{cases}$

Eliminating a, we have

$e^{1-4k} - e^{1-2k} + \ln 98 - \ln 55 = 0$

$e^{-4k} - e^{-2k} + \frac{1}{e}\ln\frac{98}{55} = 0$ — 1

$(e^{-2k})^2 - e^{-2k} + \frac{1}{e}\ln\frac{98}{55} = 0$ — 1M quadratic equation

$e^{-2k} = \dfrac{1 \pm \sqrt{1 - \frac{4}{e}\ln\frac{98}{55}}}{2}$

≈ 0.30635 or 0.69365

≈ 0.306 or 0.694 — 1A r.t. 0.306, 0.694

$\begin{cases} k \approx 0.5915 \\ a \approx 4.8401 \end{cases}$ or $\begin{cases} k \approx 0.1829 \\ a \approx 5.8929 \end{cases}$

$\begin{cases} k \approx 0.59 \\ a \approx 4.84 \end{cases}$ or $\begin{cases} k \approx 0.18 \\ a \approx 5.89 \text{ (or 5.90)} \end{cases}$ (2 d.p.) — 1A a−1 for more than 2 d.p.

(b) Using $\begin{cases} k \approx 0.59 \\ a \approx 4.84 \end{cases}$, $\ln N(7) \approx 4.80$, — 1M r.t. 4.80

$N(7) \approx 121$. — r.t. 121

Using $\begin{cases} k \approx 0.18 \\ a \approx 5.89 \end{cases}$, $\ln N(7) \approx 5.12$, — r.t. 5.12 – 5.14

$N(7) \approx 167$. (or comparing $\ln 170 \approx 5.1358$) — r.t. 167 – 170

$\therefore$ $\begin{cases} k \approx 0.18 \\ a \approx 5.89 \end{cases}$ will make the model fit for the known data. — 1A follow through

$\because$ $N(t) = e^{\ln N(t)} \approx e^{5.89 - e^{1-0.18t}}$ — 1M

$\therefore$ $N(t) \to e^{5.89} \approx 361$ as $t \to \infty$

The total possible catch of coral fish in that area since January 1, 1992 is 361 thousand tonnes. — 1A r.t. 361 – 365; *pp*−1 for wrong/missing unit

(c) (i) $\because$ $\ln N(t) = a - e^{1-kt}$

$\therefore$ $\frac{N'(t)}{N(t)} = ke^{1-kt}$

$N'(t) = k\,N(t)e^{1-kt}$ — 1

Alternatively,

$N(t) = e^{a - e^{1-kt}}$

$N'(t) = -e^{1-kt}(-k)e^{a-e^{1-kt}} = ke^{1-kt}N(t)$ — 1

(ii) $N''(t) = k[N'(t)e^{1-kt} - k\,N(t)e^{1-kt}]$

$= k^2 N(t)e^{1-kt}(e^{1-kt} - 1)$ — 1A

$\begin{cases} > 0 & \text{when } t < \frac{1}{k} \\ = 0 & \text{when } t = \frac{1}{k} \\ < 0 & \text{when } t > \frac{1}{k} \end{cases}$ — 1M

$\therefore$ $N'(t)$ is maximum at $t = \frac{1}{k}$ — 1A

≈ 5.56 — $t \in [5.47, 5.56]$

The maximum rate of change of the total catch of coral fish in that area since January 1, 1992 occurred in 1997. — 1A

$\ln N(6) \approx 4.97$, $N(6) \approx 143.6$ — $\ln N(6) \in [4.97, 4.99]$, $N(6) \in [143.6, 146.3]$

$\ln N(5) \approx 4.78$, $N(5) \approx 119.7$ — $\ln N(5) \in [4.78, 4.80]$, $N(5) \in [119.7, 122.0]$

$\therefore$ The volume of fish caught in 1997

$= [N(6) - N(5)]$ thousand tonnes — 1M

≈ 24 thousand tonnes — 1A *pp*−1 for wrong/missing unit

23. (1997 ASL-M&S Q8)

Solution	Marks	Remarks
(a) $\because$ $N(0) = 16$		
$\therefore$ $\dfrac{40}{1+b} = 16$	1M	
$b = 1.5$	1A	
$\because$ $N(7) = 17.4$		
$\therefore$ $\dfrac{40}{1+1.5e^{-7r}} = 17.4$	1M	
$e^{-7r} = \dfrac{1}{1.5}\left(\dfrac{40}{17.4}-1\right)$		
$r = \dfrac{1}{-7}\ln\left[\dfrac{1}{1.5}\left(\dfrac{40}{17.4}-1\right)\right]$	1M	
≈ 0.02	1A	
(b) $N(t) = \dfrac{40}{1+be^{-rt}}$ (or $\dfrac{40}{1+1.5e^{-0.02t}}$)		
$N'(t) = \dfrac{-40(-bre^{-rt})}{(1+be^{-rt})^2}$ (or $\dfrac{-40(-1.5)(0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)	1M+1A	
$= \dfrac{40bre^{-0.02t}}{(1+be^{-rt})^2}$ (or $\dfrac{1.2e^{-0.02t}}{(1+1.5e^{-0.02t})^2}$)		
> 0		
$\therefore$ $N(t)$ is increasing.	1	
(c) $\because$ $\lim\limits_{t\to\infty} e^{-rt} = 0$		
$\therefore$ $N_a = \lim\limits_{t\to\infty}\dfrac{40}{1+be^{-rt}}$ (or $\lim\limits_{t\to\infty}\dfrac{40}{1+1.5e^{-0.02t}}$)	1M	
$= 40$	1A	
(d) (i) $N''(t)$		
$= \dfrac{[(1+1.5e^{-0.02t})(1.2)-1.2e^{-0.02t}(2)(1.5)](1+1.5e^{-0.02t})(-0.02)e^{-0.02t}}{(1+1.5e^{-0.02t})^4}$	1M	
$= \dfrac{0.012e^{-0.02t}(3e^{-0.02t}-2)}{(1+1.5e^{-0.02t})^3}$	1A	
(ii) From (i), $N''(t)\begin{cases}>0 & \text{when } t<t_0\\ =0 & \text{when } t=t_0\\ <0 & \text{when } t>t_0\end{cases}$	1M	For Solving $N''(t) = 0$
where $t_0 = -\dfrac{1}{0.02}\ln\dfrac{2}{3} \approx 20.2733$		
$\therefore$ The rate of increase is the greatest when $t = t_0 \approx 20.2733$	1M	For checking maximum
$\because$ $N'(20) \approx 0.199999$		
$N'(21) \approx 0.199989$		
$\therefore$ The company should start to advertise on the 20th day after the first week.	1A	

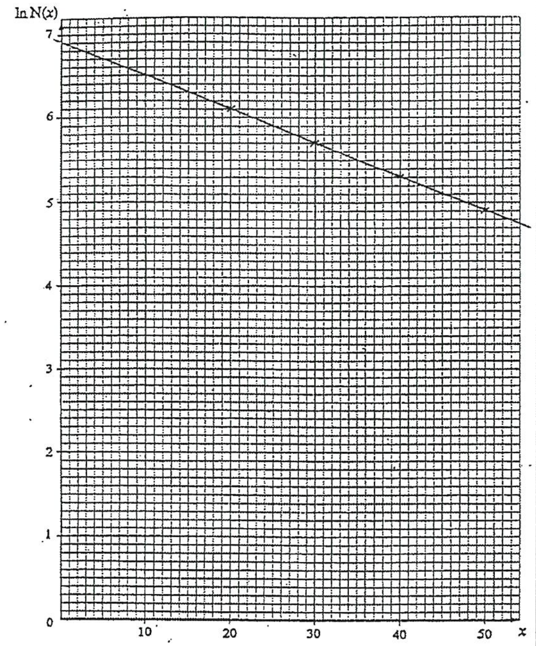
24. (1995 ASL-M&S Q8)

Solution	Marks	Remarks
(a) (i) $N(x) = ae^{-bx}$		
$\ln N(x) = \ln a + \ln e^{-bx}$	1A	
$= \ln a - bx$	1A	

(ii)

x	20	30	40	50
ln N(x)	6.11	5.71	5.31	4.91

1A — At least 1 d.p.



1A + 1A — 1A for the points, 1A for the line

Solution	Marks	Remarks
From the graph,		
$\ln a = 6.9$		Accept 6.85 - 6.95
$\therefore$ $a = 992.27$	1A	Accept 943.88 - 1043.15
$-b = \dfrac{4.91-6.11}{50-20}$		
$b = 0.04$	1A	no mark for b=slope or b=-0.04 in any calculation.

(b) $992.27e^{-0.04x} = 400$	1M	
$-0.04x = \ln\frac{400}{992.27}$		
$x = 22.7$		
<u>In general, accept</u> $ae^{-0.04x} = 400$ where $a \in (943.88, 1043.15)$ $-0.04x = \ln\frac{400}{a}$ $\Rightarrow x \in (21.5, 24.0)$	1M	
∴ The price of each CD should be \$22.7.	1A	Accept \$21.5 - \$ 24.0
(c) (i) $G(x) = 992.27(x-10)e^{-0.04x}$	1A	
<u>In general, accept</u> $G(x) = a(x-10)e^{-0.04x}$ where $a \in (943.88, 1043.15)$	1A	
(ii) $G'(x) = 992.27[e^{-0.04x}+(-0.04)(x-10)e^{-0.04x}]$	1M	
$= 992.27e^{-0.04x}[1.4-0.04x]$	1A	
<u>In general, accept</u> $G'(x) = a[e^{-0.04x}+(-0.04)(x-10)e^{-0.04x}]$ $= ae^{-0.04x}[1.4-0.04x]$ where $a \in (943.88, 1043.15)$	1M 1A	
$G'(x)=0$ when $x=35$ and	1M	for $G'(x)=0$ and solving
$G'(x)\begin{cases}>0 & \text{if } x<35\\ <0 & \text{if } x>35\end{cases}$	1M	
<u>Alternatively</u> $G''(x) = 1.59xe^{-0.04x}-95.18e^{-0.04x}$ $G''(35) = -9.750$	1M	
Therefore $G(x)$ is maximum when $x=35$. For maximum profit, the selling price for each CD should be \$35.	1A	

5. Indefinite Integrals

Learning Unit	Learning Objective
Calculus Area	
Integration with Its Applications	
7. Indefinite integrals and their applications	7.1 recognise the concept of indefinite integration 7.2 understand the basic properties of indefinite integrals and basic integration formulae 7.3 use basic integration formulae to find the indefinite integrals of algebraic functions and exponential functions 7.4 use integration by substitution to find indefinite integrals 7.5 use indefinite integration to solve problems

Section A

1. Let $f(x)$ be a continuous function such that $f'(x)=\dfrac{12x-48}{(3x^2-24x+49)^2}$ for all real numbers x.
   (a) If $f(x)$ attains its minimum value at $x=\alpha$, find α.
   (b) It is given that the extreme value of $f(x)$ is 5 . Find
   (i) $f(x)$,
   (ii) $\lim\limits_{x\to\infty} f(x)$.

(6 marks) (2018 DSE-MATH-M1 Q5)

2. (a) Express $\dfrac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$ in the form $f(x)+g(x)\ln(x^2+1)$, where $f(x)$ and $g(x)$ are polynomials.
   (b) Find $\int x^5\ln(x^2+1)dx$.

(7 marks) (2015 DSE-MATH-M1 Q8)

3. The slope of the tangent to a curve S at any point (x, y) on S is given by $\frac{dy}{dx} = \left(2x - \frac{1}{x}\right)^3$, where $x > 0$.
A point $P(1, 5)$ lies on S.
(a) Find the equation of the tangent to S at P.
(b) (i) Expand $\left(2x - \frac{1}{x}\right)^3$.
(ii) Find the equation of S for $x > 0$.
(7 marks) (2014 DSE-MATH-M1 Q3)

4. The government of a country is going to announce a new immigration policy which will last for 3 years. At the moment of the announcement, the population of the country is 8 million, After the announcement, the rate of change of the population can be modelled by

$$\frac{dx}{dt} = \frac{t\sqrt{9-t^2}}{3} \qquad (0 \le t \le 3) \ ,$$

where x is the population (in million) of the country and t is the time (in years) which has elapsed since the announcement. Find x in terms of t.
(5 marks) (2014 DSE-MATH-M1 Q5)

5. The rate of change of the value V (in million dollars) of a flat is given by $\frac{dV}{dt} = \frac{t}{\sqrt{4t+1}}$, where t is the number of years since the beginning of 2012. The value of the flat is 3 million dollars at the beginning of 2012. Find the percentage change in the value of the flat from the beginning of 2012 to the beginning of 2014.
(5 marks) (2012 DSE-MATH-M1 Q2)

6. An advertising company starts a media advertisement to recruit new members for a club. Past experience shows that the rate of change of the number of members N (in thousand) is given by

$$\frac{dN}{dt} = \frac{0.3e^{-0.2t}}{\left(1+e^{-0.2t}\right)^2} ,$$

where $t\ (\ge 0)$ is the number of weeks elapsed after the launch of the advertisement. The club has 500 members before the launch of the advertisement.
(a) Using the substitution $u = 1 + e^{-0.2t}$, express N in terms of t.
(b) Find the increase in the number of members of the club 4 weeks after the launch of the advertisement. Correct your answer to the nearest integer.
(c) Will the number of members of the club ever reach 1300 after the launch of the advertisement? Explain your answer.
(7 marks) (2012 ASL-M&S Q3)

7. A company launches a promotion plan to raise revenue. The total amount of money X (in million dollars) invested in the plan can be modelled by

$$\frac{dX}{dt} = 6\left(\frac{t}{0.2t^3+1}\right)^2 \ , \quad t \ge 0 \ ,$$

where t is the number of months elapsed since the launch of the plan.
Initially, 4 million dollars are invested in the plan.
(a) Using the substitution $u = 0.2t^3 + 1$, or otherwise, express X in terms of t.
(b) Find the number of months elapsed since the launch of the plan if a total amount of 13 million dollars are invested in the plan.
(c) If the company has a budget of 14.5 million dollars only, can the plan be run for a long time? Explain your answer.
(7 marks) (2011 ASL-M&S Q2)

8. An archaeologist models the presence of carbon-14 remaining in animal skulls fossil by $\frac{dA}{dt} = -kA$ where A (in grams) is the amount of carbon-14 present in the skull at time t (in years) and k is a constant. Let A_0 (in grams) be the original amount of carbon-14 in the skull. It is known that half of the carbon-14 will disappear after 5730 years.
(a) By expressing $\frac{dt}{dA}$ in terms of A, or otherwise, find the value of k correct to 3 significant figures.
(b) In an animal skull fossil, the archaeologist found that 30% of the original amount of carbon-14 is still present. Find the approximate age of the skull correct to the nearest ten years.
(6 marks) (2010 ASL-M&S Q3)

9. A scientist models the proportion, P, of the initial population of an endangered species of animal still surviving by

$$\frac{dP}{dt} = \frac{-0.09}{\sqrt{3t+1}} \qquad (0 \le t \le T)$$

where t is time in months since the beginning of his study, and T is the number of months elapsed for the population size to decrease to 0. It is given that when $t = 0$, $P = 1$.
(a) Find the proportion of the endangered species surviving after t months from the beginning of the study.
(b) What is the proportion of the endangered species dying off within the first 5 months of the study?
(c) Determine the value of T.
(6 marks) (2009 ASL-M&S Q2)

10. The rate of change of concentration of a drug in the blood of a patient can be modelled by

$$\frac{dx}{dt}=5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t} ,$$

where x is the concentration measured in mg/L and t is the time measured in hours after the patient has taken the drug. It is given that $x=0$ when $t=0$.

(a) Find x in terms of t .

(b) Find the concentration of the drug after a long time.

(6 marks) (2008 ASL-M&S Q3)

11. A researcher models the rate of change of the number of certain bacteria under controlled conditions by

$$\frac{dN}{dt}=\frac{800t}{(2t^2+50)^2} ,$$

where N is the number in millions of bacteria and $t(\geq 0)$ is the number of days elapsed since the start of the research. It is given that $N=4$ when $t=0$.

(a) Using the substitution $u=2t^2+50$, or otherwise, express N in terms of t .

(b) When will the number of bacteria be 6 million after the start of the research?

(7 marks) (2007 ASL-M&S Q2)

12. A researcher models the rate of change of the number of fish in a lake by

$$\frac{dN}{dt}=\frac{6}{(e^{\frac{t}{4}}+e^{-\frac{t}{4}})^2} ,$$

where N is the number in thousands of fish in the lake recorded yearly and $t(\geq 0)$ is the time measured in years from the start of the research. It is known that $N=8$ when $t=0$.

(a) Prove that $\frac{dN}{dt}=\frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$. Using the substitution $u=e^{\frac{t}{2}}+1$, or otherwise, express N in terms of t .

(b) Estimate the number of fish in the lake after a very long time.

(6 marks) (2004 ASL-M&S Q2)

13. After a fixed amount of hot liquid is poured into a vessel, the rate of change of the temperature θ of the surface of the vessel can be modelled by

$$\frac{d\theta}{dt}=\frac{12(100-t)e^{\frac{-t}{100}}}{25(1+3te^{\frac{-t}{100}})} ,$$

where θ is measured in $^{\circ}C$ and $t(\geq 0)$ is the time measured in seconds. Initially $(t=0)$, the temperature of the surface of the vessel is $16^{\circ}C$.

(a) (i) Let $u=1+3te^{\frac{-t}{100}}$, find $\frac{du}{dt}$.

(ii) Using the result of (i), or otherwise, express θ in terms of t .

(b) Will the temperature of the surface of the vessel get higher than $95^{\circ}C$? Explain your answer briefly.

(7 marks) (2003 ASL-M&S Q2)

14. An engineer conducts a test for a certain brand of air-purifier in a smoke-filled room. The percentage of smoke in the room being removed by the air-purifier is given by S%. The engineer models the rate of change of S by

$$\frac{dS}{dt}=\frac{8100t}{(3t+10)^3} ,$$

where $t\ (\geq 0)$ is measured in hours from the start of the test.

(a) Using the substitution $u=3t+10$, or otherwise, find the percentage of smoke removed from the room in the first 10 hours.

(b) If the air-purifier operates indefinitely, what will the percentage of smoke removed from the room be?

(5 marks) (2002 ASL-M&S Q4)

15. The total number of visits N to a web site increases at a rate of

$$\frac{dN}{dt}=t^{\frac{1}{3}}(8+11t^{\frac{1}{2}}) \quad (0\leq t\leq 100) ,$$

where t is the time in weeks since January 1, 1999. It is known that $N=100$ when $t=1$.

(a) Express N in terms of t .

(b) Find the total number of visits to the web site when $t=64$.

(6 marks) (1999 ASL-M&S Q4)

16. A mobile phone company plans to invite a famous singer to help to promote its products. The Executive Director of the company estimates that the rate of increase of the number of customers can modeled by

$$\frac{dx}{dt}=650e^{-0.004t} \qquad (0\le t\le 365),$$

where x is the number of customers of the company and t is the number of days which has elapsed since the start of the promotion campaign.

(a) Suppose that at the start of the campaign, the company already has 57 000 customers. Express x in terms of t.

(b) How many days after the start of the campaign will the number of customers be doubled?

(6 marks) (1998 ASL-M&S Q4)

17. A machine depreciates with time t in years. Its value \$ $V(t)$ is initially \$20 000 and will drop to \$ 0 when $t=k$ $(k\ge 0)$. The depreciation rate at time t is

$$V'(t)=200(t-15) \qquad \text{for } 0\le t\le k .$$

(a) $V(t)$ for $0\le t\le k$,

(b) the value of k, and

(c) the total depreciation in the first 5 years.

(7 marks) (1997 ASL-M&S Q4)

18. Let $y=\frac{\ln x}{x}$ $(x>0)$, find $\frac{dy}{dx}$.

Hence or otherwise, find $\int\frac{\ln x}{x^2}dx$.

(5 marks) (1996 ASL-M&S Q2)

19. The value M (in million dollars) of a house is modeled by the equation

$$\frac{dM}{dt}=\frac{1}{3t+4}+\frac{1}{\sqrt{t+25}}$$

where t is the number of years elapsed since the end of 1994. The value of the house is 3.1 million dollars at the end of 1994.

(a) Find, in terms of t, the value of the house t years after the end of 1994.

(b) Find the rise in the value of house between the end of 1994 and the end of 2000.

(7 marks) (1995 ASL-M&S Q4)

20. The rate of spread of an epidemic can be modelled by the equation

$$\frac{dx}{dt}=3t\sqrt{t^2+1},$$

where x is the number of people infected by the epidemic and t is the number of days which have elapsed since the outbreak of the epidemic. If $x=10$ when $t=0$, express x in terms of t.

(6 marks) (1994 ASL-M&S Q5)

Section B

21. In a research of the radiation intensity of a city, an expert modelled the rate of change of the radiation intensity R (in suitable units) by

$$\frac{dR}{dt}=\frac{a(30-t)+10}{(t-35)^2+b}$$

where t $(0\le t\le T)$ is the number of days elapsed since the start of the research, a, b and T are positive constants.
It is known that the intensity increased to the greatest value of 6 units at $t=35$, and then decreased to the level as at the start of the research at $t=T$. Moreover, the decrease of the intensity from $t=40$ to $t=41$ is $\ln\frac{61}{50}$ units

(a) Find the value of a.

(2 marks)

(b) Find the value of T.

(4 marks)

(c) Express R in terms of t.

(4 marks)

(d) For $0\le t\le 35$, when would the rate of change of the radiation intensity attain its greatest value?

(4 marks)

(2012 DSE-MATH-M1 Q11)

22. The manager, Mary, of a theme park starts a promotion plan to increase **the daily number of visits** to the park. The rate of change of **the daily number of visits** to the park can be modelled by

$$\frac{dN}{dt}=\frac{k(25-t)}{e^{0.04t}+4t} \qquad (t\geq 0),$$

where N is **the daily number of visits** (in hundreds) recorded at the end of a day, t is the number of days elapsed since the start of the plan and k is a positive constant.

Mary finds that at the start of the plan, $N=10$ and $\frac{dN}{dt}=50$.

(a) (i) Let $v=1+4te^{-0.04t}$, find $\frac{dv}{dt}$.

(ii) Find the value of k, and hence express N in terms of t.

(7 marks)

(b) (i) When will **the daily number of visits** attain the greatest value?

(ii) Mary claims that there will be more than 50 hundred visits on a certain day after the start of the plan. Do you agree? Explain your answer.

(3 marks)

(c) Mary's supervisor believes that **the daily number of visits** to the park will return to the original one at the start of the plan after a long period of time. Do you agree? Explain your answer.

(Hint: $\lim_{t\to\infty} te^{-0.04t}=0$.)

(2 marks)

(SAMPLE DSE-MATH-M1 Q11)

23. A biologist studied the population of fruit fly A under limited food supply. Let t be the number of days since the beginning of the experiment and N'(t) be the number of fruit fly A at time t. The biologist modelled the rate of change of the number of fruit fly A by

$$N'(t)=\frac{20}{1+he^{-kt}} \qquad (t\geq 0)$$

where h and k are positive constants.

(a) (i) Express $\ln\left(\frac{20}{N'(t)}-1\right)$ as a linear function of t.

(ii) It is given that the intercepts on the vertical axis and the horizontal axis of the graph of the linear function in (i) are 1.5 and 7.6 respectively. Find the values of h and k.

(4 marks)

(b) Take $h=4.5$ and $k=0.2$, and assume that $N(0)=50$.

(i) Let $v=h+e^{kt}$, find $\frac{dv}{dt}$.

Hence, or otherwise, find N(t).

(ii) The population of fruit fly B can be modelled by

$$M(t)=21\left(t+\frac{h}{k}e^{-kt}\right)+b,$$

where b is a constant. It is known that $M(20)=N(20)$.

(1) Find the value of b.

(2) The biologist claims that the number of fruit fly A will be smaller than that of fruit fly B for $t>20$. Do you agree? Explain your answer.

[Hint: Consider the difference between the rates of change of the two populations.]

(11 marks)

(2008 ASL-M&S Q8)

24. An airline manager, Christine, notice that the *weekly number of passengers* of the airline is declining, so she starts a promotion plan to boost the *weekly number of passengers*. She models the rate of change of the *weekly number of passengers* by

$$\frac{dx}{dt}=\frac{30t-90}{t^2-6t+11} \qquad (t\geq 0) ,$$

where x is the *weekly number of passengers* recorded at the end of a week in thousands of passengers and t is the number of weeks elapsed since the start of the plan.

Christine finds that at the start of the plan (i.e. $t=0$), the *weekly number of passenger* is 40 thousand.

(a) Let $v=t^2-6t+11$, find $\frac{dv}{dt}$.

Hence, or otherwise, express x in terms of t .

(4 marks)

(b) How many weeks after the start of the plan will the *weekly number of passengers* be the same as at the start of the plan?

(2 marks)

(c) Find the least *weekly number of passengers* after the start of the plan. Give your answer correct to the nearest thousand.

(3 marks)

(d) The week when the *weekly number of passengers* drops to the least is called the *Recovery Week*.

(i) Find the change in the *weekly number of passengers* from the *Recovery Week* to its following week. Give your answer correct to the nearest thousand.

(ii) Prove that $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t .

(iii) Christine's assistant claims that after the *Recovery Week*, the change in the *weekly number of passengers* from a certain week to its following week will be greater than 25 thousand. Do you agree? Explain your answer.

(6 marks)

(2006 ASL-M&S Q8)

25. A web administrator, David, launches a promotion plan to increase the *daily number of visits* to his web site. The rate of change of the *daily number of visits* to the web site can be modelled by

$$\frac{dN}{dt}=\frac{k(50-t)}{2e^{0.02t}+3t} ,$$

where N is the *daily number of visits* recorded at the end of a day in thousands of visits, $t(\geq 0)$ is the number of days elapsed since the start of the plan and k is a positive constant.

David finds that at the start of the plan (i.e. $t=0$), $\frac{dN}{dt}=100$ and $N=10$.

(a) (i) Let $v=2+3te^{-0.02t}$, find $\frac{dv}{dt}$.

(ii) Prove that $k=4$ and hence express N in terms of t .

(7 marks)

(b) David claims that the *daily number of visits* to his web site will be greater than 500 thousand on a certain day after the start of the plan. Do you agree? Explain your answer.

(4 marks)

(c) (i) Find $\frac{d^2N}{dt^2}$.

(ii) Describe the behaviour of N and $\frac{dN}{dt}$ during the 3rd month after the start of the plan.

(4 marks)

(2005 ASL-M&S Q9)

26. A food store manager notices that the **weekly sale** is declining, so he starts a promotion plan to boost the **weekly sale**. He models the rate of change of weekly sale G by

$$\frac{dG}{dt}=\frac{2t-8}{t^2-8t+20} \quad (t\geq 0) ,$$

where G is the **weekly sale** recorded at the end of the week in thousands of dollars and t is the number of weeks elapsed since the start of the plan. Suppose that at the start of the plan (i.e. $t=0$), the **weekly sale** is 50 thousand dollars.

(a) (i) Express G in terms of t .

(ii) At the end of which week after the start of the plan will the **weekly sale** be the same as at the start of the plan?

(5 marks)

(b) (i) At the end of which week after the start of the plan will the **weekly sale** drop to the least?

(ii) Find the increase between the **weekly sale** of the 5th and the 6th weeks after the start of the plan.

(iii) The store manager decides that once such increase of weekly sale between two consecutive weeks is less than 0.2 thousand dollars, he will terminate the promotion plan. At the end of which week after the start of the plan will the plan be terminated?

(6 marks)

(c) Let t_1 and t_2 be the roots of $\frac{d^2G}{dt^2}=0$, where $t_1<t_2$. Find t_2 .

Briefly describe the behaviour of G and $\frac{dG}{dt}$ immediately before and after t_2 .

(4 marks)

(2002 ASL-M&S Q11)

NEW

Out of syllabus

5. Indefinite Integrals

Section A

1. (2018 DSE-MATH-M1 Q5)

5. Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$ $\frac{12x-48}{(3x^2-24x+49)^2}=0$ $x=4$	1M	

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	$+$

So, $f(x)$ attains its minimum value at $x=4$. Thus, we have $\alpha=4$.	1A	
$f'(x)=0$ $\frac{12x-48}{(3x^2-24x+49)^2}=0$ $x=4$	1M	
$f''(x)$ $=\frac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$ $f''(4)$ $=12$ >0 So, $f(x)$ attains its minimum value at $x=4$. Thus, we have $\alpha=4$.	1A	

(b) (i) Let $v=3x^2-24x+49$. Then, we have $\frac{dv}{dx}=6x-24$. $f(x)$ $=\int\frac{12x-48}{(3x^2-24x+49)^2}dx$ $=\int\frac{2}{v^2}dv$	1M
$=\frac{-2}{v}+C$ $=\frac{-2}{3x^2-24x+49}+C$	
Since $f(x)$ has only one extreme value, we have $f(4)=5$. $\frac{-2}{3(4)^2-24(4)+49}+C=5$ $C=7$	1M
Thus, we have $f(x)=\frac{-2}{3x^2-24x+49}+7$.	1A
(ii) $\lim_{x\to\infty}f(x)$ $=7$	1A
	----------(6)

2. (2015 DSE-MATH-M1 Q8)

(a) $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$		
$=(x^6+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$	1M+1A	1M for product rule
$=(x^2+1)(x^4-x^2+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$	1M	
$=2x^5-2x^3+2x+6x^5\ln(x^2+1)$	1A	
(b) $(x^6+1)\ln(x^2+1)=2\int(x^5-x^3+x)\,dx+6\int x^5\ln(x^2+1)\,dx$	1M	
Note that $\int(x^5-x^3+x)\,dx=\frac{x^6}{6}-\frac{x^4}{4}+\frac{x^2}{2}+\text{constant}$.	1A	
Thus, we have $\int x^5\ln(x^2+1)\,dx=\frac{1}{6}(x^6+1)\ln(x^2+1)-\frac{x^6}{18}+\frac{x^4}{12}-\frac{x^2}{6}+\text{constant}$.	1A	
	----------(7)	

(a)	Good. Many candidates were able to apply product rule to find $\frac{d}{dx}\left((x^6+1)\ln(x^2+1)\right)$ while some candidates did not understand the definition of polynomial and simply left $(x^6+1)\frac{2x}{x^2+1}+6x^5\ln(x^2+1)$ as the final answer instead of $(2x^5-2x^3+2x)+6x^5\ln(x^2+1)$.
(b)	Fair. Many candidates employed a wrong substitution in finding $\int(x^6+1)\frac{2x}{x^2+1}dx$, and many candidates made careless mistakes in calculating the integration.

3. (2014 DSE-MATH-M1 Q3)

Solution	Marks
(a) $\left.\frac{dy}{dx}\right\vert_{(1,5)}=\left(2\cdot 1-\frac{1}{1}\right)^3$	
$=1$	1A
Hence the equation of tangent is $y-5=1(x-1)$.	
i.e. $x-y+4=0$	1A
(b) (i) $\left(2x-\frac{1}{x}\right)^3=(2x)^3-3(2x)^2\left(\frac{1}{x}\right)+3(2x)\left(\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^3$	1M
$=8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$	1A
(ii) $y=\int\left(2x-\frac{1}{x}\right)^3 dx$	
$=\int\left(8x^3-12x+\frac{6}{x}-\frac{1}{x^3}\right)dx$ by (i)	
$=2x^4-6x^2+6\ln\lvert x\rvert+\frac{1}{2x^2}+C$	1M
Since $P(1,5)$ lies on S, $5=2(1)^4-6(1)^2+6\ln\lvert 1\rvert+\frac{1}{2(1)^2}+C$.	1M
i.e. $C=\frac{17}{2}$	
Hence the equation of S is $y=2x^4-6x^2+6\ln x+\frac{1}{2x^2}+\frac{17}{2}$ for $x>0$.	1A
	(7)

(a)	Very good.
(b) (i)	Excellent.
(ii)	Satisfactory.
	Some candidates did not know $\int\frac{1}{x}dx=\ln\lvert x\rvert+C$, or wrote $\int\frac{1}{x^3}dx=-\frac{2}{x^2}$ or $\frac{1}{2x^2}$.

4. (2014 DSE-MATH-M1 Q5)

Solution	Marks	Remarks
$\frac{dx}{dt}=\frac{t\sqrt{9-t^2}}{3}$		
Let $u=9-t^2$.	} 1M	
$du=-2tdt$		
$x=\int\frac{t\sqrt{9-t^2}}{3}dt$		
$=\int\frac{u^{\frac{1}{2}}}{3}\frac{du}{-2}$	1M	OR $\int\frac{(9-t^2)^{\frac{1}{2}}}{3}\frac{d(9-t^2)}{-2}$
$=\frac{-1}{6}\cdot\frac{2u^{\frac{3}{2}}}{3}+C$	1M	
$=\frac{-1}{9}(9-t^2)^{\frac{3}{2}}+C$		
When $t=0$, $x=8$.		
$\therefore\ 8=\frac{-1}{9}(9-0)^{\frac{3}{2}}+C$	1M	
$C=11$		
i.e. $x=\frac{-1}{9}(9-t^2)^{\frac{3}{2}}+11$	1A	
	(5)	

Satisfactory.
Some candidates substituted $t=3$ and $x=8$ to determine the value of the constant of integration.

5. (2012 DSE-MATH-M1 Q2)

Solution	Marks	Remarks
Let $u=4t+1$.	1M	
$du=4dt$		
When $t=0$, $u=1$; when $t=2$, $u=9$.		
The change in the value of the flat		
$=\int_0^2\frac{t}{\sqrt{4t+1}}dt$	1M	
$=\int_1^9\frac{1}{\sqrt{u}}\cdot\frac{u-1}{4}\frac{du}{4}$		
$=\frac{1}{16}\int_1^9\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du$		
$=\frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^9$	1A	For $\frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]$
$=\frac{5}{6}$	1A	
Hence the percentage change $=\frac{\frac{5}{6}}{3}\times 100\%$		
$=27\frac{7}{9}\%$	1A	OR 27.7778%
	(5)	

Fair. Many candidates failed to find a suitable substitution or did wrong calculation in substitution. Some found the value of the flat at the beginning of 2014 instead of the percentage change.

6. (2012 ASL-M&S Q3)

(a) Let $u=1+e^{-0.2t}$.

$du=-0.2e^{-0.2t}dt$ — 1A

$N=\int\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}dt$

$N=\int\frac{0.3}{u^2}\cdot\frac{du}{-0.2}$

$=\frac{3}{2u}+C$ — 1A

$=\frac{3}{2(1+e^{-0.2t})}+C$

When $t=0$, $N=0.5$.

$\therefore\ C=\frac{-1}{4}$

i.e. $N=\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}$ — 1A

(b) $N(4)-N(0)$

$=\frac{3}{2(1+e^{-0.2\times4})}-\frac{1}{4}-0.5$ — 1M

≈ 0.284961721

Hence the increase in the number of people is 285. — 1A

(c) $\frac{dN}{dt}=\frac{0.3e^{-0.2t}}{(1+e^{-0.2t})^2}>0$ for all $t\geq 0$

Hence N is always increasing.

Withhold the last mark if this argument is missing

$\lim_{t\to\infty}N=\lim_{t\to\infty}\left[\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}\right]$

$=1.25$ — 1A

Hence the number of members will never reach 1300. — 1

(7)

OR by arguing that $e^{-0.2t}>0\Rightarrow N<\frac{3}{2}-\frac{1}{4}$

OR by arguing that $\frac{3}{2(1+e^{-0.2t})}-\frac{1}{4}=1.3$ has no real solution

Satisfactory.
Many candidates overlooked the units and did not use 0.5 to represent 500 since N was given in thousand. A number of candidates could not well explain their answer in (c) because they did not state clearly that N was an increasing function.

7. (2011 ASL-M&S Q2)

(a) $\frac{dX}{dt}=6\left(\frac{t}{0.2t^3+1}\right)^2$

$X=6\int\frac{t^2}{(0.2t^3+1)^2}dt$

Let $u=0.2t^3+1$, and therefore $du=0.6t^2\,dt$. — 1M — For $u=0.2t^3+1$ or $u=(0.2t^3+1)^2$

$\therefore\ X=6\int\frac{1}{0.6u^2}du$ — 1M

$=\frac{-10}{u}+C$ — 1A

$=\frac{-10}{0.2t^3+1}+C$

When $t=0$, $X=4$ and hence $C=14$.

i.e. $X=\frac{-10}{0.2t^3+1}+14$ — 1A

(b) $13=\frac{-10}{0.2t^3+1}+14$

$t=\sqrt[3]{45}$ months — 1A — OR 3.5569 months

(c) $X=14-\frac{10}{0.2t^3+1}<14$ for any value of t. — 1M

Hence the plan can be run for a long time. — 1A

(7)

Good.
In part (c), although most candidates found the limit of X when $t\to\infty$, the proof was incomplete without showing that the function was increasing.

8. (2010 ASL-M&S Q3)

(a) $\dfrac{dA}{dt}=-kA$

$\therefore\ \dfrac{dt}{dA}=\dfrac{-1}{kA}$ — 1A

$t=\dfrac{-1}{k}\displaystyle\int\frac{dA}{A}$

$=\dfrac{-1}{k}\ln|A|+C$ — 1A

When $t=0$, $A=A_0$ and when $t=5730$, $A=\dfrac{A_0}{2}$.

$0=\dfrac{-1}{k}\ln A_0+C$ and $5730=\dfrac{-1}{k}\ln\dfrac{A_0}{2}+C$ — 1M

$\therefore\ 5730=\dfrac{-1}{k}\ln\dfrac{A_0}{2}+\dfrac{1}{k}\ln A_0$

$=\dfrac{1}{k}\ln 2$

i.e. $k=\dfrac{\ln 2}{5730}$

$\approx 1.21\times10^{-4}$ (correct to 3 significant figures) — 1A

(b) $A=0.3A_0$

$\therefore\ t=\dfrac{-5730}{\ln 2}\ln(0.3A_0)+\dfrac{5730}{\ln 2}\ln A_0$ — 1M

$=\dfrac{5730}{\ln 2}\ln\dfrac{10}{3}$

≈ 9950 years (correct to the nearest ten years) — 1A

(6)

Fair. This question required candidates to deal with $\dfrac{dt}{dA}$ and integrating with respect to A which was different from the more familiar format of $\dfrac{dA}{dt}$ and integrating with respect to t. Part (b) was straightforward for candidates who could solve (a).

9. (2009 ASL-M&S Q2)

(a) $\dfrac{dP}{dt}=\dfrac{-0.09}{\sqrt{3t+1}}$

$P=\displaystyle\int\frac{-0.09}{\sqrt{3t+1}}\,dt$

$=-0.06\sqrt{3t+1}+C$ — 1M+1A — Withhold 1A if C was omitted

When $t=0$, $P=1$.

$\therefore\ 1=-0.06\sqrt{1}+C$

$C=1.06$

i.e. $P=-0.06\sqrt{3t+1}+1.06$ — 1A

(b) When $t=5$, $P=-0.06\sqrt{3(5)+1}+1.06=0.82$ — 1M

Thus, 18% of the population has died off. — 1A

(c) When $P=0$, $0=-0.06\sqrt{3T+1}+1.06$

$T=103\dfrac{19}{27}$ — 1A — OR 103.7037

(6)

Good. Some candidates were not clear about the concepts of definite and indefinite integrations. Some candidates were not sure about the fact that the total population is composed of ied out_ population and the urviving_ population.

10. (2008 ASL-M&S Q3)

(a) $\dfrac{dx}{dt}=5.3\left(\dfrac{1}{t+2}-\dfrac{1}{t+5}\right)+1.2e^{-0.1t}$

$x=\displaystyle\int\left[5.3\left(\frac{1}{t+2}-\frac{1}{t+5}\right)+1.2e^{-0.1t}\right]dt$ — 1M

$=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+C$ (since $t\ge 0$) — 1A — OR $5.3[\ln|t+2|-\ln|t+5|]-12e^{-0.1t}+C$

When $t=0$, $x=0$.

$\therefore 0=5.3(\ln 2-\ln 5)-12+C$ — 1M

$C=5.3(\ln 5-\ln 2)+12$

≈ 16.8563

i.e. $x=5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563$ — 1A — OR $x=\cdots+5.3\ln 2.5+12$

(b) $\displaystyle\lim_{t\to\infty}\{5.3[\ln(t+2)-\ln(t+5)]-12e^{-0.1t}+16.8563\}$ — 1M

$=5.3\displaystyle\lim_{t\to\infty}\ln\frac{t+2}{t+5}-12\lim_{t\to\infty}e^{-0.1t}+16.8563$

$=16.8563$ — 1A

i.e. the concentration of the drug after a long time = 16.8563 mg/L

(6)

Good. Some candidates could not present the mathematical notation of the limit of x properly.

11. (2007 ASL-M&S Q2)

(a) Let $u = 2t^2 + 50$.

Then, we have $\frac{du}{dt} = 4t$. — 1A

N

$= \int \frac{800t}{(2t^2+50)^2} dt$

$= \int \frac{200}{u^2} du$ — 1M

So, we have $N = \frac{-200}{u} + C$ where C is a constant. — 1A

Therefore, we have $N = \frac{-200}{2t^2+50} + C$.

Using the condition that $N = 4$ when $t = 0$, we have $4 = -4 + C$. — 1M for finding C

Hence, we have $C = 8$.

Thus, we have $N = 8 - \frac{200}{2t^2+50}$. — 1A

(b) When $N = 6$, we have $8 - \frac{200}{2t^2+50} = 6$. — 1M

So, we have $t = 5$. — 1A

The number of bacteria will be 6 million 5 days after the start of the research.

----------(7)

Good. Many candidates could handle indefinite integration, but some forgot the constant of integration.

12. (2004 ASL-M&S Q2)

(a) $\frac{dN}{dt}$

$= \frac{6}{(e^{\frac{t}{4}} + e^{\frac{-t}{4}})^2}$

$= \frac{6}{(e^{\frac{-t}{4}}(e^{\frac{t}{2}}+1))^2}$

$= \frac{6}{e^{\frac{-t}{2}}(e^{\frac{t}{2}}+1)^2}$

$= \frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2}$ — 1 must show steps

Let $u = e^{\frac{t}{2}} + 1$.

Then, we have $\frac{du}{dt} = \frac{1}{2}e^{\frac{t}{2}}$.

Also, $dt = \frac{2du}{u-1}$. Now,

N

$= \int \frac{6e^{\frac{t}{2}}}{(e^{\frac{t}{2}}+1)^2} dt$

$= \int \frac{12(u-1)}{u^2(u-1)} du$

$= \int \frac{12}{u^2} du$ — 1A accept $\frac{dN}{du} = \frac{12}{u^2}$.

So, we have $N = \frac{-12}{u} + C$ where C is a constant. — 1A

Now, $N = \frac{-12}{e^{\frac{t}{2}}+1} + C$.

Using the condition that $N = 8$ when $t = 0$, we have $8 = -6 + C$. — 1M for finding C

Hence, $C = 14$.

Thus, $N = 14 - \frac{12}{e^{\frac{t}{2}}+1}$. — 1A

(b) The required number of fish

$= \lim_{t\to\infty}(14 - \frac{12}{e^{\frac{t}{2}}+1})$

$= 14 - \lim_{t\to\infty} \frac{12}{e^{\frac{t}{2}}+1}$

$= 14$ thousands — 1A

----------(6)

Fair. Many candidates were not able to relate mathematical presentations to integrations and taking limits.

13. (2003 ASL-M&S Q2)

(a) (i) Let $u = 1 + 3te^{\frac{-t}{100}}$. Then,

$\frac{du}{dt} = 3e^{\frac{-t}{100}} - \frac{3t}{100}e^{\frac{-t}{100}}$ — 1M for Product Rule + 1A

$= \frac{3}{100}(100-t)e^{\frac{-t}{100}}$

(ii) $\theta = \int \frac{\frac{12}{25}(100-t)e^{\frac{-t}{100}}}{(1+3te^{\frac{-t}{100}})}dt$

$= \int \frac{16}{u}du$ — 1A

$= 16\ln(1+3te^{\frac{-t}{100}}) + C$

When $t = 0$, $\theta = 16$, we have $C = 16$ — 1M for finding C

$\therefore \theta = 16\ln(1+3te^{\frac{-t}{100}}) + 16$ — 1A

(b) $\frac{d\theta}{dt} = \frac{12(100-t)e^{\frac{-t}{100}}}{25(1+3te^{\frac{-t}{100}})}$

$\begin{cases} >0 & \text{if } 0 \le t < 100 \\ =0 & \text{if } t = 100 \\ <0 & \text{if } t > 100 \end{cases}$ — 1M for testing

$\therefore \theta$ attains its greatest value when $t = 100$

Note that $\theta(100) = 16\ln(1+300e^{-1}) + 16$

≈ 91.40484176

≤ 95

Thus, the temperature of the surface of the vessel will not get higher than 95°C. — 1A

---------(7)

Fair. Many candidates knew that the integration constant should not be neglected.

14. (2002 ASL-M&S Q4)

(a) $S = \int_0^{10} \frac{8100t}{(3t+10)^3}dt$. — 1A

Let $u = 3t + 10$.

$du = 3dt$.

When $t = 0$, $u = 10$.

When $t = 10$, $u = 40$.

$S = \int_{10}^{40} \frac{8100\left(\frac{u-10}{3}\right)\cdot\frac{1}{3}du}{u^3}$

$= 900\int_{10}^{40}\left(\frac{1}{u^2} - \frac{10}{u^3}\right)du$ — 1M change of variable

$= 900\left[-\frac{1}{u} + \frac{5}{u^2}\right]_{10}^{40}$

$= \frac{405}{16} = 25.3125$ — 1A

The percentage of smoke removed is 25.3125%.

$S = \int_0^{10}\frac{8100t}{(3t+10)^3}dt$	1A
$= 900\int_0^{10}\left[\frac{1}{(3t+10)^2} - \frac{10}{(3t+10)^3}\right]d(3t+10)$	1M change of variable
$= 900\left[-\frac{1}{3t+10} + \frac{5}{(3t+10)^2}\right]_0^{10}$	
$= 25.3125$	1A

$S = \int\frac{8100t}{(3t+10)^3}dt = 900\left[-\frac{1}{3t+10} + \frac{5}{(3t+10)^2}\right] + C$	1M+1M for change of variable
When $t = 0$, $S = 0$. Hence, we have $C = 45$.	
So, $S = 900\left[-\frac{1}{3T+10} + \frac{5}{(3T+10)^2}\right] + 45$.	
When $t = 10$, $S = 25.3125$.	1A

(b) $S = \int_0^T \frac{8100t}{(3t+10)^3}dt$

$= 900\left[-\frac{1}{u} + \frac{5}{u^2}\right]_{10}^{3T+10}$

$= 900\left[-\frac{1}{3T+10} + \frac{5}{(3T+10)^2} + 0.05\right]$

$\lim_{T\to\infty} S = \lim_{T\to\infty} 900\left[-\frac{1}{3T+10} + \frac{5}{(3T+10)^2} + 0.05\right]$ — 1M taking limit and in terms of T

$= 45$ — 1A

$\therefore$ 45% of smoke will be removed.

---------(5)

15. (1999 ASL-M&S Q4)

Solution	Marks	Remarks
(a) $N = \int (8t^{\frac{1}{3}} + 11t^{\frac{5}{6}})dt$	1M+1A	1M for integration $pp-1$ for missing N or dt
$= 6t^{\frac{4}{3}} + 6t^{\frac{11}{6}} + c$ for some constant c.	1A	$pp-1$ for missing c
$\because$ $N = 100$ when $t = 1$ $\therefore$ $100 = 6 + 6 + c \Rightarrow c = 88$	1A	
i.e. $N = 6t^{\frac{4}{3}} + 6t^{\frac{11}{6}} + 88$		
(b) When $t = 64$, $N = 6(64)^{\frac{4}{3}} + 6(64)^{\frac{11}{6}} + 88$ $= 13912$	1A (5)	

16. (1998 ASL-M&S Q4)

Solution	Marks	Remarks
(a) $x = \int 650e^{-0.004t}dt$	1A	$pp-1$ for missing dt
$= -162500e^{-0.004t} + c$	1A	$pp-1$ for missing c $pp-1$ for missing x
since $x = 57000$ when $t = 0$ $c = 219500$ $\therefore$ $x = 219500 - 162500e^{-0.004t}$	1M+1A	
(b) If $57000 \times 2 = 219500 - 162500e^{-0.004t}$	1M	
then $t = \frac{1}{-0.004}\ln\left(\frac{219500-114000}{162500}\right)$ ≈ 108 (or 107.9918) $\therefore$ the number of customers will be doubled in 108 days after the start of the campaign.	1A (6)	r.t. 108

17. (1997 ASL-M&S Q4)

Solution	Marks
(a) $V(t) = \int 200(t-15)dt$	1A
$= 100t^2 - 3000t + c$	1A
$\because V(0) = 20\,000$, $\therefore c = 20\,000$	1A
Hence $V(t) = 100t^2 - 3000t + 20000$ for $0 \le t \le k$.	
(b) $\because$ $V(k) = 0$ $\therefore$ $100k^2 - 3000k + 20000 = 0$	1M
$k^2 - 30k + 200 = 0$ $(k-20)(k-10) = 0$ $k = 10$ or 20 (rejected) $k = 10$	1A
(c) $V(5) - V(0)$ $= 100(5)^2 - 3000(5) + 20000 - 20000$ $= -12500$	1M
Alternatively, $\int_0^5 200(t-15)dt$ $= \left[100t^2 - 3000t\right]_0^5$ $= -12500$	1M
$\therefore$ The total depreciation in the first 5 years is \$12 500.	1A (7)

18. (1996 ASL-M&S Q2)

Solution	Marks	Remarks
$\frac{dy}{dx} = \frac{1-\ln x}{x^2}$	1M+1A	1M for quotient rule
$\int\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)dx = \frac{\ln x}{x}$ $(+c_1)$	1M	For applying anti-differentiation
$\int \frac{\ln x}{x^2}dx = \int \frac{1}{x^2}dx - \frac{\ln x}{x}$ $(+c_1)$	1A	pp-1 for missing dx more than once
$= -\frac{1+\ln x}{x} + c$ (or $-\frac{1}{x} - \frac{\ln x}{x} + c$)	1A (5)	No marks for missing c

19. (1995 ASL-M&S Q4)

(a) $M(t)=\int\left[\frac{1}{3t+4}+(t+25)^{-\frac{1}{2}}\right]dt$ — 1A — pp-1 for missing differential

$=\frac{1}{3}\ln(3t+4)+2(t+25)^{\frac{1}{2}}+c$ — 1A + 1A — 1A for integrating 1 term

$\because M(0)=3.1$

$\therefore c=3.1-\frac{1}{3}\ln 4-10$ — 1M

$=-\frac{2}{3}\ln 2-6.9$ (or -7.3621)

$M(t)=\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}-\frac{2}{3}\ln 2-6.9$ — 1A

(or $\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}-7.3621$)

The value of the house, in million dollars, t years after the end of 1994 is $\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}-\frac{2}{3}\ln 2-6.9$.

(b) The rise in the value of the house, in million dollars, between the end of 1994 and the end of 2000 is

$M(6)-M(0)$

$=\frac{1}{3}\ln 22+2\sqrt{31}-\frac{2}{3}\ln 2-6.9-3.1$ — 1M

$=\frac{1}{3}\ln\frac{11}{2}+2\sqrt{31}-10$ (or 1.7038) — 1A

Alternatively,

$M(6)-M(0)=\left[\frac{1}{3}\ln(3t+4)+2\sqrt{t+25}\right]_0^6$

$=\frac{1}{3}\ln 22+2\sqrt{31}-\frac{2}{3}\ln 2-10$ — 1M

$=\frac{1}{3}\ln\frac{11}{2}+2\sqrt{31}-10$ (or 1.7038) — 1A

(7)

20. (1994 ASL-M&S Q5)

Let $u=t^2+1$, then $du=2t\,dt$. — 1M

$x=\int 3t(t^2+1)^{\frac{1}{2}}dt$ — 1M

$=\frac{3}{2}\int u^{\frac{1}{2}}du$ — 1A

$=u^{\frac{3}{2}}+c$

$=(t^2+1)^{\frac{3}{2}}+c$ — 1A

Since $x=10$ when $t=0$, $10=(0^2+1)^{\frac{3}{2}}+c$ — 1M

$c=9$

$\therefore x=(t^2+1)^{\frac{3}{2}}+9$. — 1A

6

Section B

21. (2012 DSE-MATH-M1 Q11)

(a) When $t=35$, the intensity increased to a maximum and therefore $\frac{dR}{dt}=0$.

$\frac{a(30-35)+10}{(35-35)^2+b}=0$ — 1A

$a=2$ — 1A

(2)

(b) $\frac{dR}{dt}=\frac{2(30-t)+10}{(t-35)^2+b}$

Let $u=(t-35)^2+b$. — 1M

$du=2(t-35)dt$

$R=\int\frac{-2t+70}{(t-35)^2+b}dt$

$=\int\frac{-2t+70}{u}\frac{du}{2(t-35)}$

$=-\ln|u|+C$

$=-\ln[(t-35)^2+b]+C$ — 1A

$R|_{t=T}=R|_{t=0}$

$-\ln[(T-35)^2+b]+C=-\ln[(0-35)^2+b]+C$ — 1M

$(T-35)^2=35^2$

$T=70$ or 0 (rejected) — 1A

(4)

(c) $R\big|_{t=40} - R\big|_{t=41} = \ln\frac{61}{50}$

$-\ln[(40-35)^2+b]+C-\{-\ln[(41-35)^2+b]+C\} = \ln\frac{61}{50}$ 1M

$-\ln(25+b)+\ln(36+b) = \ln\frac{61}{50}$

$\ln\frac{36+b}{25+b} = \ln\frac{61}{50}$

$b = 25$ 1A

$\therefore \quad R = -\ln[(t-35)^2+25]+C$

$R\big|_{t=35} = 6$

$-\ln[(35-35)^2+25]+C = 6$ 1M

$C = 6+\ln 25$

i.e. $R = -\ln[(t-35)^2+25]+6+\ln 25$ 1A

(4)

(d) $\frac{dR}{dt} = \frac{2(30-t)+10}{(t-35)^2+25}$

$= \frac{70-2t}{t^2-70t+1250}$

$\frac{d^2R}{dt^2} = \frac{(t^2-70t+1250)(-2)-(70-2t)(2t-70)}{(t^2-70t+1250)^2}$ 1M+1A

$= \frac{2t^2-140t+2400}{(t^2-70t+1250)^2}$

When the rate of change of the radiation intensity attains its greatest value, $\frac{d^2R}{dt^2} = 0$.

$2t^2-140t+2400 = 0$

$t = 30$ or 40 (rejected)

t	$0 \le t < 30$	$t = 30$	$30 < t \le 35$
$\frac{d^2R}{dt^2}$	+ve	0	-ve

1M

Hence, the rate of change of the radiation intensity would attain its greatest value when $t = 30$. 1A

(4)

(a)	A common mistake was to mix up R with $\frac{dR}{dt}$. Fair. However, many candidates knew that maximum intensity implied $\frac{dR}{dt} = 0$
(b)	Poor. Some candidates were not able to choose a suitable substitution to solve for R, while others did not go on after substitution or made careless mistakes in further calculations.
(c)	Very poor. A common mistake was $R\big|_{t=41} - R\big|_{t=40} = \ln\frac{61}{50}$.
(d)	Very poor. Only a few candidates attempted this part. Among them, some forgot to square the denominator when applying quotient rule to calculate $\frac{d^2R}{dt^2}$.

22. (SAMPLE DSE-MATH-M1 Q11)

(a) (i) Let $v = 1+4te^{-0.04t}$. Then we have

$\frac{dv}{dt} = 4e^{-0.04t} - 0.16te^{-0.04t}$ 1M+1A 1M for product rule

$= 0.16e^{-0.04t}(25-t)$

(ii) When $t = 0, \frac{dN}{dt} = 50$. So we have $25k = 50$.

Thus, we have $k = 2$. 1A

$N = \int \frac{2(25-t)}{e^{0.04t}+4t} dt$

$= 2\int \frac{e^{-0.04t}(25-t)}{1+4te^{-0.04t}} dt$ 1M

$= \frac{2}{0.16}\int \frac{dv}{v}$ 1M For using (a)(i)

$= 12.5\ln|v|+C$

$= 12.5\ln(1+4te^{-0.04t})+C$

When $t = 0$, $N = 10$. So, we have $C = 10$. 1M For finding C

i.e. $N = 12.5\ln(1+4te^{-0.04t})+10$ 1A

(7)

(b) (i) $\frac{dN}{dt}=\frac{2(25-t)}{e^{0.04t}+4t}$

$\begin{cases} >0 & \text{when } 0\le t<25 \\ =0 & \text{when } t=25 \\ <0 & \text{when } t>25 \end{cases}$ 1M

So, N attains its greatest value when $t=25$. 1A

> Alternative Solution
>
> $\frac{dN}{dt}=\frac{2(25-t)}{e^{0.04t}+4t}$
>
> $\frac{dN}{dt}=0$ when $t=25$
>
> $\frac{d^2N}{dt^2}=2\left[\frac{(e^{0.04t}+4t)(-1)-(0.04e^{0.04t}+4)(25-t)}{(e^{0.04t}+4t)^2}\right]$
>
> $=-4\left[\frac{(1-0.02t)e^{0.04t}+50}{(e^{0.04t}+4t)^2}\right]$ 1M
>
> $\left.\frac{d^2N}{dt^2}\right|_{t=25}=-4\left[\frac{0.5e+50}{(e+100)^2}\right]<0$
>
> So, N attains its greatest value when $t=25$. 1A

(ii) $N(25)=12.5\ln(1+4te^{-0.04t})+10\approx 55.4>50$

Thus, the claim is agreed. 1

(3)

(c) $\lim_{t\to\infty} N=\lim_{t\to\infty}[12.5\ln(1+4te^{-0.04t})+10]$

$=12.5\ln(1+0)+10$ 1M For using $\lim_{t\to\infty} te^{-0.04t}=0$

$=10$

Thus, the belief of Mary's supervisor is agreed. 1

(2)

23. (2008 ASL-M&S Q8)

(a) (i) $N'(t)=\frac{20}{1+he^{-kt}}$ $(t\ge 0)$

$\ln\left[\frac{20}{N'(t)}-1\right]=-kt+\ln h$ 1A

(ii) $\ln h=1.5$ 1M ← Either One

$h=e^{1.5}$

≈ 4.4817 (correct to 4 d.p.) 1A

$-k=\frac{1.5-0}{0-7.6}$ ← Either One

$k=\frac{15}{76}$

≈ 0.1974 (correct to 4 d.p.) 1A

(4)

(b) (i) $v=4.5+e^{0.2t}$

$\frac{dv}{dt}=0.2e^{0.2t}$ 1A

$N(t)=\int\frac{20}{1+4.5e^{-0.2t}}dt$

$=\int\frac{100}{e^{0.2t}+4.5}(0.2e^{0.2t})dt$

$=\int\frac{100}{v}dv$ 1M

$=100\ln|v|+C$

$=100\ln(4.5+e^{0.2t})+C$ $(\because 4.5+e^{0.2t}>0)$ 1A

Since $N(0)=50$, so $C=50-100\ln 5.5$ 1M

i.e. $N(t)=100\ln\frac{4.5+e^{0.2t}}{5.5}+50$ 1A

(ii) (1) $M(20)=N(20)$

$21\left[(20)+\frac{4.5}{0.2}e^{-0.2(20)}\right]+b=100\ln\frac{4.5+e^{0.2(20)}}{5.5}+50$

$b\approx -141.2090$ 1A

(2) Consider $M'(t)-N'(t)$

$=21(1-4.5e^{-0.2t})-\frac{20}{1+4.5e^{-0.2t}}$ 1A For the 1[st] term

$=\frac{1-425.25e^{-0.4t}}{1+4.5e^{-0.2t}}$ 1A

$\therefore M'(t)-N'(t)>0$ when $e^{-0.4t}<\frac{1}{425.25}$ 1M

i.e. $t>\frac{\ln 425.25}{0.4}\approx 15.1317$ 1A

Since $M(20)=N(20)$ and $M(t)-N(t)$ is increasing when $t>20$,

so $M(t)>N(t)$ for $t>20$.

Hence the biologist's claim is correct. 1A Follow through

(11)

(a) (i)		Very good.
(a) (ii)		Very good, though some careless mistakes were found.
(b) (i)		Fair. A number of candidates could not apply substitution to do integration.
(ii) (1)		Fair. Some candidates were hindered by failing to complete (b) (i).
(2)		Poor. Not too many candidates attempted and those who attempted could not make use of the given hint.

24. (2006 ASL-M&S Q8)

(a) $\frac{dv}{dt} = 2t - 6$ — 1A

x

$= \int \frac{30t - 90}{t^2 - 6t + 11}\,dt$

$= 15\int \frac{dv}{v}$ — 1A

$= 15\ln|v| + C$

$= 15\ln(t^2 - 6t + 11) + C \quad (\because t^2 - 6t + 11 = (t-3)^2 + 2 > 0)$

Using the condition that $x = 40$ when $t = 0$, we have $C = 40 - 15\ln 11$. — 1M for finding C

Thus, we have $x = 15\ln(t^2 - 6t + 11) + 40 - 15\ln 11$. — 1A

--------(4)

(b) $15\ln(t^2 - 6t + 11) + 40 - 15\ln 11 = 40$

$15\ln(t^2 - 6t + 11) = 15\ln 11$

$t^2 - 6t + 11 = 11$ — 1M

$t(t-6) = 0$

$t = 6$ or $t = 0$ (rejected)

Therefore, we have $t = 6$. — 1A

Thus, 6 weeks after the start of the plan, the *weekly number of passengers* will be the same as at the start of the plan.

--------(2)

(c) $\frac{dx}{dt} = \frac{30(t-3)}{(t-3)^2 + 2}$

$\begin{cases} < 0 & \text{if } 0 \le t < 3 \\ = 0 & \text{if } t = 3 \\ > 0 & \text{if } t > 3 \end{cases}$ — 1M for testing + 1A

So, x attains its least value when $t = 3$.

The least *weekly number of passengers*

$= 15\ln 2 + 40 - 15\ln 11$

$= 40 - 15\ln\frac{11}{2}$

≈ 14.42877862

≈ 14 thousand — 1A

$\frac{d^2x}{dt^2} = \frac{-30(t^2 - 6t + 7)}{(t^2 - 6t + 11)^2}$

Note that $\frac{dx}{dt} = 0$ when $t = 3$.

$\left.\frac{d^2x}{dt^2}\right|_{t=3} = 15 > 0$ — 1M for testing + 1A

Note that there is only one local minimum.

So, x attains its least value when $t = 3$.

The least *weekly number of passengers*

$= 15\ln 2 + 40 - 15\ln 11$

$= 40 - 15\ln\frac{11}{2}$

≈ 14.42877862

≈ 14 thousand — 1A

--------(3)

(d) By (c), note that the end of the *Recovery Week* corresponds to $t = 3$.

(i) The required change

$= x(4) - x(3)$

$= 15\ln(4^2 - 24 + 11) - 15\ln(3^2 - 18 + 11)$ — 1M

$= 15(\ln 3 - \ln 2)$

$= 15\ln\frac{3}{2}$

≈ 6.081976622

≈ 6 thousand — 1A

The required change

$= \int_3^4 \frac{30t - 90}{t^2 - 6t + 11}\,dt$

$= 15\left[\ln(t^2 - 6t + 11)\right]_3^4$

$= 15\ln(4^2 - 24 + 11) - 15\ln(3^2 - 18 + 11)$ — 1M

$= 15(\ln 3 - \ln 2)$

$= 15\ln\frac{3}{2}$

≈ 6.081976622

≈ 6 thousand — 1A

(ii) $(t+1)^2 - 6(t+1) + 11 - 3(t^2 - 6t + 11)$

$= -2t^2 + 14t - 27$

$= -2\left(t - \frac{7}{2}\right)^2 - \frac{5}{2}$ — 1M accept using discriminant < 0

< 0 — 1

Thus, we have $(t+1)^2 - 6(t+1) + 11 < 3(t^2 - 6t + 11)$ for all t.

Solution	Marks
Note that $t^2-6t+11=(t-3)^2+2>0$.	
$\frac{(t+1)^2-6(t+1)+11}{t^2-6t+11}-3$	
$=\frac{-2t^2+14t-27}{t^2-6t+11}$	
$=\frac{-2\left(t-\frac{7}{2}\right)^2-\frac{5}{2}}{(t-3)^2+2}$	1M accept using discriminant < 0
<0	1
Thus, we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t.	

Solution	Marks
Let $f(t)=(t+1)^2-6(t+1)+11-3(t^2-6t+11)$ for all $t\geq 0$.	
$\frac{df(t)}{dt}=-4t+14$	
$\frac{df(t)}{dt}\begin{cases}>0 & \text{if } 0\leq t<\frac{7}{2}\\ =0 & \text{if } t=\frac{7}{2}\\ <0 & \text{if } t>\frac{7}{2}\end{cases}$	1M for testing
So, $f(t)$ attains its greatest value when $t=\frac{7}{2}$.	
The greatest value of $f(t)$	
$=\frac{-5}{2}$	
<0	1
Thus, we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$ for all t .	

Solution	Marks
(iii) $x(t+1)-x(t)$	
$=15\ln\left((t+1)^2-6(t+1)+11\right)-15\ln(t^2-6t+11)$	
$=15\ln\left(\frac{(t+1)^2-6(t+1)+11}{t^2-6t+11}\right)$	
$<15\ln 3$ (by (d)(ii) and $t^2-6t+11>0$)	1M for using (d)(ii) and taking ln
<25	
Thus, the claim is incorrect.	1A f.t.

Solution	Marks
By (d)(ii), we have $(t+1)^2-6(t+1)+11<3(t^2-6t+11)$	
Note that $(t+1)^2-6(t+1)+11>0$ and $3(t^2-6t+11)>0$.	
$\ln\left((t+1)^2-6(t+1)+11\right)<\ln 3+\ln(t^2-6t+11)$	1M for using (d)(ii) and taking ln
$15\ln\left((t+1)^2-6(t+1)+11\right)-15\ln(t^2-6t+11)<15\ln 3$	
$x(t+1)-x(t)<25$	
Thus, the claim is incorrect.	1A f.t.
	--------(6)

(a)		Very good
(b)		Very good.
(c)		Good. Some candidates were not able to prove that the minimum value is the least value.
(d)(i)		Very good.
(ii)		Not satisfactory. Many candidates did not know how to prove this part.
(iii)		Not satisfactory. Many candidates could not proceed due to failure in proving in (d)(ii).

25. (2005 ASL-M&S Q9)

Solution	Marks
(a) (i) Let $v=2+3te^{-0.02t}$. Then, we have	
$\frac{dv}{dt}=3e^{-0.02t}-\frac{3t}{50}e^{-0.02t}$	1M for product rule or chain rule + 1A
$=\frac{3}{50}(50-t)e^{-0.02t}$	
(ii) When $t=0$, $\frac{dN}{dt}=100$. So, we have $100=\frac{50k}{2}$.	
Thus, we have $k=4$.	1
$N=\int\frac{4(50-t)}{2e^{0.02t}+3t}dt$	
$=\frac{200}{3}\int\frac{dv}{v}$	1M for using (a)(i)
$=\frac{200}{3}\ln v+C$	
$=\frac{200}{3}\ln(2+3te^{-0.02t})+C$	1A
Note that when $t=0$, $N=10$. So, we have $C=10-\frac{200}{3}\ln 2$.	1M for finding C
Thus, we have	
$N=\frac{200}{3}\ln(2+3te^{-0.02t})+10-\frac{200}{3}\ln 2$	1A
$=\frac{200}{3}\ln(1+\frac{3te^{-0.02t}}{2})+10$	
	----------(7)
(b) $\frac{dN}{dt}=\frac{4(50-t)}{2e^{0.02t}+3t}$	
$\begin{cases}>0 & \text{if } 0\leq t<50\\ =0 & \text{if } t=50\\ <0 & \text{if } t>50\end{cases}$	1M for testing + 1A
So, N attains its greatest value when $t=50$.	
Note that $N(50)=\frac{200}{3}\ln(1+\frac{150}{2}e^{-1})+10\approx 233.5393678<500$.	1M for comparing $N(50)$ and 500
Thus, the claim is not correct.	1A f.t.

Solution	Marks
$\frac{dN}{dt}=\frac{4(50-t)}{2e^{0.02t}+3t}$	
$\frac{d^2N}{dt^2}=4\left(\frac{(2e^{0.02t}+3t)(-1)-(50-t)(\frac{2}{50}e^{0.02t}+3)}{(2e^{0.02t}+3t)^2}\right)$	
$=\frac{4}{25}\left(\frac{te^{0.02t}-100e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$	
$\frac{dN}{dt}=0$ when $t=50$. Also, when $t=50$,	
$\frac{d^2N}{dt^2}=\frac{4}{25}\left(\frac{50e-100e-3750}{(2e+150)^2}\right)$	
$=\frac{-2}{e+75}$	
<0	1M for testing + 1A
So, N attains its greatest value when $t=50$.	
Note that $N(50)=\frac{200}{3}\ln(1+\frac{150}{2}e^{-1})+10\approx 233.5393678<500$.	1M for comparing $N(50)$ and 500
Thus, the claim is not correct.	1A f.t.

---------(4)

(c) (i) $\frac{d^2N}{dt^2}=\frac{d}{dt}\left(\frac{4(50-t)}{2e^{0.02t}+3t}\right)$

$=4\left(\frac{(2e^{0.02t}+3t)(-1)-(50-t)(\frac{2}{50}e^{0.02t}+3)}{(2e^{0.02t}+3t)^2}\right)$

$=\frac{4}{25}\left(\frac{te^{0.02t}-100e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$ 1A (accept $\frac{k}{25}\left(\frac{te^{0.02t}-100e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$)

$=\frac{4}{25}\left(\frac{(t-100)e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$

(ii) Note that $\frac{d^2N}{dt^2}=\frac{4}{25}\left(\frac{(t-100)e^{0.02t}-3750}{(2e^{0.02t}+3t)^2}\right)$ (by (c)(i)).

Hence, we have $\frac{d^2N}{dt^2}<0$ for $59\le t\le 92$. — 1M for considering the sign of numerator

So, $\frac{dN}{dt}$ decreases during the 3rd month after the start of the plan. — 1A f.t. — either

Also note that $\frac{dN}{dt}=\frac{4(50-t)}{2e^{0.02t}+3t}<0$ for $59\le t\le 92$.

Therefore, N decreases during the 3rd month after the start of the plan. — 1A f.t.

---------(4)

(a) (i)		Very good.
(ii)		Fair. Many candidates could not make use of (a)(i) to express N in terms of t.
(b)		Not satisfactory. Some candidates did not know how to tackle the problem while some did not check that the stationary point is indeed a maximum point.
(c) (i)		Poor. Many candidates were not familiar with the techniques of differentiation.
(ii)		Poor. Many candidates did not know how to describe the behaviour of N and $\frac{dN}{dt}$. It is important to note that behaviour must be described with respect to t as N and $\frac{dN}{dt}$ are both functions of t.

26. (2002 ASL-M&S Q11)

(a) (i) $G=\int\frac{2t-8}{t^2-8t+20}dt$ — 1A

$=\ln(t^2-8t+20)+C$ — 1A

When $t=0$, $G=50$.

$C=50-\ln 20$ — 1A

$G=\ln(t^2-8t+20)+50-\ln 20$

(ii) For $G=50$,

$\ln(t^2-8t+20)+50-\ln 20=50$ — 1M

$t^2-8t+20=20$

$t^2-8t=0$

$t=0$ or $t=8$.

At the end of the 8th week, the weekly sale is the same as at the start of the promotion plan. — 1A

---------(5)

(b) (i) $\because\ \frac{dG}{dt}=\frac{2t-8}{t^2-8t+20}=\frac{2(t-4)}{[(t-4)^2+4]}$

$\therefore\ \frac{dG}{dt}=0$ when $t=4$

Since $\frac{dG}{dt}<0$ when $t<4$

and $\frac{dG}{dt}>0$ when $t>4$, — 1M (or: $G=\ln(t^2-8t+20)+C$ $=\ln[(t-4)^2+4]+C$)

$\therefore$ G is least at $t=4$. — 1A

At the end of the 4th week, the weekly sale is least.

(ii) $G(6)-G(5)=(\ln 8+50-\ln 20)-(\ln 5+50-\ln 20)$

$=\ln\frac{8}{5}\approx 0.4700$ (thousand dollars) — 1A $a-1$ for more than 4 d.p. or r.t. 0.470

(iii) $G(t+1)-G(t)<0.2$

$\{\ln[(t+1)^2-8(t+1)+20]+50-\ln 20\}$

$-\{\ln(t^2-8t+20)+50-\ln 20\}<0.2$ — 1M

$\ln\frac{t^2-6t+13}{t^2-8t+20}<0.2$ — 1A

$(e^{0.2}-1)t^2-(8e^{0.2}-6)t+(20e^{0.2}-13)>0$ — $0.22140t^2-3.77122t+11.42806>0$

$t<3.94316$ or $t>13.09015$ — $t<3.94315$ or $t>13.09037$

$\because\ \frac{dG}{dt}<0$ when $0<t<4$, G is decreasing

$\therefore\ t<3.94316$ is rejected.

$\therefore\ t=14$.

Thus the promotion plan will be terminated at the end of the 15th week. — 1A must show reasons

---------(6)

$G(t) - G(t-1) < 0.2$

$\{\ln(t^2 - 8t + 20) + 50 - \ln 20\} - \{\ln[(t-1)^2 - 8(t-1) + 20] + 50 - \ln 20\} < 0.2$ 1M

$\ln\dfrac{t^2 - 8t + 20}{t^2 - 10t + 29} < 0.2$ 1A

$(e^{0.2} - 1)t^2 - (10e^{0.2} - 8)t + (29e^{0.2} - 20) > 0$

$0.22140t^2 - 4.21403t + 15.42068 > 0$

$t < 4.94316$ or $t > 14.09015$

(c) $\dfrac{dG}{dt} = \dfrac{2t-8}{t^2 - 8t + 20}$

$\dfrac{d^2G}{dt^2} = \dfrac{2(t^2 - 8t + 20) - (2t-8)(2t-8)}{(t^2 - 8t + 20)^2}$ 1A

$= -\dfrac{2(t-2)(t-6)}{(t^2 - 8t + 20)^2}$

$\dfrac{d^2G}{dt^2} = 0$ when $t = 2$ or $t = 6$. $\therefore$ $t_2 = 6$ 1A

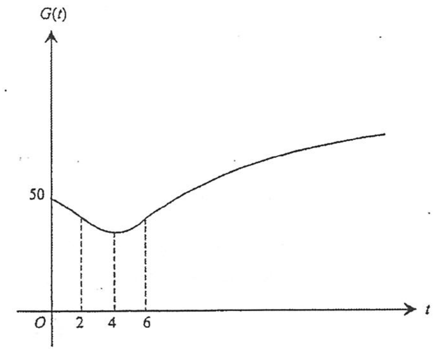
Although G keeps increasing, 1A

$\dfrac{dG}{dt}$ increases immediately before $t = 6$,

$\dfrac{dG}{dt}$ decreases immediately after $t = 6$. 1A

----------(4)

For reference only



	$G(t)$	$\Delta G(t)$
0	50.0	
1.0	49.5692	- 0.4308
2.0	49.0837	- 0.4855
3.0	48.6137	- 0.47
4.0	48.3906	- 0.2231
5.0	48.6137	+ 0.2231
6.0	49.0837	+ 0.47
7.0	49.5692	+ 0.4855
8.0	50.0	+ 0.4308
9.0	50.3716	+ 0.3716
10.0	50.6931	+ 0.3215
11.0	50.9746	+ 0.2815
12.0	51.2238	+ 0.2492
13.0	51.4469	+ 0.2231
14.0	51.6487	+ 0.2018
15.0	51.8326	+ 0.1839
16.0	52.0015	+ 0.1689
17.0	52.1576	+ 0.1561
18.0	52.3026	+ 0.145
19.0	52.438	+ 0.1354
20.0	52.5649	+ 0.1269

6. Definite Integrals

Learning Unit	Learning Objectives
Calculus Area	
Integration with Its Applications	
8. Definite integrals and their applications	8.1 recognise the concept of definite integration 8.2 recognise the Fundamental Theorem of Calculus and understand the properties of definite integrals 8.3 find the definite integrals of algebraic functions and exponential functions 8.4 use integration by substitution to find definite integrals 8.5 use definite integration to find the areas of plane figures 8.6 use definite integration to solve problems
9. Approximation of definite integrals using the trapezoidal rule	9.1 understand the trapezoidal rule and use it to estimate the values of definite integrals

Section A

1. Let m be a non-zero constant.

(a) By considering $\frac{d}{dx}(xe^{mx})$, find $\int xe^{mx}dx$.

(b) If the area of the region bounded by the curve $y=xe^{mx}$, the x-axis and the straight line $x=1$ is $\frac{1}{m}$, find m.

(7 marks) (2020 DSE-MATH-M1 Q8)

2. Define $f(x)=\frac{6-x}{x+3}$ for all $x>-3$.

(a) Prove that $f(x)$ is decreasing.

(b) Find $\lim_{x\to\infty} f(x)$.

(c) Find the exact value of the area of the region bounded by the graph of $y=f(x)$, the x-axis and the y-axis.

(6 marks) (2019 DSE-MATH-M1 Q5)

3. (a) Express $7^{\frac{-1}{\ln 7}}$ in terms of e.

(b) By considering $\frac{d}{dx}(x7^{-x})$, find $\int x7^{-x}dx$.

(c) Define $h(x)=x7^{-x}$ for all real numbers x. It is given that the equation $h'(x)=0$ has only one real root α. Find α. Also express $\int_0^{\alpha} h(x)dx$ in terms of e.

(7 marks) (2019 DSE-MATH-M1 Q8)

4. (a) By considering $\frac{d}{dx}(x\ln x)$, find $\int \ln x dx$.

(b) Find $\int \frac{\ln x}{x}dx$.

(c) Let C be the curve $y=\frac{(x-1)(\ln x-1)}{x}$, where $x>0$. Express, in terms of e, the area of the region bounded by C and the x-axis.

(7 marks) (2018 DSE-MATH-M1 Q8)

5. Define $g(x)=\frac{1}{x}\ln\left(\frac{e}{x}\right)$ for all $x>0$.

(a) Using integration by substitution, find $\int g(x)dx$.

(b) Denote the curve $y=g(x)$ by Γ.

(i) Write down the x-intercept(s) of Γ.

(ii) Find the area of the region bounded by Γ, the x-axis and the straight lines $x=1$ and $x=e^2$.

(7 marks) (2017 DSE-MATH-M1 Q8)

6. Let $f(x)=3^{2x}-10(3^x)+9$.

(a) Find $\int f(x)dx$.

(b) The equation of the curve C is $y=f(x)$. Find

(i) the two x-intercepts of C,

(ii) the exact value of the area of the region bounded by C and the x-axis.

(6 marks) (2016 DSE-MATH-M1 Q6)

7. Define $f(x)=\frac{(\ln x)^2}{x}$ for all $x>0$. Let α and β be the two roots of the equation $f'(x)=0$, where $\alpha>\beta$.

(a) Express α in terms of e. Also find β.

(b) Using integration by substitution, evaluate $\int_{\beta}^{\alpha} f(x)dx$.

(7 marks) (2016 DSE-MATH-M1 Q8)

8. Consider the curves C_1: $y=e^{2x}+e^4$ and C_2: $y=e^{x+3}+e^{x+1}$.

(a) Find the x-coordinates or the two points of intersection of C_1 and C_2.

(b) Express, in terms of e, the area of the region bounded by C_1 and C_2.

(Part b is out of Syllabus) (6 marks) (2015 DSE-MATH-M1 Q6)

9. Evaluate the following definite integrals:

(a) $\int_1^3 \frac{t+2}{t^2+4t+11}dt$,

(b) $\int_1^3 \frac{t^2+3t+9}{t^2+4t+11}dt$.

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(6 marks) (2014 DSE-MATH-M1 Q4)

10. (a) Find $\frac{d}{dx}(x\ln x)$.

(b) Use (a) to evaluate $\int_1^e \ln x dx$.

(4 marks) (2013 DSE-MATH-M1 Q5)

11. The slope of the tangent to a curve S at any point (x, y) on S is given by $\frac{dy}{dx} = e^{2x}$. Let L be the tangent to S at the point $A(0,1)$ on S.
(a) Find the equation of S.
(b) Find the equation of L.
(c) Find the area of the region bounded by S, L and the line $x = 1$.
(Part c is out of Syllabus) (7 marks) (2012 DSE-MATH-M1 Q5)

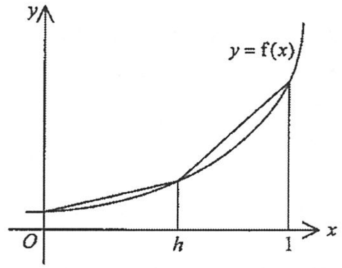
12. Consider the curve C: $y = x(x-2)^{\frac{1}{3}}$ and the straight line L that passes through the origin and is parallel to the tangent to C at $x = 3$.
(a) Find the equation of L.
(b) Find the x-coordinates of the two intersecting points of C and L.
(c) Find the area bounded by C and L.
[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]
(Part c is out of Syllabus) (8 marks) (2013 DSE-MATH-M1 Q3)

13. Consider the two curves C_1: $y = 1 - \frac{e}{e^x}$ and C_2: $y = e^x - e$.
(a) Find the x-coordinates of all the points of intersection of C_1 and C_2.
(b) Find the area of the region bounded by C_1 and C_2.
(Part b is out of Syllabus) (5 marks) (PP DSE-MATH-M1 Q5)

14. L is the tangent to the curve C: $y = x^3 + 7$ at $x = 2$.
(a) Find the equation of the tangent L.
(b) Using the result of (a), find the area bounded by the y-axis, the tangent L and the curve C.
(Part b is out of Syllabus) (7 marks) (SAMPLE DSE-MATH-M1 Q9)

15. The value $R(t)$, in thousand dollars, of a machine can be modelled by
$$R(t) = Ae^{-0.5t} + B,$$
where t (≥ 0) is the time, in years, since the machine has been purchased. At $t = 0$, its value is 500 thousand dollars and in the long run, its value is 10 thousand dollars.
(a) Find the values of A and B.
(b) The machine can generate revenue at a rate of $P'(t) = 600e^{-0.3t}$ thousand dollars per year, where t is the number since the machine has been purchased. Richard purchased the machine for his factory and used it for 5 years before he sold it. How much did he gain in this process? Correct your answer to the nearest thousand dollars.
(6 marks) (2013 ASL-M&S Q3)

16. Let $f(x) = e^{2x}$.
(a) Use trapezoidal rule with 2 intervals of equal width to find the approximate value of $\int_0^1 f(x)dx$.
(b) Evaluate the exact value of $\int_0^1 f(x)dx$.
(c) A student uses trapezoidal rule with 2 trapeziums of unequal widths to approximate $\int_0^1 f(x)dx$. The first trapezium has width h $(0 < h < 1)$ and the second trapezium has width $1 - h$ as shown below. Let A be the total area of the two trapeziums.



(i) Show that $A = \frac{e^{2h} + (1 - e^2)h + e^2}{2}$.
(ii) Find the minimum value of A.
(8 marks) (2010 ASL-M&S Q2)

17. The rate of change of the amount of water in litres flowing into a tanks can be modelled by

$$f(t)=\frac{500}{(t+2)^2e^t},$$

where $t(\geq 0)$ is the time measured in minutes.

(a) Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of water flowing into the tank from $t=1$ to $t=11$.

(b) Find $\frac{d^2f(t)}{dt^2}$.

(c) Determine whether the estimate in (a) is an over-estimate or under-estimate.

(7 marks) (2006 ASL-M&S Q3)

18. (a) Using the trapezoidal rule with 4 sub-intervals, estimate $\int_0^8 te^{\frac{t}{5}}\,dt$.

(b) A researcher modelled the rate of change of the number of certain insects under controlled conditions by

$$\frac{dx}{dt}=4te^{\frac{t}{5}}+\frac{200}{t+1},$$

where x is the number of insects and $t\ (\geq 0)$ is the time measured in weeks. It is known that $x=100$ when $t=0$.

Using the result of (a), estimate the number of insects when $t=8$.

Give your answer correct to 2 significant figures.

(7 marks) (2005 ASL-M&S Q2)

19. Suppose the rate of change of the accumulated bonus, R thousand dollars per month, for a group of salesman can be modeled by

$$R=\frac{1200}{t^2+150}\qquad(0\leq t\leq 6),$$

(a) Use the trapezoidal rule with 4 sub-intervals to estimate the total bonus for the first 6 months in 2001 .

(b) Find $\frac{d^2R}{dt^2}$.

Hence or otherwise, state with reasons whether the approximation in (a) is an overestimate or an underestimate.

(6 marks) (2001 ASL-M&S Q5)

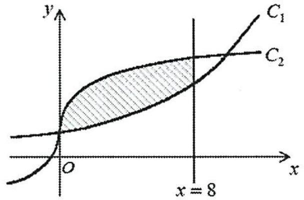
20. The figure shows the graph of the two curves

$C_1: y=e^{\frac{x}{8}}$ and

$C_2: y=1+x^{\frac{1}{3}}$.

Find the area of the shaded region.

(Out of Syllabus) (5 marks) (2000 ASL-M&S Q3)

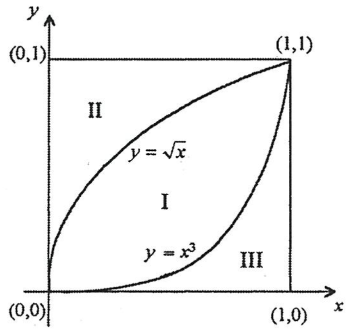


21. The figure shows a unit square target for shooting on the rectangular coordinate plane. The target is divided into three regions I , II and III by the curves $y=\sqrt{x}$ and $y=x^3$. The scores for hitting the regions I , II and III are 10 , 20 and 30 points respectively.

(a) Find the areas of the three regions.

(b) Suppose a child shoots randomly at the target twice and both shots hit the target. Find the probability that he will score 40 points.

(7 marks) (1996 ASL-M&S Q4)



22. The figure shows the graph of the two curves

$C_1: y=2^{2x}+4$ and

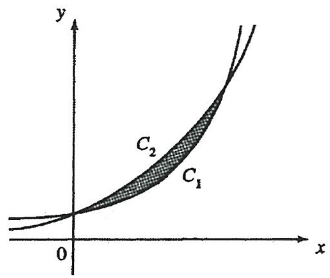
$C_2: y=5(2^x)$.

(a) Find the coordinates of the points of intersection of C_1 and C_2 .

(b) If $2^{2x}=e^{ax}$ for all x, find a.

Hence, or otherwise, find the area of the shaded region in the figure bounded by C_1 and C_2 .

(Part b is out of Syllabus) (8 marks) (1995 ASL-M&S Q6)



23. (a) Use the exponential series to find a polynomial of degree 6 which approximates $e^{\frac{-x^2}{2}}$ for x close to 0 .

Hence estimate the integral $\int_0^1 e^{\frac{-x^2}{2}}\,dx$.

(b) It is known that the area under the standard normal curve between $z=0$ and $z=a$ is $\int_0^a \frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}\,dz$. Use the result of (a) and the normal distribution table to estimate, to 3 decimal places, the value of π .

(7 marks) (1994 ASL-M&S Q6)

Section B

24. Let $f(x)=\frac{e^{0.1x}}{x}$. Define $I=\int_{0.5}^{1} f(x)dx$. In order to estimate the value of I, Ada suggests using trapezoidal rule with 5 sub-intervals while Billy suggests replacing $e^{0.1x}$ with $1+0.1x+0.005x^2$ and then evaluating the integral.

(a) Find the estimates of I according to the suggestions of Ada and Billy respectively.

(5 marks)

(b) Determine each of the two estimates in (a) is an over-estimate or an under-estimate. Explain your answer.

(6 marks)

(c) Someone claims that the difference of I and 0.746 is less than 0.002. Do you agree? Explain your answer.

(2 marks)

(2017 DSE-MATH-M1 Q11)

25. An investment consultant, Albert, predicts the total profit made by a factory in the coming year. He models the rate of change of profit (in million dollars per month) made by the factory by

$$\mathrm{A}(t)=\ln(t^2-8t+95),$$

where t ($0\le t\le 12$) is the number of months elapsed since the prediction begins. Let P_1 million dollars be the total profit made by the factory in the coming year under Albert's model.

(a) (i) Using the trapezoidal rule with 4 sub-intervals, estimate P_1.

(ii) $\frac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}$.

(4 marks)

(b) The factory manager, Christine, models the rate of change of profit (in million dollars per month) made by the factory in the coming year by

$$\mathrm{B}(t)=\frac{t+8}{\sqrt{t+3}},$$

where t ($0\le t\le 12$) is the number of months elapsed since the prediction begins. Let P_2 million dollars be the total profit made by the factory in the coming year under Christine's model.

(i) Find P_2.

(ii) Albert claims that the difference between P_1 and P_2 does not exceed 2. Do you agree? Explain your answer.

(9 marks)

(2016 DSE-MATH-M1 Q11)

26. An engineer models the rates of change of the amount of oil produced (in hundred barrels per day) by oil companies X and Y respectively by

$$f(t)=\ln(e^t-t) \quad\text{and}\quad g(t)=\frac{8t}{1+t},$$

where t $(2\le t\le 12)$ is the time measured in days.

(a) Using the trapezoidal rule with 5 subintervals, estimate the total amount of oil produced by oil company X from $t=2$ to $t=12$.

(3 marks)

(b) Determine whether the estimate in (a) is an over-estimate or an under-estimate. Explain your answer.

(3 marks)

(c) Find $\int\frac{t}{1+t}dt$.

(3 marks)

(d) The engineer claims that the total amount of oil produced by oil company X from $t=2$ to $t=12$ is less than that of oil company Y. Do you agree? Explain your answer.

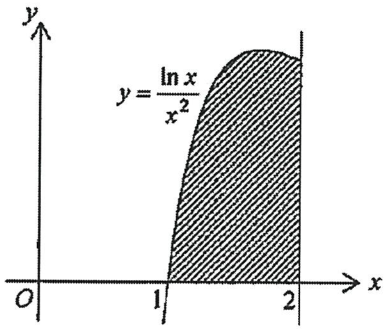
(3 marks) (2015 DSE-MATH-M1 Q11)

27. (a) (i) Find $\frac{d}{dv}(ve^{-v})$.

(ii) Using (a)(i), or otherwise, show that $\int ve^{-v}dv = -e^{-v}(1+v)+C$, where C is a constant.

(3 marks)

(b)



The figure shows a shaded region bounded by the curve $y=\frac{\ln x}{x^2}$, the line $x=2$ and the x-axis. Using a suitable substitution and the result of (a), show that the area of the shaded region is $\frac{1-\ln 2}{2}$.

(5 marks)

(c) (i) Find $\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right)$.

(ii) Using (b) and (c)(i), show that

$$\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\frac{\ln 1.3}{1.3^2}+\cdots+\frac{\ln 1.9}{1.9^2}<5-\frac{41}{8}\ln 2 .$$

(6 marks)

(2014 DSE-MATH-M1 Q10)

28. (a) Consider the function $f(x)=\ln(x^2+16)-\ln(3x+20)$ for $x>\frac{-20}{3}$.

(i) Find the range of values of x such that $f(x)<0$.

(ii) Consider the integral $I=\int_0^4 f(x)dx$.

(1) Using the trapezoidal rule with 4 subintervals, find an estimate for I .

(2) Determine whether the estimate in (1) is an over-estimate or under-estimate. Justify your answer.

(8 marks)

(b) A certain species of insects lives in a certain environment. Let $N(t)$ (in thousand) be the number of the insects at time t (in months). Assume that $N(t)$ can be treated as a differentiable function when $N(t)>0$. The birth rate and death rate of the insects at time t are equal to $10\ln(t^2+16)$ and $10\ln(3t+20)$ respectively when $N(t)>0$. It is given that $N(0)=8$.

(i) Express $N'(t)$ in terms of t when $N(t)>0$.

(ii) Jane claims that the species will not die out until $t=4$. Do you agree? Justify your answer.

(4 marks)

(2013 DSE-MATH-M1 Q10)

29. Let $P(t)$ and $C(t)$ (in suitable units) be the electric energy produced and consumed respectively in a city during the time period $[0,t]$, where t is in years and $t\geq 0$. It is known that $P'(t)=4\left(4-e^{\frac{-t}{5}}\right)$ and $C'(t)=9\left(2-e^{\frac{-t}{10}}\right)$. The redundant electric energy being generated during the time period $[0,t]$ is $R(t)$, where $R(t)=P(t)-C(t)$ and $t\geq 0$.

(a) Find t such that $R'(t)=0$.

(3 marks)

(b) Show that $R'(t)$ decreases with t .

(3 marks)

(c) Find the total redundant electric energy generated during the period when $R'(t)>0$.

(3 marks)

(d) The electric energy production is improved at $t=5$. Let $Q(t)$ be the electric energy produced during the period $[5,t]$, where $t\geq 5$, and

$$Q'(t)=\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}+9 .$$

Find the total electric energy produced for the first 3 years after the improvement.

(5 marks)

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(2013 DSE-MATH-M1 Q11)

30. Let $I=\int_1^4 \frac{1}{\sqrt{t}} e^{\frac{-t}{2}}\,dt$.

(a) (i) Use the trapezoidal rule with 6 sub-intervals to estimate I .

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Justify your answer.

(7 marks)

(b) Using a suitable substitution, show that $I=2\int_1^2 e^{\frac{-x^2}{2}}\,dx$.

(3 marks)

(c) Using the above results and the Standard Normal Distribution Table, show that $\pi<3.25$.

(3 marks)

(2012 DSE-MATH-M1 Q10)

31. An engineer models the rates of the production of an alloy in the first 10 weeks by two new machines A and B respectively by

$$\frac{dx}{dt}=\frac{61t}{(t+1)^{\frac{5}{2}}} \quad \text{and} \quad \frac{dy}{dt}=\frac{15\ln(t^2+100)}{16} \quad \text{for } 0\le t\le 10,$$

where x (in million kg) and y (in million kg) are the amount of the alloy produced by machines A and B respectively, and t (in weeks) is the time elapsed since the beginning of the production.

(a) Using the substitution $u=t+1$, find the amount of the alloy produced by machine A in the first 10 weeks.

(4 marks)

(b) Using the trapezoidal rule with 5 sub-intervals, estimate the amount of the alloy produced by machine B in the first 10 weeks.

(2 marks)

(c) The engineer uses the results of (a) and (b) to claim that machine B is more productive than machine A in the first 10 weeks. Do you agree? Explain your answer.

(4 marks)

(PP DSE-MATH-M1 Q10)

32. (a) Let $f(t)$ be a function defined for all $t\ge 0$. It is given that

$$f'(t)=e^{2bt}+ae^{bt}+8,$$

where a and b are negative constants and $f(0)=0$, $f'(0)=3$ and $f'(1)=4.73$.

(i) Find the values of a and b .

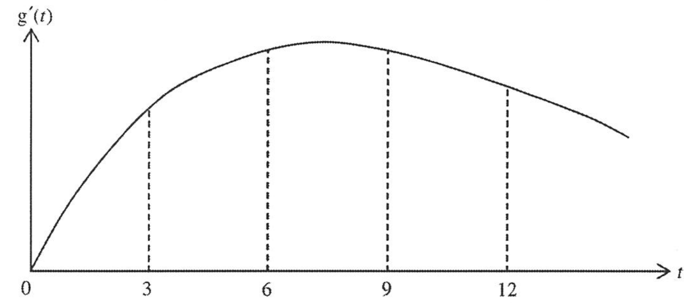
(ii) By taking $b=-0.5$, find $f(12)$.

(5 marks)

(b) Let $g(t)$ be another function defined for all $t\ge 0$. It is given that

$$g'(t)=\frac{33}{10}te^{-kt},$$

where k is a positive constant. Figure 1 shows a sketch of the graph of $g'(t)$ against t. It is given that $g'(t)$ attains the greatest value at $t=7.5$ and $g(0)=0$.



(i) Find the value of k.

(ii) Use the trapezoidal rule with four sub-intervals to estimate $g(12)$.

(6 marks)

(c) From the estimated value obtained in (b)(ii) and Figure 1, Jenny claims that $g(12)>f(12)$. Do you agree? Explain your answer.

(2 marks)

(SAMPLE DSE-MATH-M1 Q12)

33. In a certain country, the daily rate of change of the amount of oil production P , in million barrels per day, can be modelled by

$$\frac{dP}{dt}=\frac{k-3t}{1+ae^{-bt}}$$

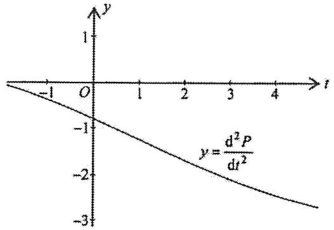
where $t\ (\geq 0)$ is the time measured in days. When $\ln\left(\dfrac{k-3t}{\dfrac{dP}{dt}}-1\right)$ is plotted against t , the graph is a straight line with slope -0.3 and the intercept on the horizontal axis 0.32 . Moreover, P attains its maximum when $t=3$.

(a) Find the values of a , b and k .

(5 marks)

(b) (i) Using trapezoidal rule with 6 subintervals, estimate the total amount of oil production from $t=0$ to $t=3$.

(ii)



The figure shows the graph of $y=\dfrac{d^2P}{dt^2}$. Using the graph, determine whether the estimation in (i) is an under-estimate or an over-estimate.

(4 marks)

(c) The daily rate of change of the demand for oil D , in million barrels per day, can be modelled by

$$\frac{dD}{dt}=1.63^{2-0.1t}$$

where $t\ (\geq 0)$ is the time measured in days.

(i) Let $y=\alpha^{\beta x}$, where α , β ($\alpha>0$, $\alpha\neq 1$ and $\beta\neq 0$) are constants. Find $\dfrac{dy}{dx}$ in terms of x .

(ii) Find the demand of oil from $t=0$ to $t=3$.

(iii) Does the overall oil production meet the overall demand of oil from $t=0$ to $t=3$? Explain your answer.

(6 marks)

(part (c)(i) is out of syllabus) (2013 ASL-M&S Q8)

34. The population size N (in trillion) of a culture of bacteria increases at the rate of $\dfrac{dN}{dt}=t\ln(2t+1)$, where $t\ (\geq 0)$ is the time measured in days.
It is given that $N=21$ when $t=0$.

(a) (i) Find $\displaystyle\int\frac{t^2}{2t+1}dt$.

(ii) Find $\dfrac{d}{dt}[t^2\ln(2t+1)]$.

(iii) Find the population of the culture of bacteria at $t=5$. Correct your answer to the nearest trillion.

(8 marks)

(b) A certain kind of drug is then added to the culture of bacteria at $t=5$. A researcher estimates that the population size M (in trillion) of the bacteria can then be modelled by

$$M=40e^{-2\lambda(t-5)}-20e^{-\lambda(t-5)}+K \qquad (5\leq t\leq 18)\ ,$$

where t is the time measured in days. K and λ are constants. It is given that $M=27$ when $t=11$.

(i) Using (a), find the value of K correct to the nearest integer.
Hence, find the value of λ correct to 1 decimal place.

(ii) By using the value of K correct to the nearest integer and the value of λ correct to 1 decimal place, determine whether M is always decreasing in this model.
Hence, explain whether the population of the bacteria will drop to 23 trillion.

(7 marks)

(2013 ASL-M&S Q9)

35. A textile factory has bought two new dyeing machines P and Q. The two machines start to operate at the same time and will emit sewage into a lake near the factory. The manager of the factory estimates the amount of sewage emitted (in tonnes) by the two machines and finds that the rates of emission of sewage by the two machines P and Q can be respectively modelled by

$$p'(t)=4.5+2t(1+6t)^{\frac{-2}{3}} \text{ and}$$

$$q'(t)=3+\ln(2t+1) ,$$

where $t\ (\geq 0)$ is the number of months that the machines have been in operation.

(a) By using a suitable substitution, find the total amount of sewage emitted by machine P in the first year of operation.

(4 marks)

(b) (i) By using the trapezoidal rule with 5 sub-intervals, estimate the total amount of sewage emitted by machine Q in the first year of operation.

(ii) The manager thinks that the amount of sewage emitted by machine Q will be less than that emitted by machine P in the first year of operation. Do you agree? Explain your answer.

(5 marks)

(c) The manager studies the relationship between the environmental protection tax R (in million dollars) paid by the factory and the amount of sewage x (in tonnes) emitted by the factory. He uses the following model:

$$R=16-ae^{-bx} ,$$

where a and b are constants.

(i) Express $\ln(16-R)$ as a linear function of x.

(ii) Given that the graph of the linear function in (c)(i) passes through the point $(-10,1)$ and the x-intercept of the graph is 90, find the values of a and b.

(iii) In addition to the sewage emitted by the machines P and Q, the other operations of the factory emit 80 tonnes of sewage annually. Using the model suggested by the manager and the values of a and b found in (c)(ii), estimate the tax paid by the factory in the first year of the operation of machines P and Q.

(6 marks)

(2012 ASL-M&S Q8)

36. The current rate of selling of a certain kind of handbags is 30 thousand per day. The sales manager decides to raise the price of the handbags. After the price of the handbags has been raised for t days, the rate of selling of handbags $r(t)$ (in thousand per day) can be modelled by

$$r(t)=20-40e^{-at}+be^{-2at} \qquad (t\geq 0),$$

where a and b are positive constants. From past experience, it is known that after the increase in the price of the handbags, the rate of selling of handbags will decrease for 9 days.

(a) Find the value of b.

(1 mark)

(b) Find the value of a correct to 1 decimal place.

(3 marks)

(c) The sales manager will start to advertise when the rate of change of the rate of selling of handbags reaches a maximum. Use the results obtained in (a) and (b) to find the rate of selling of handbags when the sales manager starts to advertise.

(4 marks)

(d) When the rate of selling of handbags drops below 18 thousand per day, the sales manager will give a 'sales warning' to his team. Use the results obtained in (a) and (b) to find

(i) the duration of the 'sales warning' period correct to the nearest day,

(ii) the average number of handbags sold per day during the 'sales warning' period correct to the nearest thousand.

(7 marks)

(2012 ASL-M&S Q9)

37. An oil tanker leaks out oil for half a day at the rate of

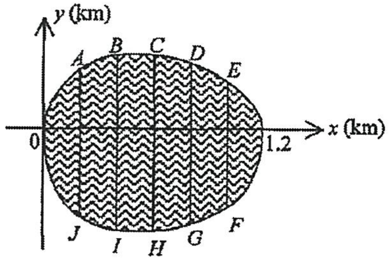
$$\frac{dV}{dt}=\frac{1}{25}e^{t^2+t+2}$$

where V is the volume of the oil (in hundred thousand m^3) leaked out and t ($0\le t\le 0.5$) is the number of days elapsed since the leakage begins.

(a) By finding a polynomial in t of degree 3 which approximates e^{t^2+t}, estimate the volume of the oil leaked out.
Is this an over-estimate or under-estimate? Explain your answer.

(6 marks)

(b) After half a day, the surface area of the ocean affected by the oil spread is as shown by the shaded region in the figure:



$A(0.2, 1.8)$ $B(0.4, 2)$ $C(0.6, 2.2)$ $D(0.8, 2.1)$ $E(1, 1.6)$
$F(1, -1.8)$ $G(0.8, -2)$ $H(0.6, -2.1)$ $I(0.4, -2.2)$ $J(0.2, -2)$

(i) Using the trapezoidal rule, estimate the surface area of the ocean affected.
Is this an over-estimate or under-estimate? Explain your answer.

(ii) Assuming that the thickness of the oil spread is uniform, estimate the thickness of the oil spread.
Is this an over-estimate or under-estimate? Explain your answer.

(5 marks)

(c) Subsequently, the oil company uses a new technology to clean up the oil spread. The rate of cleaning up the oil spread can be modelled by

$$\frac{dW}{dt}=\frac{-(W+1)^{\frac{1}{3}}}{40}$$

where W is the volume of the oil spread (in hundred thousand m^3) remained and t is the number of days elapsed since the beginning of the cleaning up.
How long will it take for all the oil spread to be cleaned up?

(4 marks)

(part (c) is out of syllabus) (2011 ASL-M&S Q8)

38. A company launches a campaign to increase the sales of a product. The monthly increase in sales (in thousand dollars) t months after the launch can be modelled by the function

$$f(t)=-250e^{2at}+300e^{at}-50$$

where a is a non-zero constant.
It is known that the monthly increase in sales attains the maximum 5 months after the launch.

(a) Find the value of a.

(3 marks)

(b) After at least T_1 months, the campaign will not increase the sales.

(i) Find the value of T_1.

(ii) Estimate the total amount of sales increased T_1 months after the launch.

(5 marks)

(c) The start up cost of the campaign is 100 thousand dollars and the running cost at time t is $\frac{100}{t+9}$ thousand dollars. The campaign will be terminated after T_2 months when the total expenditure reaches 200 thousand dollars.

(i) Express the total expenditure E (in thousand dollars) in terms of t.

(ii) Find the value of T_2.

(iii) During the period of the campaign, the manager of the company suggests replacing the campaign by a less costly plan. The monthly increase in sales (in thousand dollars) due to the plan can be modelled by the function

$$g(t)=-(t-\alpha)(t-2\alpha)\ ,\quad \alpha\le t\le 2\alpha$$

where α ($0\le\alpha\le T_2$) is the time, in months after the launching of the original campaign, of starting the plan.
In order to achieve the maximum total amount of sales increased by the plan, when should it be started? Explain your answer.

(7 marks)

(2010 ASL-M&S Q8)

39. A shop owner wants to launch two promotion plans A and B to raise the revenue. Let R and Q (in million dollars) be the respective cumulative weekly revenues of the shop after the launching of the promotion plans A and B . It is known that R and Q can be modelled by

$$\frac{dR}{dt}=\begin{cases}\ln(2t+1) & \text{when } 0\le t\le 6\\ 0 & \text{when } t>6\end{cases},$$

and

$$\frac{dQ}{dt}=\begin{cases}45t(1-t)+\dfrac{1.58}{t+1} & \text{when } 0\le t\le 1\\ \dfrac{30e^{-t}}{(3+2e^{-t})^2} & \text{when } t>1\end{cases}$$

respectively, where t is the number of weeks elapsed since the launching of a promotion plan.

(a) Suppose plan A is adopted.

(i) Using the trapezoidal rule with 6 sub-intervals, estimate the total amount of revenue in the first 6 weeks since the start of the plan.

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Explain your answer briefly.

(4 marks)

(b) Suppose plan B is adopted.

(i) Find the total amount of revenue in the first week since the start of the plan.

(ii) Using the substitution $u=3+2e^{-t}$, or otherwise, find the total amount of revenue in the first n weeks, where $n>1$, since the start of the plan. Express your answer in terms of n .

(6 marks)

(c) Which of the plans will produce more revenue in the long run? Explain your answer briefly.

(5 marks)

(2009 ASL-M&S Q9)

40. The rate of change of yearly average temperature of a city is predicted to be

$$\frac{dx}{dt}=\frac{1}{40}\sqrt{1+t^2} \qquad (t\ge 0),$$

where x is the temperature measured in °C and t is the time measured in years. It is given that $x=22$ and $t=0$.

(a) (i) Using the trapezoidal rule with 4 sub-intervals, estimate the increase of temperature from $t=0$ to $t=10$.

(ii) Determine whether this estimate is an over-estimate or an under-estimate.

(4 marks)

(b) It is known that the electricity consumption $W(x)$, in appropriate units, depends on the yearly average temperature x and is given by

$$W(x)=100(\ln x)^2-630\ln x+1960 \qquad (x\ge 22) .$$

(i) If $W(x_0)=968$, find all possible value(s) of x_0 .

(ii) Find the range of values of x while $W'(x)<0$.

(iii) Find the rate of change of electricity consumption at $t=0$.

(iv) Using (a), estimate the electricity consumption at $t=10$. Determine and explain whether the actual electricity consumption is larger than or smaller than this estimate.

(11 marks)

(2008 ASL-M&S Q9)

41. A financial analyst, Mary, models the rates of change of profit (in billion dollars) made by companies A and B respectively by

$$f(t)=\ln\left(e^t+2\right)+3 \text{ and } g(t)=\frac{8e^t}{40-t^2},$$

where t is the time measures in months.

Assume that the two models are valid for $0\le t\le 6$.

(a) (i) Using the trapezoidal rule with 6 sub-intervals, estimate the total profit made by company A from $t=0$ to $t=6$.

(ii) Find $\frac{d^2f(t)}{dt^2}$ and hence determine whether the estimate in (a)(i) is an over-estimate or an under-estimate.

(7 marks)

(b) (i) Expand $\frac{1}{40-t^2}$ in ascending powers of t as far as the term in t^4 .

(ii) Using the result of (b)(i), find the expansion of $\frac{8e^t}{40-t^2}$ in ascending powers of t as far as the term in t^4 .

(iii) Using the result of (b)(ii), estimate the total profit made by company B from $t=0$ to $t=6$.

(6 marks)

(c) Mary claims that the total profit made by company A from $t=0$ to $t=6$ is less than that of company B . Do you agree? Explain your answer.

(2 marks)

(part (b) is out of Syllabus) (2007 ASL-M&S Q8)

42. An engineer. designed a driving test to compare fuel consumption when different driving tactics are used. The rates of change of fuel consumption in litres when using driving tactics A and B can be modelled respectively by

$$f(t)=\frac{1}{4}t(15-t)e^{\frac{-t}{4}} \quad \text{and}$$

$$g(t)=\frac{1}{145}t(15-t)^2$$

where $t(\ge 0)$ is the time measured in minutes from the start of the test.

(a) Use the trapezoidal rule with 5 sub-intervals to estimate the total fuel consumption from $t=0$ to $t=15$ when using driving tactic A .

(3 marks)

(b) Use integration to find the total fuel consumption from $t=0$ to $t=15$ when using driving tactic B .

(3 marks)

(c) Find the greatest value of $f(t)$, where $0\le t\le 15$.

(5 marks)

(d) (i) Find $\frac{d^2f(t)}{dt^2}$.

(ii) By considering $\frac{d^2f(t)}{dt^2}$, can you determine whether the total fuel consumption from $t=0$ to $t=15$ when using driving tactic A will be less than that of using driving tactic B ? Explain your answer.

(4 marks)

(2004 ASL-M&S Q8)

43. According to the past production record, an oil company manager modelled the rate of change of the amount of oil production in thousand barrels by

$$f(t)=5+2^{-kt+h}\ ,$$

where h and k are positive constants and $t(\geq 0)$ is the time measured in months.

(a) Express $\ln(f(t)-5)$ as a linear function of t .

(1 marks)

(b) Given that the slope and the intercept on the vertical axis of the graph of the linear function in (a) are –0.35 and 1.39 respectively, find the values of h and k correct to 1 decimal place.

(2 marks)

(c) The manager decides to start a production improvement plan and predicts the rate of change of the amount of oil production in thousand barrels by

$$g(t)=5+\ln(t+1)+2^{-kt+h}\ ,$$

where h and k are the values obtained in (b) correct to 1 decimal place, and $t(\geq 0)$ is the time measured in months from the start of the plan.

Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of oil production in thousand barrels from $t=2$ to $t=12$.

(2 marks)

(d) It is known that $g(t)$ in (c) satisfies

$$\frac{d^2g(t)}{dt^2}=p(t)-q(t),\ \text{where}\ q(t)=\frac{1}{(t+1)^2}\ .$$

(i) If $2^t=e^{at}$ for all $t\geq 0$, find a .

(ii) Find $p(t)$.

(iii) It is known that there is no intersection between the curve $y=p(t)$ and the curve $y=q(t)$, where $2\leq t\leq 12$. Determine whether the estimate in (c) is an over-estimate or under -estimate.

(10 marks)

(2003 ASL-M&S Q8)

44. Lactic acid in large amounts is usually formed during vigorous physical exercise, which leads to fatigue. The amount of lactic acid, M , in muscles is measured in m mol/L. A student modelled the rate of change of the amount of lactic acid in his muscles during vigorous physical exercise by

$$\frac{dM}{dt}=\frac{12e^{\frac{2}{3}t}}{3+t}\qquad (0\leq t\leq 4)\ ,$$

where t is the time measured in minutes from the start of the exercise.

(a) The student used the trapezoidal rule with 5 sub-intervals to estimate the amount of lactic acid formed after the first 2.5 minutes of exercise.

(i) Find his estimate.

(ii) Find $\frac{d^2}{dt^2}\left[\frac{12e^{\frac{2}{3}t}}{3+t}\right]$ and hence determine whether his estimate is an over-estimate or an under-estimate.

(5 marks)

(b) The student re-estimated the amount of lactic acid formed by expanding $\frac{12e^{\frac{2}{3}t}}{3+t}$ as a series in ascending powers of t .

(i) Expand $\frac{1}{3+t}$ and hence find the expansion of $\frac{12e^{\frac{2}{3}t}}{3+t}$ in ascending powers of t as far as the term in t^3 .

(ii) By integrating the expansion of $\frac{12e^{\frac{2}{3}t}}{3+t}$ in (i), re-estimate the amount of lactic acid formed after the first 2.5 minutes of exercise.

(7 marks)

(c) The student wanted to predict the amount of lactic acid formed in his muscles after the first 4 minutes of exercise. He decided to use the method in (b) to estimate the amount of lactic acid formed. Briefly explain whether his method was valid.

(3 marks)

(part (b) is out of Syllabus) (2002 ASL-M&S Q9)

45. A department store has two promotion plans, F and G, designed to increase its profit, from which only one will be chosen. A marketing agent forecasts that if x hundred thousand dollars is spent on a promotion plan, the respective rates of change of its profit with respect to x can be modelled by

$$f(x)=16+4xe^{-0.25x} \text{ and } g(x)=16+\frac{6x}{\sqrt{1+8x}} .$$

(a) Suppose that promotion plan F is adopted.

(i) Show that $f(x)\le f(4)$ for $x>0$.

(ii) If six hundred thousand dollars is spent on the plan, use the trapezoidal rule with 6 sub-intervals to estimate the expected increase in profit to the nearest hundred thousand dollars.

(6 marks)

(b) Suppose that promotion plan G is adopted.

(i) Show that $g(x)$ is strictly increasing for $x>0$.
As x tends to infinity, what value would $g(x)$ tend to?

(ii) If six hundred thousand dollars is spent on the plan, use the substitution $u=\sqrt{1+8x}$, or otherwise, to find the expected increase in profit to the nearest hundred thousand dollars.

(7 marks)

(c) The manager of the department store notices that if six hundred thousand dollars is spent on promotion, plan F will result in a bigger profit than G . Determine which plan will eventually result in a bigger profit if the amount spent on promotion increases indefinitely. Explain your answer briefly.

(2 marks)

(2000 ASL-M&S Q9)

46. In a 100 m race, the speeds, S_A m/s and S_B m/s , of two athletes A and B respectively can be modelled by the functions

$$S_A=\frac{256}{9625}\left(\frac{1}{3}t^3-\frac{47}{4}t^2+120t\right)$$

$$\text{and } S_B=\frac{183}{50}te^{-kt} ,$$

where k is a positive constant and t is the time measured from the start in seconds.
It is known that A finishes the race in 12.5 seconds and during the race, A and B attain their respective top speeds at the same time.

(a) Find the top speed of A during the race.

(3 marks)

(b) Find the value of k .

(3 marks)

(c) Suppose the model for B is valid for $0\le t\le 12.5$. Use the trapezoidal rule with 5 sub-intervals to estimate the distance covered by B in 12.5 seconds.

(3 marks)

(d) Find $\frac{d^2S_B}{dt^2}$. Hence or otherwise, state with reasons whether B finishes the race ahead of A or not.

(3 marks)

(e) In the same race, the speed, S_C m/s , of another athlete C is modelled by

$$S_C=\frac{50[\ln(t+2)-\ln 2]}{t+2} .$$

Determine whether or not C is the last one to finish the race among the three athletes.

(3 marks)

(1999 ASL-M&S Q8)

47. Mr. Lee has a fish farm in Sai Kung. Last week, the fish in his farm were affected by a certain disease. An expert told Mr. Lee that the number N of fish in his farm could be modelled by the function

$$N = \frac{5000e^{\lambda t}}{t} \qquad (0 < t < 120) \ ,$$

where λ is a constant and t is the number of days elapsed since the disease began to spread.

(a) Suppose that the numbers of fish will be the same when $t = 15$ and $t = 95$.

(i) Find the value of λ .

(ii) How many days after the start of the spread of the disease will the number of fish decrease to the minimum?

(8 marks)

(b) The day that the number of fish decreased to the minimum is called the *Recovery Day*. It is suggested that from the *Recovery Day*, the fish will begin to gain weight according to the model

$$\frac{dW}{ds} = \frac{3}{5}(e^{-\frac{s}{20}} - e^{-\frac{s}{10}}) \qquad (0 < s < 60) \ .$$

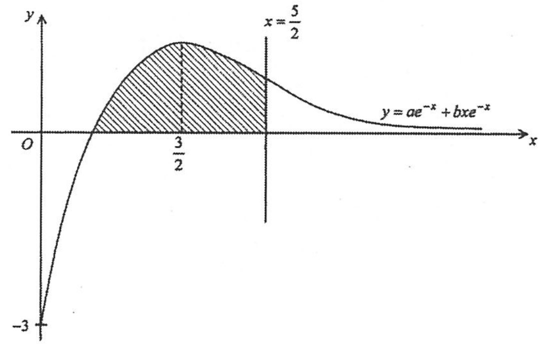
where s is the number of days elapsed since the *Recovery Day* and W is the mean weight of the fish in kg. Find the increase in mean weight of the fish in the first 15 days from the *Recovery Day*. How long will it take for the mean weight of the fish to increase 0.5 kg from the *Recovery Day*?

(7 marks)

(1998 ASL-M&S Q8)

48. The curve in the figure represents the graph of $y = ae^{-x} + bxe^{-x}$ for $x \geq 0$, where a and b are constants. The y-intercept of the curve is -3 and y attains its maximum when $x = \frac{3}{2}$.

Define $I = \int_{\frac{1}{2}}^{\frac{5}{2}} e^{-x}dx$ and $J = \int_{\frac{1}{2}}^{\frac{5}{2}} xe^{-x}dx$.



(a) Evaluate I .

(2 marks)

(b) Find the values of a and b .

(4 marks)

(c) Find the x-intercept and the coordinates of the point(s) of inflection of the curve.

(4 marks)

(d) Let A be the area of the shaded region in Figure 1 bounded by the curve, the x-axis and the line $x = \frac{5}{2}$. Let J_0 be an estimate of J obtained by using the trapezoidal rule with 4 sub-intervals.

A student uses $A_0 = aI + bJ_0$ to estimate A .

(i) Find A_0 .

(ii) The student made the following argument:

Since $\frac{d^2y}{dx^2} < 0$ for $\frac{1}{2} < x < \frac{5}{2}$, the curve is concave downward in the interval, therefore J_0 is an underestimate of J , and hence A_0 is an underestimate of A .

Determine whether the student's argument is correct or not. Explain your answer briefly.

(5 marks)

(part (c) is out of Syllabus) (1998 ASL-M&S Q9)

49. Let $y = x^x$, $I = \int_1^2 x^x dx$ and $J = \int_1^2 x^x \ln x dx$.

(a) Using logarithmic differentiation, find $\frac{dy}{dx}$.

(2 marks)

(b) By finding $\frac{d^2y}{dx^2}$, state whether I would be overestimated or underestimated if the trapezoidal rule is used to estimate I. Explain your answer briefly.

(3 marks)

(c) Using (a) or otherwise, show that $I + J = 3$.

(2 marks)

(d) Let J_0 be an estimate of J obtained by using the trapezoidal rule with 5 sub-intervals.

(i) Find J_0 .

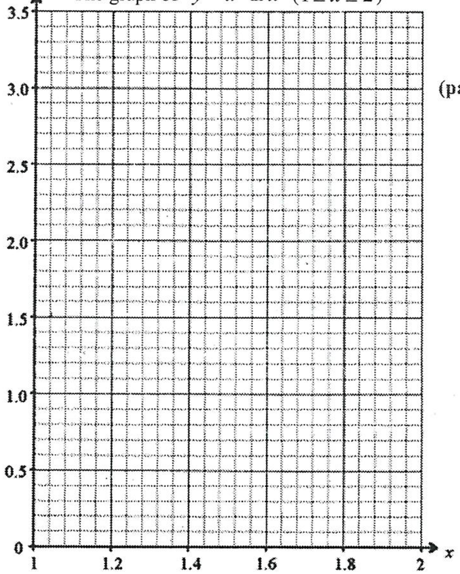
(ii) Plot the graph of $y = x^x \ln x$ on the graph paper. Hence state whether J_0 is an overestimate or underestimate of J. Explain your answer briefly.

(iii) How can the estimation be improved if the trapezoidal rule is applied again to estimate J?

(iv) Let $I_0 = 3 - J_0$. State whether I_0 is an overestimate or underestimate of I. Explain your answer briefly.

(8 marks)

The graph of $y = x^x \ln x$ $(1 \le x \le 2)$



(1997 ASL-M&S Q10)

(part (a) is out of syllabus)

50. The population size P of a species of reptiles living in a jungle increases at a rate of

$$\frac{dP}{dt} = 5e^{\frac{t^2}{10}} - 2t \qquad (t \ge 0),$$

where t is the time in month. It is known that $P = 10$ when $t = 0$.

(a) Use the trapezoidal rule with 6 sub-intervals to estimate $\int_0^6 e^{\frac{t^2}{10}} dt$.

Hence estimate P , to the nearest integer, at $t = 6$.

(7 marks)

(b) A chemical plant was recently built near the jungle. Pollution from the plant affects the growth of the population of the reptiles from $t = 6$ onwards. An ecologist suggests that the population size of the species of reptiles can then be approximated by

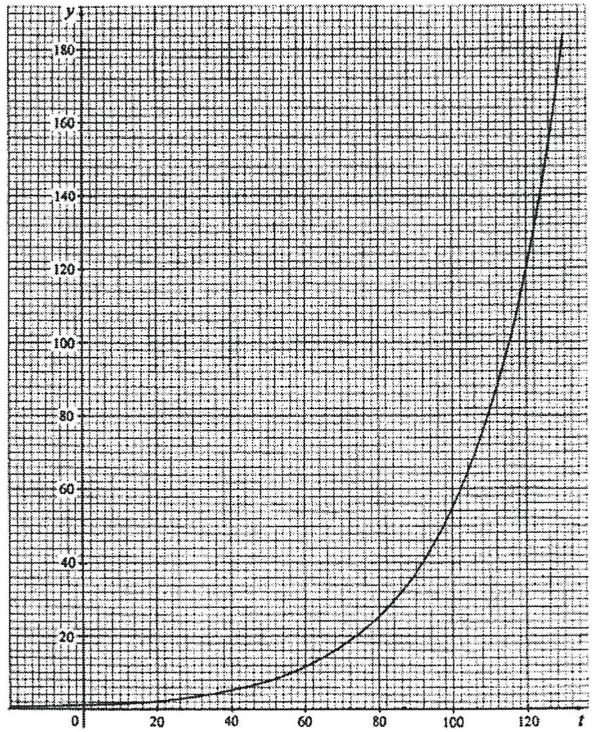
$$P = kte^{-0.04t} - 50 \qquad (t \ge 6).$$

(i) Using (a), find the value of k correct to 1 decimal place.

(ii) Determine the time at which the population size will attain its maximum. Hence find the maximum population size correct to the nearest integer.

(iii) Use the graph in the figure to find the value of t , correct to the nearest integer, when the species of reptiles becomes extinct due to pollution.

The graph of $y = e^{0.04t}$



(8 marks)

(1996 ASL-M&S Q9)

51. The monthly cost $C(t)$ at time t of operating a certain machine in a factory can be modelled by

$$C(t) = ae^{bt} - 1 \qquad (0 < t \leq 36),$$

where t is in month and $C(t)$ is in thousand dollars.

Table 2 shows the values of $C(t)$ when $t = 1, 2, 3, 4$.

Table

t	1	2	3	4
$C(t)$	1.21	1.44	1.70	1.98

(a) (i) Express $\ln[C(t)+1]$ as a linear function of t.

(ii) Use the table and a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(iii) Using the values of a and b found in (a)(ii), estimate the monthly cost of operating this machine when $t = 36$.

(8 marks)

(b) The monthly income $P(t)$ generated by this machine at time t can be modelled by

$$P(t) = 439 - e^{0.2t} \qquad (0 < t \leq 36),$$

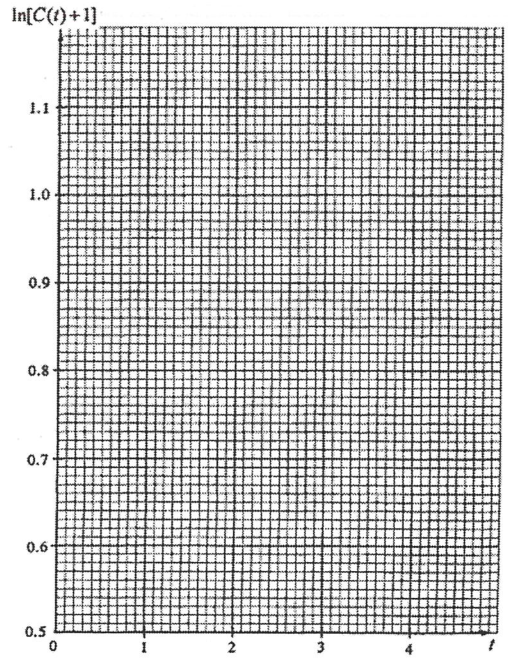
where t is in month and $P(t)$ is in thousand dollars.

The factory will stop using this machine when the monthly cost of operation exceeds the monthly income.

(i) Find the value of t when the factory stops using this machine. Give the answer correct to the nearest integer.

(ii) What is the total profit generated by this machine? Give the answer correct to the nearest thousand dollars.

(7 marks)



(1996 ASL-M&S Q10)

52. Let $f(x)=\frac{1}{\sqrt{1-x^2}}$ where $0\le x\le\frac{1}{2}$, and $I=\int_0^{\frac{1}{2}} f(x)dx$.

(a) (i) Find the estimate I_1 of I using the trapezoidal rule with 5 sub-intervals.

(ii) Find $f'(x)$ and $f''(x)$.

(iii) Using (a)(ii) or otherwise, state whether in (a)(i) is an over-estimate or under-estimate of I . Explain your answer briefly.

(7 marks)

(b) (i) Using the binomial expansion to find a polynomial $p(x)$ of degree 6 which approximates $f(x)$ for $0\le x\le\frac{1}{2}$.

Let $I_2=\int_0^{\frac{1}{2}} p(x)dx$. Find I_2 .

(ii) State whether I_2 in (b)(i) is an over-estimate or under-estimate of I . Explain your answer briefly.

(8 marks)

(part (b) is out of Syllabus) (1995 ASL-M&S Q7)

53. A chemical plant discharges pollutant to be a lake at an unknown rate of $r(t)$ units per month, where t is the number of months that the plant has been in operation.

Suppose that $r(0)=0$.

The government measured $r(t)$ once every two months and reported the following figures:

t	2	4	6	8
$r(t)$	11	32	59	90

(a) Use the trapezoidal rule to estimate the total amount of pollutant which entered the lake in the first 8 months of the plant's operation.

(2 mark)

(b) An environmental scientist suggests that

$$r(t)=at^b,$$

where a and b are constants.

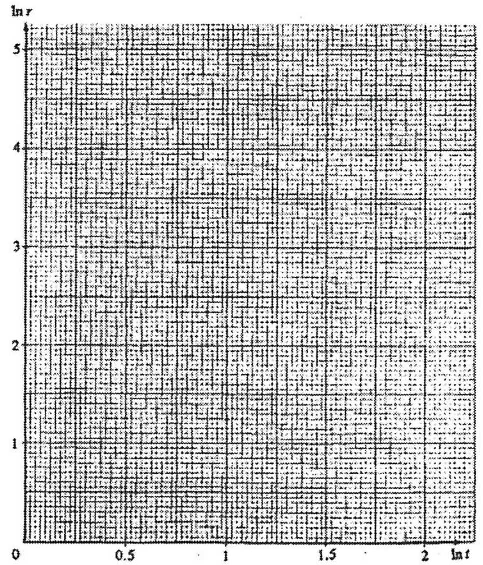
(i) Use a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(ii) Based on this scientist's model, estimate the total amount of pollutant, correct to 1 decimal place, which entered the lake in the first 8 months of the plant's operation.

(8 mark)

(c) It is known that no life can survive when 1000 units of pollutant have entered the lake. Adopting the scientist's model in (b), how long does it take for the pollutant from the plant to destroy all life in the lake? Give your answer correct to the nearest month.

(5 mark)



(1994 ASL-M&S Q10)

2021 DSE Q11

Let $f(x)=\left(\dfrac{x}{2-x}\right)^{\frac{1}{2}}$, where $0\le x\le 1$.

(a) Find $f'(x)$ and $f''(x)$. (3 marks)

(b) Define $J=\int_0^{0.5} f(x)\,dx$ and $K=\int_{0.5}^{1} f(x)\,dx$.

(i) Using the trapezoidal rule with 5 sub-intervals, estimate J .

(ii) Using the fact that $\int_0^1 f(x)\,dx=\dfrac{\pi-2}{2}$ and the result of (b)(i), estimate K .

(iii) Someone claims that $\dfrac{J}{K}<0.44$. Do you agree? Explain your answer.

(8 marks)

6. Definite Integrals

Section A

1. (2020 DSE-MATH-M1 Q8)

8. (a) $\dfrac{d}{dx}(xe^{mx})$		
$=mxe^{mx}+e^{mx}$	1A	
So, we have $xe^{mx}=\dfrac{1}{m}\left(\dfrac{d}{dx}(xe^{mx})-e^{mx}\right)$.		
$\int xe^{mx}\,dx$		
$=\dfrac{1}{m}\left(xe^{mx}-\int e^{mx}\,dx\right)$	1M	
$=\dfrac{xe^{mx}}{m}-\dfrac{e^{mx}}{m^2}+\text{constant}$	1A	
(b) Note that the x-intercept of the curve $y=xe^{mx}$ is 0 .		
$\int_0^1 xe^{mx}\,dx=\dfrac{1}{m}$	1M	
$\left[\dfrac{xe^{mx}}{m}-\dfrac{e^{mx}}{m^2}\right]_0^1=\dfrac{1}{m}$	1M	for using the result of (a)
$\dfrac{e^m}{m}-\dfrac{e^m}{m^2}+\dfrac{1}{m^2}=\dfrac{1}{m}$		
$me^m-e^m-m+1=0$		
$(m-1)(e^m-1)=0$	1M	
$m=1$ or $m=0$ (rejected)		
Thus, we have $m=1$.	1A	
	----------(7)	

2. (2019 DSE-MATH-M1 Q5)

(a) For all $x > -3$,

$\mathrm{f}'(x)$

$= \dfrac{(x+3)(-1)-(6-x)(1)}{(x+3)^2}$

$= \dfrac{-9}{(x+3)^2}$

< 0

Thus, $\mathrm{f}(x)$ is decreasing. — 1

> Note that $\mathrm{f}(x) = \dfrac{9}{x+3} - 1$ for all $x > -3$.
> Thus, $\mathrm{f}(x)$ is decreasing. — 1

(b) $\lim\limits_{x\to\infty} \mathrm{f}(x)$

$= \lim\limits_{x\to\infty} \dfrac{\frac{6}{x}-1}{1+\frac{3}{x}}$

$= -1$ — 1A

> $\lim\limits_{x\to\infty} \mathrm{f}(x)$
> $= \lim\limits_{x\to\infty} \left(\dfrac{9}{x+3} - 1\right)$
> $= -1$ — 1A

(c) For $y = 0$, we have $x = 6$.

The required area

$= \int_0^6 \mathrm{f}(x)\,dx$ — 1M

$= \int_0^6 \dfrac{6-x}{x+3}\,dx$

$= \int_0^6 \left(\dfrac{9}{x+3} - 1\right)dx$ — 1M

$= [9\ln(x+3) - x]_0^6$ — 1M

$= 9\ln 3 - 6$ — 1A

> For $y = 0$, we have $x = 6$.
> The required area
> $= \int_0^6 \mathrm{f}(x)\,dx$ — 1M
> $= \int_0^6 \dfrac{6-x}{x+3}\,dx$
> $= \int_3^9 \dfrac{6-(u-3)}{u}\,du$ (by letting $u = x+3$) — 1M
> $= \int_3^9 \left(\dfrac{9}{u} - 1\right)du$
> $= [9\ln u - u]_3^9$ — 1M
> $= 9\ln 3 - 6$ — 1A

----------(6)

(a)	Good. Some candidates were unable to show that $\mathrm{f}'(x) < 0$ to complete the proof.
(b)	Good. Some candidates were unable to consider $\lim\limits_{x\to\infty} \mathrm{f}(x) = \lim\limits_{x\to\infty} \dfrac{\frac{6}{x}-1}{1+\frac{3}{x}}$ to obtain the required limit.
(c)	Good. Many candidates were able to use integration to obtain the required area, but some candidates were unable to give the answer in exact value.

3. (2019 DSE-MATH-M1 Q8)

(a) $\ln 7^{\frac{-1}{\ln 7}}$

$= \dfrac{-1}{\ln 7}(\ln 7)$

$= -1$

$7^{\frac{-1}{\ln 7}}$

$= e^{-1}$ — 1A

$= \dfrac{1}{e}$

(b) $\dfrac{d}{dx}(x7^{-x})$

$= 7^{-x} - x(7^{-x}\ln 7)$ — 1M — for $\dfrac{d}{dx}(7^{-x}) = -7^{-x}\ln 7$

So, we have $x7^{-x} = \dfrac{1}{\ln 7}\left(7^{-x} - \dfrac{d}{dx}(x7^{-x})\right)$.

$\int x7^{-x}\,dx$

$= \dfrac{1}{\ln 7}\left(\int 7^{-x}\,dx - x7^{-x}\right)$ — 1M

$= \dfrac{1}{\ln 7}\left(\dfrac{-7^{-x}}{\ln 7} - x7^{-x}\right) + \text{constant}$ — 1A

$= \dfrac{-1}{\ln 7}\left(\dfrac{1}{\ln 7} + x\right)7^{-x} + \text{constant}$

(c) For $\mathrm{h}'(x) = 0$, we have $7^{-x}(1 - x\ln 7) = 0$.

So, we have $\alpha = \dfrac{1}{\ln 7}$. — 1A — r.t. 0.5139

$\int_0^{\alpha} \mathrm{h}(x)\,dx$

$= \left[\dfrac{-1}{\ln 7}\left(\dfrac{1}{\ln 7} + x\right)7^{-x}\right]_0^{\frac{1}{\ln 7}}$ — 1M

$= \dfrac{-1}{\ln 7}\left(\dfrac{2\left(7^{\frac{-1}{\ln 7}}\right)}{\ln 7} - \dfrac{1}{\ln 7}\right)$

$= \dfrac{1}{(\ln 7)^2}\left(1 - \dfrac{2}{e}\right)$ (by (a)) — 1A

$= \dfrac{e-2}{e(\ln 7)^2}$

----------(7)

(a)	Good. Many candidates were able to obtain the required answer by taking logarithms.
(b)	Good. Many candidates were able to obtain the required indefinite integral by considering $\dfrac{d}{dx}(x7^{-x})$.
(c)	Fair. Some candidates were able to obtain the required definite integral by using the result of (b).

4. (2018 DSE-MATH-M1 Q8)

Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$

$\frac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	$+$

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

> $f'(x)=0$
>
> $\frac{12x-48}{(3x^2-24x+49)^2}=0$ 1M
>
> $x=4$
>
> $f''(x)$
>
> $=\frac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$
>
> $f''(4)$
>
> $=12$
>
> >0
>
> So, $f(x)$ attains its minimum value at $x=4$.
>
> Thus, we have $\alpha=4$. 1A

(b) (i) Let $v=3x^2-24x+49$. Then, we have $\frac{dv}{dx}=6x-24$.

$f(x)$

$=\int\frac{12x-48}{(3x^2-24x+49)^2}dx$

$=\int\frac{2}{v^2}dv$ 1M

$=\frac{-2}{v}+C$

$=\frac{-2}{3x^2-24x+49}+C$

Since $f(x)$ has only one extreme value, we have $f(4)=5$. 1M

$\frac{-2}{3(4)^2-24(4)+49}+C=5$

$C=7$

Thus, we have $f(x)=\frac{-2}{3x^2-24x+49}+7$. 1A

(ii) $\lim_{x\to\infty}f(x)$

$=7$ 1A

----------(6)

(a)	Very good. Over 85% of the candidates were able to find the value of α .
(b) (i)	Good. Many candidates were able to find $f(x)$ by indefinite integral but some candidates were unable to use a suitable substitution.
(ii)	Fair. Only some candidates were able to find the constant of integration in (b)(i), and thus the required limit.

5. (2017 DSE-MATH-M1 Q8)

(a) Let $u=\ln x$. 1M

So, we have $\frac{du}{dx}=\frac{1}{x}$.

$\int g(x)dx$

$=\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$

$=\int\left(\frac{1}{x}(1-\ln x)\right)dx$

$=\int(1-u)du$ 1M

$=u-\frac{1}{2}u^2+\text{constant}$

$=\ln x-\frac{1}{2}(\ln x)^2+\text{constant}$ 1A

> Let $u=\ln\left(\frac{e}{x}\right)$. 1M
>
> Then, we have $\frac{du}{dx}=\frac{-1}{x}$.
>
> $\int g(x)dx$
>
> $=\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$
>
> $=\int -u\,du$ 1M
>
> $=\frac{-1}{2}u^2+\text{constant}$
>
> $=\frac{-1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2+\text{constant}$ 1A

(b) (i) e 1A

(ii) The required area

$=\int_1^e g(x)dx+\int_e^{e^2}-g(x)dx$ 1M

$=\left[\ln x-\frac{1}{2}(\ln x)^2\right]_1^e+\left[-\ln x+\frac{1}{2}(\ln x)^2\right]_e^{e^2}$ (by (a)) 1M for using (a)

$=\frac{1}{2}+\frac{1}{2}$

$=1$ 1A

> The required area
>
> $=\int_1^e g(x)dx+\int_e^{e^2}-g(x)dx$ 1M
>
> $=\left[\frac{-1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2\right]_1^e+\left[\frac{1}{2}\left(\ln\left(\frac{e}{x}\right)\right)^2\right]_e^{e^2}$ (by (a)) 1M for using (a)
>
> $=\frac{1}{2}+\frac{1}{2}$
>
> $=1$ 1A

----------(7)

(a)	Very good. Most candidates were able to use a correct substitution in finding $\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$.
(b) (i)	Very good. Many candidates were able to write down the x-intercept of Γ . However, some candidates wrongly gave $(e,0)$ instead of e as the answer.
(ii)	Fair. Many candidates were unable to note that part of Γ lies above the x-axis while part of Γ lies below the x-axis.

6. (2016 DSE-MATH-M1 Q6)

(a) $\int f(x)\,dx$ $=\int (3^{2x}-10(3^x)+9)\,dx$ $=\frac{3^{2x}}{2\ln 3}-\frac{10(3^x)}{\ln 3}+9x+\text{constant}$	1M+1A	1M for $\int a^x dx=\frac{a^x}{\ln a}+\text{constant}$
(b) (i) $3^{2x}-10(3^x)+9=0$ $(3^x)^2-10(3^x)+9=0$	1M	
$3^x=1$ or $3^x=9$ $x=0$ or $x=2$ Thus, the x-intercepts are 0 and 2 .	1A	for both
(ii) The area of the region bounded by C and the x-axis $=-\int_0^2 f(x)\,dx$	1M	
$=-\left[\frac{3^{2x}}{2\ln 3}-\frac{10(3^x)}{\ln 3}+9x\right]_0^2$ (by (a)) $=-\left(\frac{81}{2\ln 3}-\frac{90}{\ln 3}+18\right)+\left(\frac{1}{2\ln 3}-\frac{10}{\ln 3}\right)$ $=\frac{40}{\ln 3}-18$	1A	
	----------(6)	

(a)	Fair. Some candidates wrongly evaluated the indefinite integral $\int 3^x dx$ as $\ln 3(3^x)+\text{constant}$ instead of $\frac{3^x}{\ln 3}+\text{constant}$.
(b) (i)	Very good. More than 70% of the candidates were able to find the two x-intercepts of C , while a small number of candidates were unable to write a quadratic equation in 3^x .
(ii)	Fair. Although many candidates were able to use the results of (a) and (b)(i) to find the area of the required region, they were unable to give the answer in exact value. $\frac{e^6}{2}-2e^4-\frac{e^2}{2}$.

7. (2016 DSE-MATH-M1 Q8)

(a) $f'(x)$ $=\frac{x\left(2(\ln x)\frac{1}{x}\right)-(\ln x)^2}{x^2}$	1M	for quotient rule
$=\frac{2\ln x-(\ln x)^2}{x^2}$ $=\frac{(2-\ln x)(\ln x)}{x^2}$ $f'(x)=0$ $\ln x=2$ or $\ln x=0$ $x=e^2$ or $x=1$ $\alpha=e^2$ and $\beta=1$	1A+1A	
(b) Let $u=\ln x$.	1M	
Then, we have $\frac{du}{dx}=\frac{1}{x}$. $\int_\beta^\alpha f(x)\,dx$ $=\int_1^{e^2}\frac{(\ln x)^2}{x}dx$ $=\int_0^2 u^2\,du$	1A	
$=\left[\frac{u^3}{3}\right]_0^2$	1M	
$=\frac{8}{3}$ ≈ 2.666666667 ≈ 2.6667	1A	r.t. 2.6667
	----------(7)	

(a)	Very good. More than 60% of the candidates were able to apply quotient rule or product rule to find $f'(x)$ and hence find the values of α and β by solving the equation $f'(x)=0$, while some candidates wrongly wrote the value of β as 0 instead of 1 .
(b)	Good. Many candidates employed a suitable substitution in evaluating the definite integral $\int_1^{e^2}\frac{(\ln x)^2}{x}dx$.

8. (2015 DSE-MATH-M1 Q6)

Solution	Marks
(a) $e^{2x}+e^4=e^{x+3}+e^{x+1}$	
$(e^x)^2-(e^3+e)e^x+e^4=0$	1M
$(e^x-e)(e^x-e^3)=0$	
$e^x=e$ or $e^x=e^3$	
$x=1$ or $x=3$	1A
Thus, the x-coordinates are 1 and 3.	
(b) The area of the region bounded by C_1 and C_2	
$=\int_1^3 (e^{x+3}+e^{x+1}-(e^{2x}+e^4))\,dx$	1M+1A
$=\left[e^{x+3}+e^{x+1}-\frac{e^{2x}}{2}-e^4x\right]_1^3$	1M
$=\frac{e^6}{2}-2e^4-\frac{e^2}{2}$	1A
	----------(6)

(a)	Good. Many candidates were able to find the x-coordinates of the two points of intersection of C_1 and C_2, while some candidates failed to write a quadratic equation in e^x.
(b)	Good. Some candidates failed to give a simplified answer and left an absolute value sign in the answer, and some candidates got a wrong answer $-\frac{e^6}{2}+2e^4+\frac{e^2}{2}$ instead of $\frac{e^6}{2}-2e^4-\frac{e^2}{2}$.

9. (2014 DSE-MATH-M1 Q4)

Solution	Marks	Remarks
(a) Let $u=t^2+4t+11$.	} 1A	
$du=(2t+4)dt$		
When $t=1, u=16$; when $t=3, u=32$.		
$\int_1^3 \frac{t+2}{t^2+4t+11}dt=\int_{16}^{32}\frac{1}{u}\frac{du}{2}$	1M	OR $\frac{1}{2}\int_{t=1}^{3}\frac{d(t^2+4t+11)}{t^2+4t+11}$
$=\frac{1}{2}[\ln\lvert u\rvert]_{16}^{32}$	1A	
$=\frac{\ln 32-\ln 16}{2}$	1A	
$=\frac{\ln 2}{2}$		OR 0.3466
(b) $\int_1^3 \frac{t^2+3t+9}{t^2+4t+11}dt=\int_1^3\left(1-\frac{t+2}{t^2+4t+11}\right)dt$	1M	
$=[t]_1^3-\int_1^3\frac{t+2}{t^2+4t+11}dt$		
$=2-\frac{\ln 2}{2}$	1A	OR 1.6534
	(6)	

(a)	Very good.
(b)	Poor. Many candidates seemed to have no idea about how to solve the problem.

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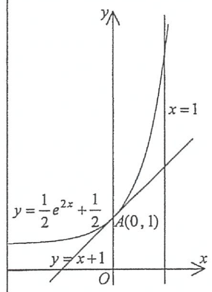
10. (2013 DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) $\frac{d}{dx}(x\ln x)=(1)\ln x+x\left(\frac{1}{x}\right)$		
$=\ln x+1$	1A	
(b) $\ln x=\frac{d}{dx}(x\ln x)-1$		
$\int_1^e \ln x\,dx=[x\ln x]_1^e-\int_1^e 1\,dx$	1M	
$=e\ln e-\ln 1-[x]_1^e$	1A	For x
$=1$	1A	
	(4)	

(a)	Excellent.
(b)	Satisfactory. Some candidates failed to use the result of (a), while some others wrote $x\ln x$ instead of $[x\ln x]_1^e$.

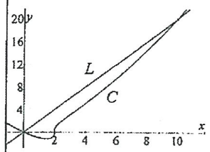
11. (2012 DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) $\frac{dy}{dx}=e^{2x}$		
$y=\frac{1}{2}e^{2x}+C$	1A	
Since $A(0,1)$ lies on S, we have $1=\frac{1}{2}e^{2(0)}+C$.	1M	
i.e. $C=\frac{1}{2}$		
Hence the equation of S is $y=\frac{1}{2}e^{2x}+\frac{1}{2}$.	1A	
(b) At $A(0,1)$, $\frac{dy}{dx}=e^{2(0)}=1$.		
Hence the equation of L is $y-1=1(x-0)$.	1M	
i.e. $y=x+1$	1A	
(c) The area of the region bounded by S, L and the line $x=1$		
$=\int_0^1\left[\left(\frac{1}{2}e^{2x}+\frac{1}{2}\right)-(x+1)\right]dx$	1M	1M for $A=\int_0^1(y_1-y_2)dx$
$=\left[\frac{1}{4}e^{2x}-\frac{1}{2}x^2-\frac{1}{2}x\right]_0^1$		
$=\frac{e^2-5}{4}$	1A	OR 0.5973
	(7)	



(a)	Satisfactory. Some candidates omitted the constant of integration or wrote $\int e^{2x}dx=2e^{2x}+C$ while others mixed S with L.
(b)	Satisfactory. Some candidates treated e^{2x} as the slope of L and wrote $y=e^{2x}x+1$ as the equation of L.
(c)	Poor. Some candidates regarded $y=e^{2x}$ as the equation of S.

12. (2013 DSE-MATH-M1 Q3)

Solution	Marks	Remarks
(a) $y=x(x-2)^{\frac{1}{3}}$		
$\frac{dy}{dx}=(x-2)^{\frac{1}{3}}+\frac{1}{3}(x-2)^{\frac{-2}{3}}x$	1M	For product rule
When $x=3$, $\frac{dy}{dx}=2$.		
Hence the equation of L is $y=2x$.	1A	
(b) Solving C and L :		
$x(x-2)^{\frac{1}{3}}=2x$	1M	
$x\left[(x-2)^{\frac{1}{3}}-2\right]=0$		
$x=0$ or 10	1A	
(c) The area bounded by L and C		
$=\int_0^{10}\left[2x-x(x-2)^{\frac{1}{3}}\right]dx$	1M	
$=\int_0^{10}2x\,dx-\int_0^{10}x(x-2)^{\frac{1}{3}}dx$		
Let $u=x-2$ and so $du=dx$. When $x=0$, $u=-2$; when $x=10$, $u=8$.	1M	
$\therefore$ the area bounded by L and C		
$=\int_0^{10}2x\,dx-\int_{-2}^{8}(u+2)u^{\frac{1}{3}}du$		
$=[x^2]_0^{10}-\int_{-2}^{8}\left(u^{\frac{4}{3}}+2u^{\frac{1}{3}}\right)du$		
$=100-\left[\frac{3}{7}u^{\frac{7}{3}}+\frac{3}{2}u^{\frac{4}{3}}\right]_{-2}^{8}$	1M	For the primitive function
$=\frac{148+9\sqrt[3]{2}}{7}$	1A	OR 22.7628
	(8)	

(a)	Good. Some candidates found the equation of the tangent to C at $x=3$ instead of the equation of L .
(b)	Good. Some candidates did not know how to solve equations with fraction exponents or missed out the root $x=0$ by dividing both sides of an equation by x .
(c)	Fair. Most candidates made mistakes in finding correct primitive functions or calculating the final answer.

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13. (PP DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) $1-\frac{e}{e^x}=e^x-e$		
$(e^x)^2-(e+1)e^x+e=0$	1A	
$e^x=1$ or e		
$x=0$ or 1	1A	
(b) The area of the region bounded by C_1 and C_2		
$=\int_0^1\left[1-\frac{e}{e^x}-(e^x-e)\right]dx$	1M	For lower and upper limits
$=\left[x+e\cdot e^{-x}-e^x+ex\right]_0^1$	1M	Accept $\left[e^x-ex-x-e\cdot e^{-x}\right]_0^1$
$=1+1-e+e-e+1$		
$=3-e$	1A	
	(5)	

(a)	平平。 部分學生不懂分解 $e^{2x}-(e+1)e^x+e=0$ 。
(b)	平平。 部分學生沒有說明積分法正確的上下限。

14. (SAMPLE DSE-MATH-M1 Q9)

Solution	Marks	Remarks
(a) $y=x^3+7$		
$\frac{dy}{dx}=3x^2$	1A	
$\left.\frac{dy}{dx}\right\|_{x=2}=12$		
Hence L is $y-15=12(x-2)$	1M	For point-slope form
i.e. $y=12x-9$	1A	
(b) The area $=\int_0^2\left(x^3+7-12x+9\right)dx$	1M+1M	1M for $A=\int_a^b(y_1-y_2)dx$ 1M for using (a)
$=\left[\frac{x^4}{4}-6x^2+16x\right]_0^2$	1A	For $\frac{x^4}{4}-6x^2+16x$
$=12$	1A	
	(7)	

15. (2013 ASL-M&S Q3)

(a) $R(t)=Ae^{-0.5t}+B$	1M	
$R(t)\to 10$ when $t\to\infty$ $\therefore\ B=10$	1A	
$R(0)=500$ $500=A+B$ $\therefore\ A=490$	1A	
(b) $\int_0^5 P'(t)\,dt+R(5)-R(0)$	1M	
$=\int_0^5 600e^{-0.3t}dt+\left[490e^{-0.5(5)}+10\right]-500$		
$=[-2000e^{-0.3t}]_0^5+490e^{-2.5}-490$	1A	For $[-2000e^{-0.3t}]_0^5$
$=-2000e^{-1.5}+490e^{-2.5}+1510$ ≈ 1104 Hence Richard gains 1104 thousand dollars in the process.	1A	
	(6)	

Good.
In (b), some candidates did not consider the depreciation of the value of the machine in five years.

16. (2010 ASL-M&S Q2)

(a) $\int_0^1 f(x)dx\approx\frac{0.5}{2}(1+e^2+2e)$	1M	
$=\frac{(e+1)^2}{4}$	1A	OR $\frac{e^2+2e+1}{4}$ OR 3.4564
(b) $\int_0^1 f(x)dx=\left[\frac{e^{2x}}{2}\right]_0^1$ $=\frac{e^2-1}{2}$	1A	
(c) (i) $A=\frac{(1+e^{2h})h}{2}+\frac{(e^{2h}+e^2)(1-h)}{2}$ $=\frac{e^{2h}+(1-e^2)h+e^2}{2}$	1	Follow through
(ii) $\frac{dA}{dh}=\frac{2e^{2h}+1-e^2}{2}$	1A	
$\frac{dA}{dh}=0$ when $h=\frac{1}{2}\ln\frac{e^2-1}{2}$	1A	OR 0.5807
$\frac{d^2A}{dh^2}=2e^{2h}>0$	1M	OR by using sign test
Hence A is minimum when $h=\frac{1}{2}\ln\frac{e^2-1}{2}$.		
The minimum value of A is $\frac{3e^2-1}{4}+\frac{1-e^2}{4}\ln\frac{e^2-1}{2}$.	1A	OR 3.4367
	(8)	

Very good. Candidates knew the trapezoidal rule very well. Nevertheless, many candidates ignored the requirement for exact value in (b) and some candidates were not able to make use of the formula of area of trapezium in solving (c).

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17. (2006 ASL-M&S Q3)

(a) The total amount $=\int_1^{11} f(t)\,dt$	1A can be absorbed
$\approx\frac{11-1}{10}\left(f(1)+f(11)+2\left(f(3)+f(5)+f(7)+f(9)\right)\right)$	1M for trapezoidal rule
≈ 22.57906572 ≈ 22.5791 litres	1A $a-1$ for r.t. 22.579
(b) $f(t)=\frac{500}{(t+2)^2e^t}$	
$\frac{df(t)}{dt}=\frac{-500\left(2(t+2)e^t+(t+2)^2e^t\right)}{(t+2)^4e^{2t}}$ $=\frac{-500(t+4)}{(t+2)^3e^t}$	1M for quotient rule
$\frac{d^2f(t)}{dt^2}=-500\left(\frac{(t+2)^3e^t-(t+4)\left(3(t+2)^2e^t+(t+2)^3e^t\right)}{(t+2)^6e^{2t}}\right)$ $=-500\left(\frac{t+2-(t+4)(t+5)}{(t+2)^4e^t}\right)$	
$=500\left(\frac{t^2+8t+18}{(t+2)^4e^t}\right)$	1A or equivalent
$f(t)=500(t+2)^{-2}e^{-t}$	
$\frac{df(t)}{dt}=500(-2)(t+2)^{-3}e^{-t}+500(t+2)^{-2}(-1)e^{-t}$ $=-1000(t+2)^{-3}e^{-t}-500(t+2)^{-2}e^{-t}$	1M for product rule
$\frac{d^2f(t)}{dt^2}=3000(t+2)^{-4}e^{-t}+1000(t+2)^{-3}e^{-t}+1000(t+2)^{-3}e^{-t}+500(t+2)^{-2}e^{-t}$ $=3000(t+2)^{-4}e^{-t}+2000(t+2)^{-3}e^{-t}+500(t+2)^{-2}e^{-t}$	1A or equivalent
(c) Note that $\frac{d^2f(t)}{dt^2}>0$ for all $1\le t\le 11$.	1M for considering the sign of $\frac{d^2f(t)}{dt^2}$
So, $f(t)$ is concave upward on $[1,11]$. Thus, the estimate in (a) is an over-estimate.	1A f.t.
	----------(7)

Fair. Some candidates could not find the second derivative. Some candidates could not make use of the second derivative to determine whether the trapezoidal rule gives an over-estimate or under-estimate.

18. (2005 ASL-M&S Q2)

Solution	Marks
(a) $\int_0^8 te^{\frac{t}{5}}\,dt$	
$\approx \frac{8-0}{2(4)}\left(0+8e^{\frac{8}{5}}+2(2e^{\frac{2}{5}}+4e^{\frac{4}{5}}+6e^{\frac{6}{5}})\right)$	1M for trapezoidal rule
≈ 103.2372887	
≈ 103.2373	1A $a-1$ for r.t. 103.237
(b) $\int_0^8 \frac{dx}{dt}dt = \int_0^8\left(4te^{\frac{t}{5}}+\frac{200}{t+1}\right)dt$	1M for considering $\int_0^8 \frac{dx}{dt}dt$
$x(8)-x(0) = \int_0^8\left(4te^{\frac{t}{5}}+\frac{200}{t+1}\right)dt$	1A
$x(8)-x(0) = 4\int_0^8 te^{\frac{t}{5}}dt + 200\int_0^8 \frac{dt}{t+1}$	
$x(8)-100 \approx 4(103.2372887)+200\int_0^8 \frac{dt}{t+1}$ (by (a))	1M for using (a)
Note that $\int_0^8 \frac{dt}{t+1}$	
$=[\ln(t+1)]_0^8$	1A for $\int\frac{dt}{t+1}=\ln(t+1)+C$
$=\ln 9$	
So, we have $x(8) \approx 952.3940702 \approx 950$ (correct to 2 significant figures).	1A
Thus, the required number is 950.	-----------(7)

Fair. Some candidates were still confusing definite integrals with indefinite integrals.

19. (2001 ASL-M&S Q5)

(a)

t	0	1.5	3	4.5	6
R	8	7.88177	7.54717	7.04846	6.45161

Solution	Marks
	1M correct to 4 d.p.
$\int_0^6 R\,dt \approx \frac{1.5}{2}[8+6.45161+2(7.88177+7.54717+7.04846)]$	1M
≈ 44.5548	1A
$\therefore$ The total bonus for the first 6 months is 44.5548 thousand dollars.	
(b) $\frac{dR}{dt} = \frac{-2400t}{(t^2+150)^2}$	1A
$\frac{d^2R}{dt^2} = \frac{7200(t^2-50)}{(t^2+150)^3}$	1A
<0 for $0 \le t \le 6$	
$\therefore$ The graph of R is concave downward in the interval $0 \le t \le 6$.	
The approximation in (a) is an underestimate.	1M
	-----------(6)

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20. (2000 ASL-M&S Q3)

Solution	Marks
Area of the shaded region $= \int_0^8 (1+x^{\frac{1}{3}}-e^{\frac{x}{8}})\,dx$	1A integrand (accept $e^{\frac{x}{8}}-1-x^{\frac{1}{3}}$); 1A limits; ($pp-1$ for missing dx)
$=\left[x+\frac{3}{4}x^{\frac{4}{3}}-8e^{\frac{x}{8}}\right]_0^8$	1A for $x+\frac{3}{4}x^{\frac{4}{3}}$; 1A for $-8e^{\frac{x}{8}}$
≈ 6.2537 (or $28-8e$)	1A $a-1$ for r.t. 6.254
	-----(5)

21. (1996 ASL-M&S Q4)

Solution	Marks	
(a) Area of regions I & III $= \int_0^1 \sqrt{x}dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$ $\left(=\frac{2}{3}\right)$	1A	Or 0.6667
Area of region III $= \int_0^1 x^3dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$	1A	Or 0.25
Area of region II $= 1-\frac{2}{3} = \frac{1}{3}$	1A	Or 0.3333
Area of region I $= \frac{2}{3}-\frac{1}{4} = \frac{5}{12}$	1A	Or 0.4167
(b) Probability of scoring 40 points $= 2\times\frac{5}{12}\times\frac{1}{4}+(\frac{1}{3})^2$	1M+1M	1M for $2\times\frac{5}{12}\times\frac{1}{4}+p$; 1M for $p+(\frac{1}{3})^2$
$=\frac{23}{72}$ (or 0.3194)	1A	
	(7)	

22. (1995 ASL-M&S Q6)

Solution	Marks
(a) $2^{2x}+4 = 5(2^x)$	
$(2^x)^2-5(2^x)+4 = 0$	
$(2^x-4)(2^x-1) = 0$	1M
$2^x = 4$ or 1	
$x = 2$ or 0	1A
$\therefore$ The intersection points are (0,5) and (2,20)	1A
(b) If $2^x = e^{ax}$ for all values of x, then $a = \ln 2$.	1A
Area $= \int_0^2 [5(2^x) - 2^{2x} - 4]\,dx$	1A
$= \int_0^2 [5e^{x\ln 2} - e^{x\ln 4} - 4]\,dx$	1M
$= \frac{5}{\ln 2}[e^{x\ln 2}]_0^2 - \frac{1}{\ln 4}[e^{x\ln 4}]_0^2 - 4[x]_0^2$	1M
$= 15\left(\frac{1}{\ln 2}-\frac{1}{\ln 4}\right) - 8$	
$= \frac{15}{2\ln 2} - 8$ (or 2.8202)	1A
	(8)

23. (1994 ASL-M&S Q6)

(a) Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ for $x = 0$,

$e^{-\frac{x^2}{2}} = 1 + (-\frac{x^2}{2}) + \frac{1}{2}(-\frac{x^2}{2})^2 + \frac{1}{6}(-\frac{x^2}{2})^3$ 1M

$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$ for $x = 0$. 1A

$\int_0^1 e^{-\frac{x^2}{2}} dx = \int_0^1 (1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48})\, dx$

$= \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right]_0^1$ 1M

$= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336}$

$= 0.8554$ 1A

(b) From the normal distribution table,

$\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{x^2}{2}} dx = 0.3413$ 1A

Hence $\frac{1}{\sqrt{2\pi}} \times 0.8554 = 0.3413$ 1M

$\therefore \pi = \frac{0.8554^2}{2 \times 0.3413^2} = 3.141$ 1A 3.140 for using exact value of (a)

7

Section B

24. (2017 DSE-MATH-M1 Q11)

(a) According to the suggestion by Ada,

I

$\approx \frac{1}{2}\left(\frac{1-0.5}{5}\right)(f(0.5) + f(1) + 2(f(0.6) + f(0.7) + f(0.8) + f(0.9)))$ 1M

[redacted]

≈ 0.7476 1A r.t. 0.7476

According to the suggestion by Billy,

I

$\approx \int_{0.5}^1 \left(\frac{1}{x} + 0.1 + 0.005x\right) dx$ 1M

$= \left[\ln x + 0.1x + 0.0025x^2\right]_{0.5}^1$ 1M

$= \ln 2 + 0.051875$ 1A r.t. 0.7450

[redacted]

≈ 0.7450

----------(5)

(b) $f(x)$

$= \frac{e^{0.1x}}{x}$

$f'(x)$

$= \frac{0.1e^{0.1x}}{x^2}(x - 10)$ 1M

$f''(x)$

$= \frac{0.01e^{0.1x}}{x^3}(x^2 - 20x + 200)$ 1A

$= \frac{0.01e^{0.1x}}{x^3}((x-10)^2 + 100)$ 1M

> 0 for $0.5 \le x \le 1$

Thus, the estimate suggested by Ada is an over-estimate. 1A f.t.

$e^{0.1x} = 1 + 0.1x + \frac{(0.1x)^2}{2!} + \frac{(0.1x)^3}{3!} + \cdots$ 1M

$e^{0.1x} > 1 + 0.1x + 0.005x^2$ for $0.5 \le x \le 1$

$I > \int_{0.5}^1 \left(\frac{1}{x} + 0.1 + 0.005x\right) dx$

Thus, the estimate suggested by Billy is an under-estimate. 1A f.t.

----------(6)

(c) $0.7450 < I < 0.7476$ 1M

$-0.0010 < I - 0.746 < 0.0016$

So, we have $-0.002 < I - 0.746 < 0.002$.

Thus, the claim is agreed. 1A f.t.

$I - 0.746 < 0.7476 - 0.746 = 0.0016$ $0.746 - I < 0.746 - 0.7450 = 0.0010$ So, the difference of I and 0.746 is less than 0.002 . Thus, the claim is agreed.	1M 1A	 f.t.

----------(2)

(a)	Very good. Most candidates were able to use correct sub-intervals when applying the trapezoidal rule to find an estimate of I.
(b)	Fair. Many candidates were unable to find $\frac{d^2f(t)}{dt^2}$ correctly, hence they were unable to determine the nature of the estimate according to the suggestion of Ada in (a).
(c)	Poor. Most candidates did not prove that one of the estimates in (a) is an over-estimate while the other is an under-estimate, hence they were unable to finish the argument.

25. (2016 DSE-MATH-M1 Q11)

Solution	Marks	Remarks
(a) (i) P_1		
$=\int_0^{12} A(t)\,dt$		
$\approx \frac{1}{2}\left(\frac{12-0}{4}\right)(A(0)+A(12)+2(A(3)+A(6)+A(9)))$	1M	
≈ 54.61085671		
≈ 54.6109	1A	r.t. 54.6109
(ii) $\frac{dA(t)}{dt}$		
$=\frac{2t-8}{t^2-8t+95}$	1A	
$\frac{d^2A(t)}{dt^2}$		
$=\frac{2(t^2-8t+95)-(2t-8)^2}{(t^2-8t+95)^2}$		
$=\frac{-2t^2+16t+126}{(t^2-8t+95)^2}$	1A	
$=\frac{-2(t^2-8t-63)}{(t^2-8t+95)^2}$		
	----------(4)	
(b) (i) Let $u=t+3$.	1M	
Then, we have $\frac{du}{dt}=1$.		
P_2		
$=\int_0^{12} B(t)\,dt$	1M	
$=\int_0^{12} \frac{t+8}{\sqrt{t+3}}\,dt$		
$=\int_3^{15} \frac{u-3+8}{\sqrt{u}}\,du$	1A	
$=\int_3^{15} (u^{\frac{1}{2}}+5u^{\frac{-1}{2}})\,du$		
$=\left[\frac{2}{3}u^{\frac{3}{2}}+10u^{\frac{1}{2}}\right]_3^{15}$	1M	
$=20\sqrt{15}-12\sqrt{3}$	1A	r.t. 56.6751
≈ 56.67505723		
≈ 56.6751		

Solution	Marks	Remarks
(b) (ii) $\frac{d^2A(t)}{dt^2}=\frac{-2[t-(4-\sqrt{79})][t-(4+\sqrt{79})]}{(t^2-8t+95)^2}$	1M	1M for considering $\frac{d^2A(t)}{dt^2}$
Note that $4-\sqrt{79}<0$ and $4+\sqrt{79}>12$.		
Therefore, we have $\frac{(t-(4-\sqrt{79}))(t-(4+\sqrt{79}))}{(t^2-8t+95)^2}<0$ for $0\le t\le 12$.		
Hence, we have $\frac{d^2A(t)}{dt^2}>0$ for $0\le t\le 12$.	1A	f.t.
So, the estimate of P_1 is an over-estimate. $P_1<54.61085671$.		
P_2-P_1		
$=20\sqrt{15}-12\sqrt{3}-P_1$		
$>20\sqrt{15}-12\sqrt{3}-54.61085671$	1M	
≈ 2.064200523		
>2		
Thus, the claim is disagreed.	1A	f.t.
	----------(9)	

(a) (i)	Very good. More than 60% of the candidates were able to find the correct answer using trapezoidal rule. However, a small number of candidates were unable to use the correct sub-intervals when applying the trapezoidal rule.
(ii)	Good. Many candidates were able to find $\frac{dA(t)}{dt}$ by quotient rule, but some candidates were unable to simplify $\frac{d^2A(t)}{dt^2}$.
(b) (i)	Very good. Most candidates were able to formulate and evaluate the definite integral $\int_0^{12}\frac{t+8}{\sqrt{t+3}}dt$ by using a suitable substitution.
(ii)	Poor. Most candidates just mentioned $\frac{d^2A(t)}{dt^2}>0$ without proof. They showed difficulties in using inequality to express the relation between P_1 and its over-estimate, hence unable to complete the argument.

26. (2015 DSE-MATH-M1 Q11)

(a) The total amount of oil produced by oil company X

$= \int_2^{12} \mathrm{f}(t)\,\mathrm{d}t$ — 1M

$\approx \frac{1}{2}\left(\frac{12-2}{5}\right)(\mathrm{f}(2)+\mathrm{f}(12)+2(\mathrm{f}(4)+\mathrm{f}(6)+\mathrm{f}(8)+\mathrm{f}(10)))$ — 1M

≈ 69.49587529

≈ 69.4959 hundred barrels — 1A — r.t. 69.4959

----------(3)

(b) $\frac{\mathrm{d}\mathrm{f}(t)}{\mathrm{d}t}$

$= \frac{e^t-1}{e^t-t}$ — 1A

$\frac{\mathrm{d}^2\mathrm{f}(t)}{\mathrm{d}t^2}$

$= \frac{(e^t-t)e^t-(e^t-1)(e^t-1)}{(e^t-t)^2}$ — 1A

$= \frac{e^t(2-t)-1}{(e^t-t)^2}$

< 0 (since $2 \le t \le 12$)

Thus, the estimate in (a) is an under-estimate. — 1A — f.t.

----------(3)

(c) Let $u = 1+t$. Then, we have $\frac{\mathrm{d}u}{\mathrm{d}t} = 1$. — 1M

$\int \frac{t}{1+t}\mathrm{d}t$

$= \int \frac{u-1}{u}\mathrm{d}u$

$= \int \left(1-\frac{1}{u}\right)\mathrm{d}u$ — 1A

$= u - \ln u + \text{constant}$

$= t - \ln(1+t) + \text{constant}$ — 1A

Note that $\frac{t}{1+t} = 1 - \frac{1}{1+t}$.	1A
$\int \frac{t}{1+t}\mathrm{d}t$	
$= \int \left(1-\frac{1}{1+t}\right)\mathrm{d}t$	1M
$= t - \ln(1+t) + \text{constant}$	1A

----------(3)

(d) The total amount of oil produced by oil company Y

$= 8\int_2^{12} \frac{t}{1+t}\mathrm{d}t$

$= 8\left[t - \ln(1+t)\right]_2^{12}$ (by (c)) — 1M — for using the result of (c)

≈ 68.26930345 — 1A

< 69.49587529

By (b), the claim is disagreed. — 1A — f.t.

----------(3)

(a)	Good. Some candidates did not formulate the required amount as a definite integral, and some candidates did not use the correct number of sub-intervals when applying the trapezoidal rule.
(b)	Fair. Many candidates failed to find $\frac{\mathrm{d}^2\mathrm{f}(t)}{\mathrm{d}t^2}$ correctly, as a result, many candidates failed to determine the nature of the estimate in (a).
(c)	Very good. Most candidates were able to find the indefinite integral by using the method of substitution.
(d)	Good. Many candidates were able to find the correct total amount of oil produced by company Y for comparison. However, some candidates failed to show that the estimate in (a) is an under-estimate and hence could not complete the argument.

DSE Mathematics Module 1 — 6. Definite Integrals

27. (2014 DSE-MATH-M1 Q10)

(a) (i) $\frac{\mathrm{d}}{\mathrm{d}v}(ve^{-v}) = e^{-v} - ve^{-v}$ — 1A

(ii) $ve^{-v} = e^{-v} - \frac{\mathrm{d}}{\mathrm{d}v}ve^{-v}$

$\int ve^{-v}\mathrm{d}v = \int e^{-v}\mathrm{d}v - ve^{-v}$ — 1M

$= -e^{-v} - ve^{-v} + C$

$= -e^{-v}(1+v) + C$ — 1

(3)

(b) The area of the shaded region $= \int_1^2 \frac{\ln x}{x^2}\mathrm{d}x$ — 1A

Let $x = e^u$.

$\mathrm{d}x = e^u\mathrm{d}u$

When $x=1$, $u=0$; when $x=2$, $u=\ln 2$ — 1A — OR $u = \ln x$

$\therefore$ the area $= \int_0^{\ln 2} \frac{u}{e^{2u}} \cdot e^u\mathrm{d}u$ — 1M

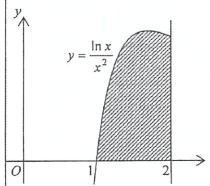
$= \int_0^{\ln 2} ue^{-u}\mathrm{d}u$

$= \left[-e^{-u}(1+u)\right]_0^{\ln 2}$ by (a) — 1M

$= \frac{-1}{2}(1+\ln 2)+1$

$= \frac{1-\ln 2}{2}$ — 1

(5)



(c) (i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{x^2}\right) = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{(x^2)^2}$ — 1M

$= \frac{1-2\ln x}{x^3}$

$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\ln x}{x^2}\right) = \frac{x^3 \cdot \frac{-2}{x} - (1-2\ln x)3x^2}{x^6}$

$= \frac{6\ln x - 5}{x^4}$ — 1A

(ii) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\ln x}{x^2}\right) < 0$ when $x < e^{\frac{5}{6}} \approx 2.30098$ — 1A — OR when $1 \le x \le 2$

Hence the trapezoidal rule will underestimate $\int_1^2 \frac{\ln x}{x^2}\mathrm{d}x$. — 1A

Consider the trapezoidal rule with 10 intervals.

$\therefore \frac{1}{2}\cdot\frac{1}{10}\left[\frac{\ln 1}{1^2} + 2\left(\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\cdots+\frac{\ln 1.9}{1.9^2}\right)+\frac{\ln 2}{2^2}\right] < \frac{1-\ln 2}{2}$ — 1M — For L.H.S.

$0 + 2\left(\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\cdots+\frac{\ln 1.9}{1.9^2}\right)+\frac{\ln 2}{4} < 10 - 10\ln 2$

$\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\frac{\ln 1.3}{1.3^2}+\cdots+\frac{\ln 1.9}{1.9^2} < 5 - \frac{41}{8}\ln 2$ — 1

(6)

(a)	(i)	Excellent.
	(ii)	Fair. Some candidates wrote nothing but only the expression provided in the question.
(b)		Fair. Some candidates failed to write the correct integral for the area. Some omitted details showing why $[-e^{-u}(1+u)]_0^{\ln 2} = \frac{1-\ln 2}{2}$.
(c)	(i)	Satisfactory. Careless mistakes prevented some candidates from obtaining the correct answer.
	(ii)	Very poor. Most candidates did not attempt this part.

28. (2013 DSE-MATH-M1 Q10)

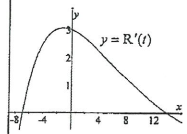
Solution	Marks	Remarks
(a) (i) $\ln(x^2+16)-\ln(3x+20)<0$		
$\ln(x^2+16)<\ln(3x+20)$		
$x^2+16<3x+20$	1A	
$x^2-3x-4<0$		
$-1<x<4$	1A	
(ii) (1) $I=\int_0^4[\ln(x^2+16)-\ln(3x+20)]dx$		
$\approx\frac{1}{2}[-0.223143551+0+2(-0.302280871-0.262364264-0.148420005)]$	1M	OR $\frac{1}{2}\{(\ln16-\ln20)+0+2[(\ln17-\ln23)+(\ln20-\ln26)+(\ln25-\ln29)]\}$
≈-0.824636917		
≈-0.8246	1A	
(2) $f(x)=\ln(x^2+16)-\ln(3x+20)$		
$f'(x)=\frac{2x}{x^2+16}-\frac{3}{3x+20}$	1M+1A	OR $\frac{3x^2+40x-48}{(x^2+16)(3x+20)}$
$f''(x)=2\cdot\frac{(x^2+16)-x(2x)}{(x^2+16)^2}-3\cdot\frac{(-1)\cdot3}{(3x+20)^2}$	1M	
$=\frac{2(4+x)(4-x)}{(x^2+16)^2}+\frac{9}{(3x+20)^2}$		
>0 for $0\le x\le4$		
Hence the estimate in (1) is an over-estimate.	1A	Follow through
	(8)	
(b) (i) $N'(t)=10\ln(t^2+16)-10\ln(3t+20)$	1A	
(ii) Assume that Jane's claim is true: the species will not die out until $t=4$, i.e. $N(t)>0$ for $0\le t\le4$.		
$N(4)-N(0)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$	1A	
$N(4)-8<-8.24636917$ (since the estimate is an over-estimate)	1M	
$N(4)<0$		
Hence Jane's claim is false and cannot be agreed with.	1A	Follow through
	(4)	

(a) (i)		Satisfactory. Some candidates were not able to solve the inequality $\ln(x^2+16)-\ln(3x+20)<0$. Some considered $[\ln(x^2+16)-\ln(3x+20)]'<0$ instead.
(ii)	(1)	Satisfactory. Some candidates did not apply the formula for trapezoidal rule correctly. Some found the absolute value of I instead.
	(2)	Satisfactory. After obtaining $f'(x)$, many candidates got a point $x_0\in[0,4]$ such that $f(x)$ would decrease on $[0,x_0)$ and increase on $(x_0,4]$, and then claimed immediately that the estimate in (1) was an over-estimate. Among those who were able to find $f''(x)$, only few showed that $f''(x)>0$ for all $x\in[0,4]$ correctly.
(b) (i)		Poor. A common mistake was to write $N(t)=10\ln(t^2+16)-10\ln(3t+20)$ and then to differentiate both sides of it.
(ii)		Very poor. Common mistakes included writing $N(t)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$ and failing to apply the result of (a)(ii)(2) in additional to that of (a)(ii)(1).

DSE Mathematics Module 1 — 6. Definite Integrals

29. (2013 DSE-MATH-M1 Q11)

Solution	Marks	Remarks
(a) $R'(t)=0$		
$P'(t)-C'(t)=0$		
$4(4-e^{\frac{-t}{5}})-9(2-e^{\frac{-t}{10}})=0$	1A	
$-4\left(e^{\frac{-t}{10}}\right)^2+9e^{\frac{-t}{10}}-2=0$	1M	For $e^{\frac{-t}{5}}=\left(e^{\frac{-t}{10}}\right)^2$
$e^{\frac{-t}{10}}=0.25$ or 2		
$t=20\ln2$ or $-10\ln2$ (rejected as $t\ge0$)	1A	OR $t\approx13.8629$
	(3)	
(b) $R'(t)=-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2$		
$R''(t)=\frac{4}{5}e^{\frac{-t}{5}}-\frac{9}{10}e^{\frac{-t}{10}}$	1A	
$=\frac{1}{10}e^{\frac{-t}{10}}\left(8e^{\frac{-t}{10}}-9\right)$	1M	
<0 for $t\ge0$ (since $e^{\frac{-t}{10}}\le1$ for $t\ge0$)		
Therefore $R'(t)$ decreases with t.	1	
	(3)	



Solution	Marks	Remarks
(c) By (a) and (b), $R'(t)>0$ when $0\le t<20\ln2$.		
The total redundant electric energy generated during the period when $R'(t)>0$		
$=\int_0^{20\ln2}\left(-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2\right)dt$	1M	For lower and upper limits
$=\left[20e^{\frac{-t}{5}}-90e^{\frac{-t}{10}}-2t\right]_0^{20\ln2}$	1A	For primitive function
$=48.75-40\ln2$	1A	OR 21.0241
	(3)	
(d) Consider $\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt$		
Let $u=\ln(t^2+2t+3)$.	1M	
$du=\frac{2t+2}{t^2+2t+3}dt$	1A	
When $t=5$, $u=\ln38$; when $t=8$, $u=\ln83$.		
$\therefore\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt=\int_{\ln38}^{\ln83}u^3\frac{du}{2}$	1A	For $\frac{u^3}{2}$
$=\frac{1}{8}\left[u^4\right]_{\ln38}^{\ln83}$		
$=\frac{1}{8}[(\ln83)^4-(\ln38)^4]$		
Hence the total electric energy produced for the first 3 years after the improvement		
$=\int_5^8\left[\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}+9\right]dt$	1A	
$=\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt+\int_5^8 9\,dt$		
$=\frac{1}{8}[(\ln83)^4-(\ln38)^4]+[9t]_5^8$		
$=\frac{1}{8}[(\ln83)^4-(\ln38)^4]+27$	1A	OR 52.7730
	(5)	

(a)	Fair. Some candidates confused $R(t)$ with $R'(t)$, or found $R(t)=P(t)-C(t)$ by integration first and then obtained the expression for $R'(t)=P'(t)-C'(t)$ by differentiation. Many candidates failed to make use of knowledge about quadratic equations to solve for t. Some got wrong answers such as '$e^{\frac{-t}{5}}=0.25$ or 2' or did not reject $t=-10\ln 2$.
(b)	Very poor. Many candidates failed to find $R''(t)$ correctly. Among those who were able to find $R''(t)$, only few provided sufficient reasons to conclude that '$R'(t)$ decreases with t'.
(c)	Very poor. Common mistakes included putting wrong values as limits of the definite integral involved and getting wrong primitives of its integrand.
(d)	Poor. Few candidates were able to use correctly the method of substitution for integration. Among them, some put wrong values as limits of definite integrals, while some others missed out the term $\int_5^8 9\,\mathrm{d}t$ or wrote $\int_{\ln 38}^{\ln 83} 9\,\mathrm{d}t$.

30. (2012 DSE-MATH-M1 Q10)

(a) (i)

$$I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,\mathrm{d}t$$

$$=\frac{1}{2}\cdot\frac{4-1}{6}\left[\frac{1}{\sqrt{1}}e^{\frac{-1}{2}}+\frac{1}{\sqrt{4}}e^{\frac{-4}{2}}+2\left(\frac{1}{\sqrt{1.5}}e^{\frac{-1.5}{2}}+\frac{1}{\sqrt{2}}e^{\frac{-2}{2}}+\frac{1}{\sqrt{2.5}}e^{\frac{-2.5}{2}}+\frac{1}{\sqrt{3}}e^{\frac{-3}{2}}+\frac{1}{\sqrt{3.5}}e^{\frac{-3.5}{2}}\right)\right]$$ 1M

≈ 0.692913377

≈ 0.6929 1A

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)=\frac{-1}{2}t^{\frac{-3}{2}}e^{\frac{-t}{2}}+t^{\frac{-1}{2}}\cdot\frac{-1}{2}e^{\frac{-t}{2}}$$ 1M+1A

$$=\frac{-1}{2}e^{\frac{-t}{2}}\left(t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)=\frac{-1}{2}\left[e^{\frac{-t}{2}}\left(\frac{-3}{2}t^{\frac{-5}{2}}+\frac{-1}{2}t^{\frac{-3}{2}}\right)+\frac{-1}{2}e^{\frac{-t}{2}}\left(t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)\right]$$ 1M+1A

$$=\frac{1}{4}e^{\frac{-t}{2}}\left(3t^{\frac{-5}{2}}+2t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)$$

>0 for $1\le t\le 4$.

Hence the estimation in (i) is an over-estimate. 1

(7)

(b) Let $t=x^2$. 1M

$\mathrm{d}t=2x\mathrm{d}x$

When $t=1$, $x=1$; when $t=4$, $x=2$. } 1A

$$I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,\mathrm{d}t$$

$$=\int_1^2 \frac{1}{x}e^{\frac{-x^2}{2}}2x\mathrm{d}x$$

$$=2\int_1^2 e^{\frac{-x^2}{2}}\,\mathrm{d}x$$ 1

(3)

(c) $2\int_1^2 e^{\frac{-x^2}{2}}\,\mathrm{d}x<0.692913377$ 1M

$2\sqrt{2\pi}\int_1^2 \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\,\mathrm{d}x<0.692913377$

$2\sqrt{2\pi}(0.4772-0.3413)<0.692913377$ 1A For 0.4772 and 0.3413

$\pi<3.249593152$

$\therefore\ \pi<3.25$ 1

(3)

(a)	(i)	Good. Many candidates applied the trapezoidal rule correctly.
	(ii)	Poor. Many candidates used $\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)$ instead of $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)$ to determine whether the estimate in (i) is an over-estimate or under-estimate.
(b)		Fair. Many candidates used wrong substitutions.
(c)		Very poor. Only a few candidates attempted this part. Among them, some wrote $I\approx 0.692913377$ instead of $I<0.692913377$.

31. (PP DSE-MATH-M1 Q10)

(a) $\dfrac{dx}{dt}=\dfrac{61t}{(t+1)^{\frac{5}{2}}}$

Let $u=t+1$ and hence $du=dt$.

The amount of alloy produced by A

$=\displaystyle\int_0^{10}\frac{61t}{(t+1)^{\frac{5}{2}}}\,dt$ — 1A

$=\displaystyle\int_1^{11}\frac{61(u-1)}{u^{\frac{5}{2}}}\,du$

$=\displaystyle\int_1^{11}\left(61u^{\frac{-3}{2}}-61u^{\frac{-5}{2}}\right)du$ — 1M

$=\left[-122u^{\frac{-1}{2}}+\dfrac{122}{3}u^{\frac{-3}{2}}\right]_1^{11}$ — 1A — For primitive function

Alternative Solution

$x=\displaystyle\int\frac{61t}{(t+1)^{\frac{5}{2}}}\,dt$

$=\displaystyle\int\frac{61(u-1)}{u^{\frac{5}{2}}}\,du$ — 1A

$=\displaystyle\int\left(61u^{\frac{-3}{2}}-61u^{\frac{-5}{2}}\right)du$ — 1M

$=-122u^{\frac{-1}{2}}+\dfrac{122}{3}u^{\frac{-3}{2}}+C$

$=-122(t+1)^{\frac{-1}{2}}+\dfrac{122}{3}(t+1)^{\frac{-3}{2}}+C$ — 1A

The amount of alloy produced by A

$=\left[-122(10+1)^{\frac{-1}{2}}+\dfrac{122}{3}(10+1)^{\frac{-3}{2}}+C\right]-\left[-122+\dfrac{122}{3}+C\right]$

≈ 45.6636 — 1A — OR $=\dfrac{244}{3}-\dfrac{3904}{33\sqrt{11}}$

(4)

(b) The amount of alloy produced by B

$=\displaystyle\int_0^{10}\frac{15\ln(t^2+100)}{16}\,dt$

$\approx\dfrac{2}{2}\cdot\dfrac{15}{16}\{\ln(0+100)+\ln(10^2+100)+2[\ln(2^2+100)$
$+\ln(4^2+100)+\ln(6^2+100)+\ln(8^2+100)]\}$ — 1M

≈ 45.6792 — 1A

(2)

(c) $\dfrac{d}{dt}\left(\dfrac{dy}{dt}\right)=\dfrac{d}{dt}\dfrac{15\ln(t^2+100)}{16}$

$=\dfrac{15t}{8(t^2+100)}$ — 1A

$\dfrac{d^2}{dt^2}\left(\dfrac{dy}{dt}\right)=\dfrac{15}{8}\cdot\dfrac{(t^2+100)-t(2t)}{(t^2+100)^2}$

$=\dfrac{15(100-t^2)}{8(t^2+100)^2}$ — 1A

$\therefore\ \dfrac{d^2}{dt^2}\left(\dfrac{dy}{dt}\right)>0$ for $0<t<10$

Thus, 45.6792 is an over-estimate of the amount of alloy produced by B. — 1A

Hence it is uncertain whether machine B is more productive than machine A by the results of (a) and (b). The engineer cannot be agreed with. — 1A

(4)

(a)	平平。 部分學生未能正確運用代換法。
(b)	平平。 部分學生未能正確運用梯形法則公式。
(c)	甚差。 部分同學對凹凸曲線概念不了解。

32. (SAMPLE DSE-MATH-M1 Q12)

(a) (i) $f'(0)=e^{2b(0)}+ae^{b(0)}+8=3$

$a=-6$ — 1A

$f'(1)=e^{2b(1)}+ae^{b(1)}+8=4.73$

$e^{2b}-6e^{b}+3.27=0$

$e^{b}=0.606258159$ or 5.393741841 — 1M

$b=\ln 0.606258159$ or $\ln 5.393741841$

$=-0.50044937$ or 1.685239363 (rejected)

i.e. $b=-0.5004$ — 1A

(ii) $f(12)-f(0)=\int_0^{12} f'(t)\,dt$

$f(12)-0=\int_0^{12}(e^{-t}-6e^{-0.5t}+8)\,dt$

$=\left[-e^{-t}+12e^{-0.5t}+8t\right]_0^{12}$ — 1M

$f(12)=85.02973888$

$=85.0297$ — 1A

(5)

(b) (i) $g'(t)=\frac{33}{10}te^{-kt}$

$g''(t)=\frac{33}{10}e^{-kt}(1-kt)$ — 1A

Since $g'(t)$ is greatest when $t=7.5$, $g''(7.5)=0$ — 1M

$\frac{33}{10}e^{-7.5k}(1-7.5k)=0$

$k=\frac{2}{15}$ — 1A

(ii)

t	0	3	6	9	12
$g'(t)$	0	6.63617	8.89671	8.94547	7.99510

$g(12)-g(0)=\int_0^{12} g'(t)\,dt$ — 1M

$g(12)-0\approx\frac{3}{2}[0+7.99510+2(6.63617+8.89671+8.94547)]$ — 1M For RHS

$g(12)\approx 85.427703$

≈ 85.4277 — 1A

(6)

(c) Agree. Since the graph of $g'(t)$ is concave downward for $0\le t\le 12$, the estimated value obtained in b(ii) is under-estimated and the estimate 85.4277 is greater than 85.0297 in a(iii). — 1M For pointing out that (b)(ii) is under-estimated

Hence $g(12)>f(12)$. — 1

(2)

33. (2013 ASL-M&S Q8)

(a) $\frac{dP}{dt}=\frac{k-3t}{1+ae^{-bt}}$

$\ln\left(\frac{k-3t}{\frac{dP}{dt}}-1\right)=-bt+\ln a$ — 1A

$\because$ slope $=-0.3$

$\therefore$ $b=0.3$ — 1A

$\because$ intercept on the horizontal axis $=0.32$

$\therefore$ $0=-(0.3)(0.32)+\ln a$

$a\approx 1.100759064$

≈ 1.1008 — 1A OR $e^{0.096}$

When $t=3$ P attains maximum and hence $\frac{dP}{dt}=0$. — 1M

$\frac{k-3(3)}{1+(1.100759064)e^{-(0.3)(3)}}=0$

$k=9$ — 1A

(5)

(b) (i) $P=\int_0^3\frac{9-3t}{1+1.100759064e^{-0.3t}}\,dt$

$\approx\frac{0.5}{2}[4.284165735+0+2(3.851225403+3.30494319+2.644142541$
$+1.870196654+0.986866929)]$ — 1M

≈ 7.3997 million barrels — 1A

(ii) From the graph $\frac{d^2P}{dt^2}$ is decreasing for $0<t<3$. — 1A

Thus, $\frac{d^3P}{dt^3}<0$ for $0<t<3$ and hence the estimation is under-estimate. — 1

(4)

(c) (i) $y=\alpha^{\beta x}$

$\ln y=\beta x\ln\alpha$ — 1M

$\frac{1}{y}\cdot\frac{dy}{dx}=\beta\ln\alpha$

$\frac{dy}{dx}=\beta\alpha^{\beta x}\ln\alpha$ — 1A

(ii) $\therefore$ $\int\alpha^{\beta x}\,dx=\frac{1}{\beta\ln\alpha}\alpha^{\beta x}+C$ ------------------------------ (*)

$D=\int_0^3 1.63^{2-0.1t}\,dt$

$=1.63^2\left[\frac{1}{-0.1\ln 1.63}1.63^{-0.1t}\right]_0^3$ by (*) — 1M

≈ 7.414075736

≈ 7.4141 (million barrels) — 1A

(iii) The amount of oil production is approximately 7.3997 million barrels which is an underestimate. Compare with (c)(ii), we cannot conclude that whether the overall oil production meets the overall demand of oil. — 1M, 1

(6)

(a)		Good.
(b)	(i)	Good.
	(ii)	Satisfactory. Some candidates were not able to gather relevant information from the graph to support that $\frac{d^2P}{dt^2}$ is decreasing for $0<t<3$.
(c)		Satisfactory.

34. (2013 ASL-M&S Q9)

(a) (i) Let $u=2t+1$. **1A**

$\therefore\ t=\frac{u-1}{2}$

$dt=\frac{1}{2}du$

$\therefore\ \int\frac{t^2}{2t+1}dt=\int\frac{1}{u}\left(\frac{u-1}{2}\right)^2\frac{1}{2}du$

$=\frac{1}{8}\int\left(u-2+\frac{1}{u}\right)du$ **1A**

$=\frac{u^2}{16}-\frac{u}{4}+\frac{1}{8}\ln|u|+C$

$=\frac{(2t+1)^2}{16}-\frac{2t+1}{4}+\frac{1}{8}\ln|2t+1|+C$ **1A**

(ii) $\frac{d}{dt}[t^2\ln(2t+1)]=2t\ln(2t+1)+\frac{2t^2}{2t+1}$ **1A**

(iii) $\therefore\ t\ln(2t+1)=\frac{1}{2}\cdot\frac{d}{dt}t^2\ln(2t+1)-\frac{t^2}{2t+1}$

$\int t\ln(2t+1)\,dt=\frac{1}{2}t^2\ln(2t+1)-\int\frac{t^2}{2t+1}dt$ **1M**

$=\frac{1}{2}t^2\ln(2t+1)-\frac{(2t+1)^2}{16}+\frac{2t+1}{4}-\frac{1}{8}\ln|2t+1|-C$ by (a)(i) **1M**

$N|_{t=5}-N|_{t=0}=\int_0^5 t\ln(2t+1)\,dt$ **1M**

$N|_{t=5}-21=\left[\frac{1}{2}t^2\ln(2t+1)-\frac{(2t+1)^2}{16}+\frac{2t+1}{4}-\frac{1}{8}\ln|2t+1|\right]_0^5$

$N|_{t=5}\approx 45.673954$

Hence the population of the culture of bacteria is approximately 46 trillions. **1A**

(8)

(b) (i) By (a)(iii), $45.673954=40e^{-2\lambda(5-5)}-20e^{-\lambda(5-5)}+K$

$K\approx 25.673954$

≈ 26 **1A**

$27=40e^{-2\lambda(11-5)}-20e^{-\lambda(11-5)}+25.673954$ **1M**

$40e^{-12\lambda}-20e^{-6\lambda}-1.326046=0$

$e^{-6\lambda}=0.559275201$ or -0.059275201 (rejected)

$\lambda\approx 0.1$ **1A**

(ii) $M=40e^{-0.2(t-5)}-20e^{-0.1(t-5)}+26$

$M'=-8e^{-0.2(t-5)}+2e^{-0.1(t-5)}$ **1M**

$=-2e^{-0.2(t-5)}[4-e^{0.1(t-5)}]$

<0 since $e^{0.1(t-5)}\le e^{1.3}<4$ for $t\le 18$ **1M**

Thus, M is always decreasing for $t\le 18$. **1**

Since we have $M\approx 23.5203$ when $t=18$, the population of the bacteria will not drop to 23 trillion. **1**

(7)

(a)	(i)(ii)		Very good.
	(iii)		Satisfactory. Some candidates were not able to make use of the results of (i) and (ii) to integrate $t\ln(2t+1)$.
(b)			Fair.

35. (2012 ASL-M&S Q8)

(a) Let $u=1+6t$. **1A**

$du=6dt$

When $t=0$, $u=1$; when $t=12$, $u=73$

$\int_0^{12}\left[4.5+2t(1+6t)^{\frac{-2}{3}}\right]dt$

$=\int_1^{73}\left(4.5+\frac{u-1}{3}u^{\frac{-2}{3}}\right)\frac{du}{6}$ **1M** For integrand

$=\int_1^{73}\left(\frac{3}{4}+\frac{1}{18}u^{\frac{1}{3}}-\frac{1}{18}u^{\frac{-2}{3}}\right)du$

$=\left[\frac{3u}{4}+\frac{1}{24}u^{\frac{4}{3}}-\frac{1}{6}u^{\frac{1}{3}}\right]_1^{73}$ **1A** For primitive function

≈ 66.14060019

$\therefore$ the total amount of sewage emitted by machine $P\approx 66.1406$ tonnes. **1A** OR $\frac{433+23\sqrt[3]{73}}{8}$ tonnes

(4)

(b) (i) $\int_0^{12}[3+\ln(2t+1)]\,dt$

$=\frac{12-0}{2(5)}[3+\ln 1+3+\ln 25+2(3+\ln 5.8+3+\ln 10.6+3+\ln 15.4+3+\ln 20.2)]$ **1M**

≈ 63.52367987

$\therefore$ the total amount of sewage emitted by machine $Q\approx 63.5237$ tonnes. **1A**

(ii) $q'(t)=\frac{2}{2t+1}$

$q''(t)=\frac{-4}{(2t+1)^2}<0$ for all $t\ge 0$ **1A** For $\frac{-4}{(2t+1)^2}$

Hence the estimate in (b)(i) is an under-estimate. **1M**

Therefore we cannot conclude that the amount of sewage emitted by Q will be less than that by P and so the manager cannot be agreed with. **1**

(5)

(c) (i) $R=16-ae^{-bx}$

$\ln(16-R)=\ln a-bx$ **1A**

(ii) $\begin{cases}1=\ln a+10b\\0=\ln a-90b\end{cases}$ **1M**

Solving, $a=e^{0.9}$ and $b=0.01$ **1A+1A** OR $a\approx 2.4596$

(iii) Total amount of sewage

$\approx 80+66.14060019+63.52367987$ **1M**

$=209.6642801$

Hence $R=16-e^{0.9}e^{-0.01(209.6642801)}$

≈ 15.69779292

i.e. the tax paid is 15.6978 million dollars. **1A**

(6)

(a)			Satisfactory. Many candidates were able to use substitution appropriately but some did not change the limits of integration accordingly.
(b)	(i)		Good.
	(ii)		Fair. Some candidates did not realise that the second derivative of the rate of emission should be q'' rather than q'.
(c)	(i)		Very good.
	(ii)		Good. Some candidates failed to see that a was not the intercept of the function $\ln(16-R)$ when expressed as a linear function of x.
	(iii)		Very poor. Many candidates were not able to comprehend the given situation and failed to state that the total amount of sewage was '80 + result of (a) + result of (b)(i)'.

36. (2012 ASL-M&S Q9)

(a) $\mathrm{r}(t) = 20 - 40e^{-at} + be^{-2at}$

$\because \mathrm{r}(0) = 20 - 40e^0 + be^0 = 30$

$\therefore b = 50$ — 1A

(1)

(b) $\mathrm{r}'(t) < 0$ for 9 days

$40ae^{-at} - 100ae^{-2at} < 0$ for $t < 9$ — 1M

$20ae^{-2at}(2e^{at} - 5) < 0$

$e^{at} < 2.5$

$t < \frac{\ln 2.5}{a}$ — 1A

$\therefore \frac{\ln 2.5}{a} = 9$

i.e. $a \approx 0.1$ (correct to 1 decimal place) — 1A

(3)

(c) The rate of change of the rate of selling of handbags is $\mathrm{r}'(t) = 4e^{-0.1t} - 10e^{-0.2t}$.

$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{r}'(t) = -0.4e^{-0.1t} + 2e^{-0.2t}$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{r}'(t) = 0$ when $0.4e^{-0.1t} = 2e^{-0.2t}$ — 1M

$e^{0.1t} = 5$

$t = 10\ln 5$ — 1A — OR 16.0944

$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{r}'(t) = 0.04e^{-0.1t} - 0.4e^{-0.2t}$

When $t = 10\ln 5$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{r}'(t) = -0.008 < 0$ — 1M — OR by using sign test

Hence $\mathrm{r}'(t)$ is maximum when $t = 10\ln 5$

$\mathrm{r}(10\ln 5) = 20 - 40e^{-0.1(10\ln 5)} + 50e^{-0.2(10\ln 5)} = 14$

The rate of selling $= 14$ thousand per day — 1A — OR 14000 per day

(4)

(d) (i) $\mathrm{r}(t) = 20 - 40e^{-0.1t} + 50e^{-0.2t} < 18$ — 1M

$25e^{-0.2t} - 20e^{-0.1t} + 1 < 0$

$0.053589838 < e^{-0.1t} < 0.746410161$

$2.924800155 < t < 29.26395809$ — 1A

$\because 29.26395809 - 2.924800155 = 26.33915794$

$\therefore$ the 'sales warning' will last for 26 days. — 1A

(ii) Number of handbags sold (in thousand) during the 'sales warning' period

$= \int_{2.92480155}^{29.26395809} (20 - 40e^{-0.1t} + 50e^{-0.2t})\,\mathrm{d}t$ — 1M

$= [20t + 400e^{-0.1t} - 250e^{-0.2t}]_{2.924800155}^{29.26395809}$

≈ 388.2190941 — 1A — Accept 388.2191

$\frac{388.2190941}{26.33915794} \approx 14.7392$ — 1M

Hence the average number of handbags sold per day is 15 thousand. — 1A — OR 15000

(7)

(a)		Very good.
(b)		Satisfactory. Many candidates used an equation rather than an inequality to solve for the value of a.
(c)		Fair. Some candidates overlooked that the given condition was for the rate of change of the rate of selling. When consider the maximum rate of change, candidates should set the second derivative $\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$ zero.
(d) (i)		Poor. Many candidates were not able to handle the quadratic inequality.
(ii)		Fair. Many candidates were not able to get the correct answer due to errors made in the previous parts.

37. (2011 ASL-M&S Q8)

(a) $e^{t^2+t} = 1 + (t^2+t) + \frac{(t^2+t)^2}{2} + \frac{(t^2+t)^3}{3!} + \cdots$ — 1M

$= = 1 + t^2 + t + \frac{2t^3 + t^2 + \cdots}{2} + \frac{t^3 + \cdots}{6} + \cdots$

$= 1 + t + \frac{3t^2}{2} + \frac{7t^3}{6} + \cdots$ — 1A

$V = \int_0^{\frac{1}{2}} \frac{1}{25} e^{t^2+t+2}\,\mathrm{d}t$

$\approx \frac{e^2}{25} \int_0^{\frac{1}{2}} \left(1 + t + \frac{3t^2}{2} + \frac{7t^3}{6}\right)\mathrm{d}t$ — 1M

$= \frac{e^2}{25}\left[t + \frac{t^2}{2} + \frac{t^3}{2} + \frac{7t^4}{24}\right]_0^{\frac{1}{2}}$

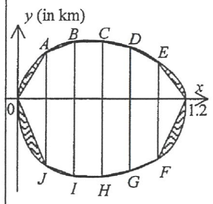
$= \frac{271}{9600}e^2$ hundred thousand m^3 — 1A — OR 0.2086 hund. th. m^3 OR 20858.6896 m^3

Since for $t > 0$, $e^{t^2+t} = 1 + t + \frac{3t^2}{2} + \frac{7t^3}{6} +$ positive terms, — 1M

$e^{t^2+t} > 1 + t + \frac{3t^2}{2} + \frac{7t^3}{6}$

Hence the estimation is an under-estimate. — 1A

(6)



(b) (i) Area $\approx \frac{0.2}{2}[0 + 2(3.8 + 4.2 + 4.3 + 4.1 + 3.4) + 0]$ — 1M

$= 3.96\ \mathrm{km}^2$ — 1A

Since the upper half of the curve is concave downwards and the lower half is concave upwards, the estimation is an under-estimate. — 1A

(ii) Thickness $\approx \frac{20858.6896}{3.96 \times 1000^2}$ m — OR 5.2673×10^{-3} m OR 5.2673 mm

≈ 0.0053 m — 1A

Since both the numerator and denominator are under-estimates, we cannot determine whether the thickness is an over- or under-estimate. — 1A

(5)

(c) $\dfrac{dW}{dt}=\dfrac{-(W+1)^{\frac{1}{3}}}{40}$

$\dfrac{dt}{dW}=-40(W+1)^{\frac{-1}{3}}$ 1M

$t=-40\int (W+1)^{\frac{-1}{3}}\,dW$

$=-60(W+1)^{\frac{2}{3}}+C$ 1A

When $t=0$, $W=\dfrac{271}{9600}e^2$. 1M

$\therefore\ 0=-60\left(\dfrac{271}{9600}e^2+1\right)^{\frac{2}{3}}+C$

$C=68.07743296$

Hence $t=-60(W+1)^{\frac{2}{3}}+68.07743296$

When $W=0$, $t\approx 8.0774$. 1A

Thus all the oil spread will be cleaned up after 8.0774 days.

(4)

(a)		Satisfactory. In order to arrive at a polynomial in t of degree 3, the exponential function had to be expanded to the fourth term which is $\dfrac{(t^2+t)^3}{3!}$. Many candidates only considered three terms and hence did not meet the requirement.
(b) (i)		Fair. Candidates' concept about concave and convex curves was unclear.
(ii)		Fair. Some candidates did not know how to conclude when the denominator and numerator of a fraction were both under-estimates.
(c)		Poor. Many candidates were unable to deal with $\dfrac{dW}{dt}$ when it was expressed as a function of W rather than a function of t .

38. (2010 ASL-M&S Q8)

(a) $f'(t)=-500ae^{2at}+300ae^{at}$ 1A

Since $f(t)$ attains maximum when $t=5$, $f'(5)=0$ 1M

$-500ae^{10a}+300ae^{5a}=0$

$a=0.2\ln 0.6$ 1A OR -0.1022

(3)

(b) (i) $-250e^{0.4T_1\ln 0.6}+300e^{0.2T_1\ln 0.6}-50=0$ 1M

$e^{0.2T_1\ln 0.6}=0.2$ or 1 (rejected as $T_1>0$)

$T_1=\dfrac{5\ln 0.2}{\ln 0.6}$ 1A OR 15.7533

(ii) The total amount of sales increased

$=\int_0^{T_1}(-250e^{2at}+300e^{at}-50)\,dt$ 1M

$=\left[\dfrac{-125e^{0.4t\ln 0.6}}{0.2\ln 0.6}+\dfrac{300e^{0.2t\ln 0.6}}{0.2\ln 0.6}-50t\right]_0^{\frac{5\ln 0.2}{\ln 0.6}}$ 1A

$=\dfrac{-125}{0.2\ln 0.6}(0.2^2-1)+\dfrac{300}{0.2\ln 0.6}(0.2-1)-50\left(\dfrac{5\ln 0.2}{\ln 0.6}\right)$

$=\dfrac{-600+250\ln 5}{\ln 0.6}$ thousand dollars 1A OR 386.9041 thousand dollars OR \$ 386904.0876

(5)

(c) (i) $E=100+\int\dfrac{100}{t+9}\,dt$

$=100+100\ln(t+9)+C$ 1A

When $t=0, E=100$.

$100=100+100\ln 9+C$ 1M

$C=-100\ln 9$

$\therefore\ E=100[\ln(t+9)+1-\ln 9]$ 1A

(ii) $200=100\ln(t+9)+100-100\ln 9$

$T_2=9(e-1)$ 1A OR 15.4645

(iii) Total sales increased

$=\int_\alpha^{2\alpha}-(t-\alpha)(t-2\alpha)\,dt$ 1M

$=\int_\alpha^{2\alpha}(-t^2+3\alpha t-2\alpha^2)\,dt$

$=\left[\dfrac{-t^3}{3}+\dfrac{3\alpha t^2}{2}-2\alpha^2 t\right]_\alpha^{2\alpha}$

$=\dfrac{\alpha^3}{6}$ 1A

Hence the maximum total increase of sales can be achieved when

$\alpha=T_2$ 1M

$=9(e-1)$

Hence the plan should be started $9(e-1)$ months after the launching of the campaign.

(7)

(a)		Very good.
(b) (i)		Good. Some candidates could not make use of the given condition $f(T_1)=0$ to solve for T_1 .
(ii)		Fair. Some candidates had difficulty in performing integration involving exponential functions.
(c) (i)		Poor. Many candidates could not express the total expenditure E , which is the sum of a fixed cost and the integrated total of a variable cost.
(ii)		Poor, since it depends on (c)(i).
(iii)		Very poor. Very few candidates attempted this part.

39. (2009 ASL-M&S Q9)

(a) (i) $R_6 = \int_0^6 \ln(2t+1)\,dt$

$\approx \frac{1}{2}\{\ln(2\cdot 0+1)+2[\ln(2\cdot 1+1)+\ln(2\cdot 2+1)+\ln(2\cdot 3+1)+\ln(2\cdot 4+1)+\ln(2\cdot 5+1)]+\ln(2\cdot 6+1)\}$ 1M

$=10.53155488$

The total amount of revenue in the first 6 weeks is 10.5316 million dollars. 1A

(ii) Let $f(t)=\ln(2t+1)$

$f'(t)=\frac{2}{2t+1}$

$f''(t)=\frac{-4}{(2t+1)^2}$ 1A

<0 for $0\le t\le 6$

$\therefore$ $f(t)$ is concave downward for $0\le t\le 6$.

Hence the estimate in (a)(i) is an under-estimate. 1A Follow through

(4)

(b) (i) $Q_1 = \int_0^1\left[45t(1-t)+\frac{1.58}{t+1}\right]dt$ 1A

$=\left[45\left(\frac{t^2}{2}-\frac{t^3}{3}\right)+1.58\ln|t+1|\right]_0^1$

$=\frac{15}{2}+1.58\ln 2$

≈ 8.595172545

The total amount of revenue in the first week is 8.5952 million dollars. 1A

(ii) $Q_n = Q_1+\int_1^n \frac{30e^{-t}}{(3+2e^{-t})^2}dt$ 1M

Let $u=3+2e^{-t}$

$du=-2e^{-t}dt$

$\therefore$ $Q_n=Q_1+\int_{3+2e^{-1}}^{3+2e^{-n}}\frac{-15}{u^2}du$ 1M For $\frac{-15}{u^2}$

$=Q_1+\left[\frac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$ 1A For $\left[\frac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$

$=\frac{15}{2}+1.58\ln 2+\frac{15}{3+2e^{-n}}-\frac{15}{3+2e^{-1}}$

Hence the total amount of revenue in the first n weeks is

$\left(\frac{15}{2}+1.58\ln 2+\frac{15}{3+2e^{-n}}-\frac{15}{3+2e^{-1}}\right)$ million dollars, where $n>1$. 1A Accept $4.5799+\frac{15}{3+2e^{-n}}$

(6)

(c) For $n>6$, $R_n=R_6+\int_6^n 0\,dt\approx 10.5316$ (by (a)(i)) 1M For $\int_6^n 0\,dt=0$

When $n\to\infty$, $e^{-n}\to 0$ and so $Q_n\to 4.5799+\frac{15}{3+0}=9.5799$ 1M For $e^{-n}\to 0$

Therefore, over a long period of time, plan A produces approximately 10.5316 million dollars and plan B produces 9.5799 million dollars of revenue. } 1A

Moreover, the revenue of plan A is even an under-estimate. 1M

Hence, plan A will produce more revenue over a long period of time. 1A Follow through

(5)

(a) (i)		Good.
(ii)		Poor. The poor performance was rather unexpected since applying the concept of concave and convex curves should be quite standard.
(b) (i)		Good.
(ii)		Poor. The problem might look unfamiliar. Many candidates did not realize that the lower and upper limits of the integral should be 1 and n , and Q_1 should be added to the integral to get Q_n .
(c)		Very poor. Most candidates got the wrong conclusion due to mistakes made in the previous parts.

DSE Mathematics Module 1 6. Definite Integrals

40. (2008 ASL-M&S Q9)

(a) (i) $\int_0^{10}\frac{1}{40}\sqrt{1+t^2}\,dt$

$\approx \frac{10}{2(4)}\cdot\frac{1}{40}\left(\sqrt{1+0^2}+2\sqrt{1+2.5^2}+2\sqrt{1+5^2}+2\sqrt{1+7.5^2}+\sqrt{1+10^2}\right)$ 1M

≈ 1.305182044

≈ 1.3052 1A

So the increase of temperature is about 1.3052 °C .

(ii) $\frac{d}{dt}\left(\frac{1}{40}\sqrt{1+t^2}\right)=\frac{t}{40\sqrt{1+t^2}}$

$\frac{d^2}{dt^2}\left(\frac{1}{40}\sqrt{1+t^2}\right)=\frac{1}{40(1+t^2)^{\frac{3}{2}}}$ 1A

>0

Hence it is an over-estimate. 1A Follow through

(4)

(b) (i) $100(\ln x_0)^2-630\ln x_0+1960=968$

$50(\ln x_0)^2-315\ln x_0+496=0$ 1A

$\ln x_0=\frac{31}{10}$ or $\frac{16}{5}$

$x_0\approx 22.1980$ or 24.5325 1A

(ii) $W'(x)=\frac{200\ln x}{x}-\frac{630}{x}$ 1A

$\therefore W'(x)<0$ when $200\ln x-630<0$ $(\because x\ge 22)$

$\therefore \ln x<3.15$

i.e. $22\le x<e^{3.15}\approx 23.3361$ 1A Accept $x<23.3361$

(iii) $\frac{dW}{dt}=\frac{dW}{dx}\cdot\frac{dx}{dt}$

$=\frac{200\ln x-630}{x}\cdot\frac{\sqrt{1+t^2}}{40}$ 1M

When $t=0$, $x=22$.

$\therefore \left.\frac{dW}{dt}\right|_{t=0}=\frac{200\ln 22-630}{22}\cdot\frac{\sqrt{1+0}}{40}$ 1M

≈ -0.0134 1A

Hence the rate of change of electricity consumption at $t=0$ is -0.0134 units per year.

(iv) The electricity consumption at $t=10$ is approximately

$W(22+1.305182044)$ 1M

$=100(\ln 23.305182044)^2-630\ln 23.305182044+1960$

≈ 967.7502 units 1A

Since the estimate in (a)(i) is an over-estimate, the actual temperature when $t=10$ is $x<23.305182044$.

Moreover, $W(x)$ is decreasing for $22\le x<23.3361$ by (b)(ii). } 1M

Therefore the actual electricity consumption is larger than this estimate. 1A Follow through

(11)

(a) (i)		Very good.
(ii)		Poor. Many candidates were not aware that the second derivative of the given equation is $\frac{d^3x}{dt^3}$ rather than $\frac{d^2x}{dt^2}$.
(b) (i) (ii)		Good.
(iii)		Poor. Many candidates did not realise that $\frac{dW}{dt}$ should be found and some failed to apply chain rule to find $\frac{dW}{dt}$.
(iv)		Very poor. Most candidates could not interpret their own mathematical findings and hence failed to make use of the results to make judgement.

41. (2007 ASL-M&S Q8)

(a) (i) The total profit made by company A

$= \int_0^6 f(t)\,dt$ — 1A withhold 1A for omitting this step

$\approx \frac{1}{2}\left(f(0)+f(6)+2\left(f(1)+f(2)+f(3)+f(4)+f(5)\right)\right)$ — 1M for trapezoidal rule

[redacted]

≈ 37.4871 billion dollars — 1A $a-1$ for r.t. 37.487

(ii) $f(t)=\ln(e^t+2)+3$

$\frac{df(t)}{dt}=\frac{e^t}{e^t+2}$ — 1A

$\frac{d^2f(t)}{dt^2}$

$=\frac{(e^t+2)e^t-e^t(e^t)}{(e^t+2)^2}$

$=\frac{2e^t}{(e^t+2)^2}$ — 1A

Since $\frac{d^2f(t)}{dt^2}>0$, $f(t)$ is concave upward for $0\le t\le 6$. — 1M

Thus, the estimate in (a)(i) is an over-estimate. — 1A f.t.

----------(7)

(b) (i) $\frac{1}{40-t^2}=\frac{1}{40}\left(1+\frac{t^2}{40}+\frac{t^4}{1600}+\cdots\right)=\frac{1}{40}+\frac{1}{1600}t^2+\frac{1}{64000}t^4+\cdots$ — 1A pp−1 for omitting ' … '

(ii) Note that $e^t=1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\cdots$. Hence, we have — 1M for any four terms correct

$\frac{8e^t}{40-t^2}$

$=8\left(\frac{1}{40}+\frac{1}{1600}t^2+\frac{1}{64000}t^4+\cdots\right)\left(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\cdots\right)$

$=\frac{1}{5}+\frac{1}{5}t+\frac{21}{200}t^2+\frac{23}{600}t^3+\frac{263}{24000}t^4+\cdots$ — 1A pp−1 for omitting ' … '

(iii) The total profit made by company B

$=\int_0^6 g(t)\,dt$

$\approx \int_0^6\left(\frac{1}{5}+\frac{1}{5}t+\frac{21}{200}t^2+\frac{23}{600}t^3+\frac{263}{24000}t^4\right)dt$ — 1M

$=\left[\frac{1}{5}t+\frac{1}{10}t^2+\frac{7}{200}t^3+\frac{23}{2400}t^4+\frac{263}{120000}t^5\right]_0^6$ — 1A for correct integration

$=41.8224$ billion dollars — 1A $a-1$ for r.t. 41.822

----------(6)

(c) Since the estimate in (b)(iii) is an under-estimate, we have — 1A

$\int_0^6 f(t)\,dt<37.4871<41.8224<\int_0^6 g(t)\,dt$.

Thus, Mary's claim is correct. — 1A f.t.

----------(2)

(a) (i)		Good. Most candidates could apply the trapezoidal rule.
(ii)		Fair.
(b) (i)		Good.
(ii)		Fair. Some candidates could not expand the exponential function.
(iii)		Fair. Failing to get the correct result was mainly due to the performance of the previous parts.
(c)		Poor. Many candidates did not attempt this part and others could not make use of the concept of over- and under-estimate of a mathematical model to explain the underlying meaning.

DSE Mathematics Module 1 **6. Definite Integrals**

42. (2004 ASL-M&S Q8)

(a) The total fuel consumption

$=\int_0^{15} f(t)\,dt$ — 1A withhold 1A for omitting this step

$\approx\frac{15-0}{10}\left(f(0)+f(15)+2\left(f(3)+f(6)+f(9)+f(12)\right)\right)$ — 1M for trapezoidal rule

[redacted]

≈ 27.4036 litres — 1A $a-1$ for r.t. 27.404

----------(3)

(b) The total fuel consumption

[redacted]

$=\int_0^{15}\frac{1}{145}t(15-t)^2\,dt$

$=\frac{1}{145}\int_0^{15}(225t-30t^2+t^3)\,dt$ — 1A

$=\frac{1}{145}\left[\frac{225t^2}{2}-10t^3+\frac{t^4}{4}\right]_0^{15}$ — 1A for correct integration

$=\frac{3375}{116}$ litres — 1A

[redacted]

≈ 29.0948 litres — $a-1$ for r.t. 29.095

----------(3)

(c) $f(t)=\frac{1}{4}t(15-t)e^{\frac{-t}{4}}$

$\frac{df(t)}{dt}=\frac{1}{4}(15-2t)e^{\frac{-t}{4}}-\frac{1}{16}t(15-t)e^{\frac{-t}{4}}$ — 1M for Product Rule or Chain Rule

$=\frac{1}{16}(t^2-23t+60)e^{\frac{-t}{4}}$ — 1A must be simplified

$=\frac{1}{16}(t-3)(t-20)e^{\frac{-t}{4}}$

$\frac{df(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<3\\ =0 & \text{if } t=3\\ <0 & \text{if } 3<t\le 15\end{cases}$ — 1M for testing + 1A

So, we have

the greatest value

$=f(3)$

$=9e^{\frac{-3}{4}}$ — 1A provided the testing is correct

[redacted]

≈ 4.2513 — $a-1$ for r.t. 4.251

$f(t)=\frac{1}{4}t(15-t)e^{\frac{-t}{4}}$	
$\frac{df(t)}{dt}=\frac{1}{4}(15-2t)e^{\frac{-t}{4}}-\frac{1}{16}t(15-t)e^{\frac{-t}{4}}$	1M for Product Rule or Chain Rule
$=\frac{1}{16}(t^2-23t+60)e^{\frac{-t}{4}}$	1A must be simplified
$=\frac{1}{16}(t-3)(t-20)e^{\frac{-t}{4}}$	
For $\frac{df(t)}{dt}=0$, we have $t=3$ or $t=20$ (rejected since $0\le t\le 15$).	
$\frac{d^2f(t)}{dt^2}=\frac{-1}{64}(t-3)(t-20)e^{\frac{-t}{4}}+\frac{1}{16}(2t-23)e^{\frac{-t}{4}}$	
$=\frac{-1}{64}(t^2-31t+152)e^{\frac{-t}{4}}$	
$\left.\frac{d^2f(t)}{dt^2}\right\rvert_{t=3}=\frac{-17}{16}e^{\frac{-3}{4}}$ [redacted] <0	1M for testing + 1A
So, we have the greatest value $=f(3)$ $=9e^{\frac{-3}{4}}$ [redacted] ≈ 4.2513	1A provided the testing is correct $a-1$ for r.t. 4.251
	----------(5)

(d) (i) $\frac{d^2f(t)}{dt^2}$

$=\frac{-1}{64}(t-3)(t-20)e^{\frac{-t}{4}}+\frac{1}{16}(2t-23)e^{\frac{-t}{4}}$

$=\frac{-1}{64}(t^2-31t+152)e^{\frac{-t}{4}}$ — 1A must be simplified

(ii) $\left.\frac{d^2f(t)}{dt^2}\right\rvert_{t=0}=\frac{-19}{8}<0$

$\left.\frac{d^2f(t)}{dt^2}\right\rvert_{t=15}=\frac{11}{8}e^{\frac{-15}{4}}>0$

} 1M for testing two values of t in [0, 15] or for factorizing $\frac{d^2f(t)}{dt^2}e^{\frac{t}{4}}$

Therefore, by considering $\frac{d^2f(t)}{dt^2}$, we cannot determine whether the estimate in (a) is an over-estimate or an under-estimate. } 1A

Thus, by considering $\frac{d^2f(t)}{dt^2}$, we cannot determine whether the total fuel consumption from $t=0$ to $t=15$ when using driving tactic A will be less than that of using driving tactic B. } 1M

----------(4)

(a/b/c)		Good.
(d)(i)		Fair.
(ii)		Poor. Very few candidates were able to explain their answers correctly.

DSE Mathematics Module 1 — 6. Definite Integrals

43. (2003 ASL-M&S Q8)

(a) $f(t)=5+2^{-kt+h}$
$\ln(f(t)-5)=-(k\ln 2)t+h\ln 2$ — 1A do not accept $-k\ln 2t+h\ln 2$
----------(1)

(b) $-k\ln 2=-0.35$
~~$k\approx 0.504943264$~~
$k\approx 0.5$ (correct to 1 decimal place) — 1A

$h\ln 2=1.39$
~~$h\approx 2.005346107$~~
$h\approx 2.0$ (correct to 1 decimal place) — 1A
----------(2)

(c) The total amount
$=\int_2^{12}g(t)dt$
$\approx\frac{12-2}{10}\left(g(2)+g(12)+2\left(g(4)+g(6)+g(8)+g(10)\right)\right)$ — 1M for trapezoidal rule
~~≈ 75.77699747~~
≈ 75.7770 thousand barrels — 1A
----------(2)

(d) (i) $2^t=e^{at}$ for all $t\ge 0$
$t\ln 2=at$ for all $t\ge 0$
$a=\ln 2$ — 1A accept $a\approx 0.6931$

(ii) $g(t)=5+\ln(t+1)+2^{\frac{-t}{2}+2}$
$=5+\ln(t+1)+4e^{\left(\frac{-\ln 2}{2}\right)t}$
$\frac{dg(t)}{dt}=\frac{1}{t+1}+(4)\left(\frac{-\ln 2}{2}\right)e^{\left(\frac{-\ln 2}{2}\right)t}$ — 1A for the first term + 1M for Chain Rule
$=\frac{1}{t+1}-2(\ln 2)e^{\left(\frac{-\ln 2}{2}\right)t}$ — 1M for the second term
$\frac{d^2g(t)}{dt^2}=\frac{-1}{(t+1)^2}-(2\ln 2)\left(\frac{-\ln 2}{2}\right)e^{\left(\frac{-\ln 2}{2}\right)t}$
$=(\ln 2)^2e^{\left(\frac{-\ln 2}{2}\right)t}-\frac{1}{(t+1)^2}$ — 1A for all being correct
$=(\ln 2)^2 2^{\frac{-t}{2}}-\frac{1}{(t+1)^2}$
$\therefore\ p(t)=(\ln 2)^2 2^{\frac{-t}{2}}$ — 1A accept $p(t)=(\ln 2)^2e^{\left(\frac{-\ln 2}{2}\right)t}$

(iii) $\because$ $p(2) = (\ln 2)^2 2^{-1} \approx 0.240226506 \approx 0.2402$

$q(2) = \frac{1}{9} \approx 0.111111111 \approx 0.1111$ — 1M for testing

$\therefore$ $p(2) > q(2)$ — 1A

It is known that $y = p(t)$ and $y = q(t)$ have no intersection, where $2 \le t \le 12$.

So, we have $p(t) > q(t)$ for all $2 \le t \le 12$.

$\therefore$ $\frac{d^2 g(t)}{dt^2} > 0$ on $[2, 12]$ — 1M

Thus, the estimate is an over-estimate of I. — 1M

----------(10)

(a/b/c) (d)		Good. Fair. Some candidates were unable to perform differentiation involving 'ln' function. Very few candidates were able to explain why the estimate was an over-estimate of I.

44. (2002 ASL-M&S Q9)

(a) (i)

t	0	0.5	1.0	1.5	2	2.5
$\frac{dM}{dt}$	4	4.78496	5.84320	7.24875	9.10480	11.55161

$$M = \int_0^{2.5} \frac{12e^{\frac{2}{3}t}}{3+t}\,dt \approx \frac{0.5}{2}[4 + 11.55161 + 2(4.78496 + 5.8432 + 7.24875 + 9.1048)]$$ — 1M

$= 17.3788$ (m mol/L) — 1A $a-1$ for r.t. 17.379

(ii) $\because$ $\frac{dM}{dt} = \frac{12e^{\frac{2}{3}t}}{3+t}$,

$$\frac{d}{dt}\left(\frac{12e^{\frac{2}{3}t}}{3+t}\right) = 12\left[\frac{2}{3}\cdot\frac{e^{\frac{2}{3}t}}{3+t} - \frac{e^{\frac{2}{3}t}}{(3+t)^2}\right] = \frac{4(3+2t)e^{\frac{2}{3}t}}{(3+t)^2}$$ — 1A need not simplify

and $$\frac{d^2}{dt^2}\left(\frac{12e^{\frac{2}{3}t}}{3+t}\right) = \frac{8(9+6t+2t^2)}{3(3+t)^3}e^{\frac{2}{3}t}$$ — 1A need simplification

$\therefore$ $\frac{d^2}{dt^2}\left(\frac{dM}{dt}\right) > 0$ (for $0 \le t \le 2.5$)

So, $\frac{dM}{dt}$ is concave upward on $[0, 2.5]$.

Hence it is over-estimate. — 1

----------(5)

(b) (i) $\frac{1}{3+t} = \frac{1}{3}(1 - \frac{1}{3}t + \frac{1}{9}t^2 - \frac{1}{27}t^3 + \cdots)$ — 1A

$= \frac{1}{3} - \frac{1}{9}t + \frac{1}{27}t^2 - \frac{1}{81}t^3 + \cdots$

$e^{\frac{2}{3}t} = 1 + \frac{2}{3}t + \frac{1}{2!}(\frac{2}{3}t)^2 + \frac{1}{3!}(\frac{2}{3}t)^3 + \cdots$ — 1M any three terms

$= 1 + \frac{2}{3}t + \frac{2}{9}t^2 + \frac{4}{81}t^3 + \cdots$ — 1A

$\frac{12e^{\frac{2}{3}t}}{3+t} = 12(\frac{1}{3} - \frac{1}{9}t + \frac{1}{27}t^2 - \frac{1}{81}t^3 + \cdots)(1 + \frac{2}{3}t + \frac{2}{9}t^2 + \frac{4}{81}t^3 + \cdots)$

$= 4 + \frac{4}{3}t + \frac{4}{9}t^2 + \frac{4}{81}t^3 + \cdots$ — 1A for the first three terms or the term t^3; 1A for all being correct

(ii) $\int_0^{2.5} \frac{12e^{\frac{2}{3}t}}{3+t}\,dt \approx \int_0^{2.5} (4 + \frac{4}{3}t + \frac{4}{9}t^2 + \frac{4}{81}t^3)\,dt$

$= \left[4t + \frac{2}{3}t^2 + \frac{4}{27}t^3 + \frac{1}{81}t^4\right]_0^{2.5}$ — 1M

$= 16.9637$ (m mol/L) — 1A $a-1$ for r.t. 16.964

----------(7)

(c) The expansion is valid only when

$-1 < \frac{t}{3} < 1$ — 2A

$-3 < t < 3$

Hence $0 \le t < 3$ (as $t \ge 0$)

$\therefore$ this method is not valid to estimate the amount of lactic acid for $t \ge 3$. — 1A

----------(3)

45. (2000 ASL-M&S Q9)

(a) (i) $f(x)=16+4xe^{-0.25x}$

$f'(x)=4e^{-0.25x}(1-0.25x)$ — 1M attempting to find f'; 1A

$\begin{cases} >0 & \text{if } 0<x<4 \\ =0 & \text{if } x=4 \\ <0 & \text{if } x>4 \end{cases}$ — accept considering $f'(x)=e^{-0.25x}(0.25x-2)$

$\therefore$ $f(x)\le f(4)$ for $x>0$. — 1 follow through

(ii)

x	0	1	2	3	4	5	6
f(x)	16 (16)	19.1152 (19.1)	20.8522 (20.9)	21.6684 (21.7)	21.8861 (21.9)	21.7301 (21.7)	21.3551 (21.4)

1A correct to 1 d.p.

$\int_0^6 f(x)dx$

$\approx \frac{1}{2}[16+21.3551+2(19.1152+20.8523+21.6684+21.8861+21.7301)]$ — 1M

≈ 124

$\therefore$ The expected increase in profit is 124 hundred thousand dollars. — 1A a−1 for r.t. 124; pp−1 for wrong/missing unit

(b) (i) $g(x)=16+\frac{6x}{\sqrt{1+8x}}$

$g'(x)=\dfrac{6\sqrt{1+8x}-\dfrac{6x\cdot 8}{2\sqrt{1+8x}}}{1+8x}$ — 1A

$=\dfrac{6(1+4x)}{(1+8x)^{\frac{3}{2}}}$

>0 for $x>0$. — 1

$\therefore$ $g(x)$ is strictly increasing for $x>0$.

$\because$ $\lim\limits_{x\to\infty}\left(16+\frac{6x}{\sqrt{1+8x}}\right)=\lim\limits_{x\to\infty}\left(16+\dfrac{6\sqrt{x}}{\sqrt{\frac{1}{x}+8}}\right)$

$\therefore$ $g(x)\to\infty$ as $x\to\infty$ — 1A

(ii) Let $u=\sqrt{1+8x}$, then $u^2=1+8x$, $2udu=8dx$

$\int_0^6 g(x)dx=\int_0^6\left(16+\frac{6x}{\sqrt{1+8x}}\right)dx$ (or $\int_0^6 16dx+\int_0^6\frac{6x}{\sqrt{1+8x}}dx$)

$=\int_1^7\left(16+\frac{6(u^2-1)}{8u}\right)\frac{1}{4}udu$ (or $[16x]_0^6+\int_1^7\frac{6(u^2-1)}{8u}\frac{1}{4}udu$) — 1A integrand; 1A limits

$=\int_1^7\left(\frac{3}{16}u^2+4u-\frac{3}{16}\right)du$ (or $96+\int_1^7\left(\frac{3}{16}u^2-\frac{3}{16}\right)du$)

$=\left[\frac{1}{16}u^3+2u^2-\frac{3}{16}u\right]_1^7$ (or $96+\left[\frac{1}{16}u^3-\frac{3}{16}u\right]_1^7$) — 1A ignore limits

$=116\frac{1}{4}$

≈ 116

$\therefore$ The expected increase in profit is 116 hundred thousand dollars. — 1A a−1 for r.t. 116; pp−1 for wrong/missing unit

(c) From (a)(i), $f(x)\le f(4)$ (≈ 21.8861) for $x>0$.
i.e. $f(x)$ is bounded above by $f(4)$.
From (b)(i), $g(x)$ increases to infinity as x increases to infinity. — 1M

$\because$ $f(x)>0$ and $g(x)>0$ for $x>0$,
the area under the graph of $g(x)$ will be greater than that of $f(x)$ as x increases indefinitely.

$\therefore$ Plan G will eventually result in a bigger profit. — 1A

46. (1999 ASL-M&S Q8)

(a) $S_A = \frac{256}{9625}\left(\frac{1}{3}t^3 - \frac{47}{4}t^2 + 120t\right)$

$\frac{dS_A}{dt} = \frac{256}{9625}\left(t^2 - \frac{47}{2}t + 120\right)$ — 1A

$= \frac{128}{9625}(t-16)(2t-15)$

$\frac{dS_A}{dt}\begin{cases} >0 & \text{when } 0 \le t < \frac{15}{2} \\ =0 & \text{when } t = \frac{15}{2} \\ <0 & \text{when } \frac{15}{2} < t \le 12.5 \end{cases}$ — 1M

$\therefore$ A attains its top speed at $t = \frac{15}{2}$ (or 7.5)

Top speed of $A = \frac{256}{9625}\left[\frac{1}{3}\left(\frac{15}{2}\right)^3 - \frac{47}{4}\left(\frac{15}{2}\right)^2 + 120\left(\frac{15}{2}\right)\right]$ m/s

≈ 10.0987 m/s — 1A — a−1 for r.t. 10.099; pp−1 for missing unit

(b) $S_B = \frac{183}{50}te^{-kt}$

$\frac{dS_B}{dt} = \frac{183}{50}e^{-kt}(1-kt)$ — 1A

$\because$ $k > 0$

$\therefore$ $\frac{dS_B}{dt}\begin{cases} >0 & \text{when } 0 \le t < \frac{1}{k} \\ =0 & \text{when } t = \frac{1}{k} \\ <0 & \text{when } t > \frac{1}{k} \end{cases}$ — 1M — for solving $\frac{dS_B}{dt} = 0$ or sub. $t = \frac{15}{2}$ into $\frac{dS_B}{dt} = 0$

B attains its top speed at $t = \frac{1}{k}$.

From (a), $\frac{1}{k} = \frac{15}{2}$

$k = \frac{2}{15}$ (or 0.1333) — 1A — a−1 for r.t. 0.133

(c)

t	0	2.5	5	7.5	10	12.5
S_B	0	6.55626 (6.5563)	9.39553 (9.3955)	10.09829 (10.0983)	9.64766 (9.6477)	8.64106 (8.6411)

1M — correct to 4 d.p.

The distance covered by B in 12.5 seconds

$= \int_0^{12.5} S_B dt$ m

$\approx \frac{2.5}{2}[0 + 8.64106 + 2(6.55626 + 9.39553 + 10.09829 + 9.64766)]$ m — 1M

≈ 100.0457 m — 1A — accept 100.0457 to 100.0699; a−1 for r.t. 3 d.p.

(d) $\frac{d^2S_B}{dt^2} = \frac{183}{50}k^2e^{-kt}(t - \frac{2}{k})$ (or $\frac{183}{50}ke^{-kt}(kt-2)$) — 1M

$= \frac{122}{1875}e^{-\frac{2t}{15}}(t-15)$ (or $\frac{61}{125}e^{-\frac{2t}{15}}(\frac{2}{15}t-2)$)

< 0 for $0 \le t \le 12.5$

$\therefore$ The graph of S_B is concave downward for $0 \le t \le 12.5$.

i.e., The estimated distance covered by B in (c) is underestimated. — 1M

Hence B covers more than 100 m in 12.5 seconds.

B finishes the race ahead of A. — 1

(e) $\int_0^{12.5} \frac{50[\ln(t+2) - \ln 2]}{t+2} dt$

$= \int_0^{12.5} \frac{25[\ln\frac{t+2}{2}]}{\frac{t+2}{2}} dt$ (or $50\int_0^{12.5}\left(\frac{\ln(t+2)}{t+2} - \frac{\ln 2}{t+2}\right)dt$)

$= 25\left[\left(\ln\frac{t+2}{2}\right)^2\right]_0^{12.5}$ (or $50\left[\frac{(\ln(t+2))^2}{2} - \ln 2\ln(t+2)\right]_0^{12.5}$) — 1A

≈ 98.1092 — 1A

$\therefore$ C covers only 98.1092 m but both A and B finish the race in 12.5 seconds. C is the last one to finish the race among the three athletes. — 1

Alternatively,

$\int_0^x \frac{50[\ln(t+2) - \ln 2]}{t+2} dt = 25\left[\left(\ln\frac{t+2}{2}\right)^2\right]_0^x$ — 1A

If $25\left(\ln\frac{x+2}{2}\right)^2 = 100$

then $\ln\frac{x+2}{2} = 2$

$x \approx 12.78$ — 1A

$\therefore$ C needs 12.78 seconds to finish the race but both A and B finish the race within 12.5 seconds. C is the last one to finish the race among the three athletes. — 1

47. (1998 ASL-M&S Q8)

(a) (i) If $\frac{5000e^{15\lambda}}{15}=\frac{5000e^{95\lambda}}{95}$ **1A**

then $e^{80\lambda}=\frac{19}{3}$

$\lambda=\frac{1}{80}\ln(\frac{19}{3})$ **1M+1A**

≈ 0.0231

(ii) $N=\frac{5000e^{\lambda t}}{t}\approx\frac{5000e^{0.0231t}}{t}$

$\frac{dN}{dt}=5000\left(\frac{\lambda te^{\lambda t}-e^{\lambda t}}{t^2}\right)$ **1M+1A**

$=\frac{5000e^{\lambda t}(\lambda t-1)}{t^2}$

$\begin{cases}<0 & \text{when } 0<t<\frac{1}{\lambda}\\ =0 & \text{when } t=\frac{1}{\lambda} \quad (\approx 43.3410)\\ >0 & \text{when } \frac{1}{\lambda}<t<120\end{cases}$ **1M+1A**

$\therefore$ N attains its minimum when $t\approx 43.3410$ **1A** r.t. 43
(The number of fish decreased to the minimum in about 43 days after the spread of the disease.)

(b) $\int_0^{15}\frac{dW}{ds}ds$

$=\int_0^{15}\frac{3}{50}(e^{-\frac{s}{20}}-e^{-\frac{s}{10}})ds$ **1A**

$=\frac{3}{50}\left[-20e^{-\frac{s}{20}}+10e^{-\frac{s}{10}}\right]_0^{15}$ **1A**

≈ 0.1670 **1A**

$\therefore$ The increase in the mean weight of fish in the first 15 days is 0.1670 kg.

If $\int_0^a\frac{dW}{ds}ds=0.5$,

then $\frac{3}{50}\left[-20e^{-\frac{s}{20}}+10e^{-\frac{s}{10}}\right]_0^a=0.5$ **1M**

$10e^{-\frac{a}{10}}-20e^{-\frac{a}{20}}=\frac{25}{3}-10$

$\left(e^{-\frac{a}{20}}\right)^2-2\left(e^{-\frac{a}{20}}\right)+\frac{1}{6}=0$ **1M**

$e^{-\frac{a}{20}}\approx 0.0871$ or 1.9129 **1A**

$a\approx 48.8073$ or -12.9721 (rej.) **1A**

$\therefore$ It takes about 49 days for the mean weight of the fish to increase 0.5 kg from the *Recovery Day*.

48. (1998 ASL-M&S Q9)

(a) $I=\int_{0.5}^{2.5}e^{-x}dx$

$=\left[-e^{-x}\right]_{0.5}^{2.5}$ **1A**

$=e^{-0.5}-e^{-2.5}$ **1A**

≈ 0.5244 (0.524446)

(b) $y=ae^{-x}+bxe^{-x}$

$\because$ y-intercept is -3

$\therefore$ $a=-3$ **1A**

$y'=-ae^{-x}+be^{-x}-bxe^{-x}$ **1A** neglecting the value of a

$=(-a+b-bx)e^{-x}$

$\because$ y attains its maximum when $x=\frac{3}{2}$

$\therefore$ $-a+b-\frac{3}{2}b=0$ **1M**

$3-\frac{1}{2}b=0$

$b=6$ **1A**

Hence $y=-3e^{-x}+6xe^{-x}$

(c) If $y=0$, $3e^{-x}(2x-1)=0$

$x=\frac{1}{2}$ **1A**

$\therefore$ The x-intercept of the curve is $\frac{1}{2}$.

$y'=9e^{-x}-6xe^{-x}$

$y''=-9e^{-x}-6e^{-x}+6xe^{-x}$

$=-15e^{-x}+6xe^{-x}$ **1M**

$=3(2x-5)e^{-x}$

$\therefore$ $y''\begin{cases}<0 & \text{if } 0\le x<\frac{5}{2}\\ =0 & \text{if } x=\frac{5}{2}\\ >0 & \text{if } x>\frac{5}{2}\end{cases}$ **1M**

The point of inflection is $(\frac{5}{2},12e^{\frac{-5}{2}})$ [or $(\frac{5}{2},0.9850)$] **1A**

(d) (i)

x	0.5	1	1.5	2	2.5
xe^{-x}	0.303265	0.367879	0.334695	0.270671	0.205212

$J_0\approx\frac{0.5}{2}[0.303265+0.205212+2(0.367879+0.334695+0.270671)]$ **1M+1A**

≈ 0.6137 (0.613742)

$A_0\approx -3\times 0.524446+6\times 0.613742$

≈ 2.1091 (2.109114) **1A**

(ii) The argument is not correct because the trapezoidal rule was used to approximate the value of J only.
The convexity of the function xe^{-x} should be considered instead of the function $-3e^{-x}+6xe^{-x}$. **1A+1** 1A for either reason, 1 for both

49. (1997 ASL-M&S Q10)

(a) $y = x^x$

$\ln y = x \ln x$ — 1A

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = x^x(1+\ln x)$ — 1A

(b) $\frac{d^2y}{dx^2} = x^x \frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}x^x$

$= x^x \cdot \frac{1}{x} + (1+\ln x)x^x(1+\ln x)$ — 1A

$= x^{x-1} + x^x(1+\ln x)^2$

> 0 for $1 \le x \le 2$ — 1A

y is concave upward (or convex) for $1 \le x \le 2$

$\therefore$ I would be overestimated if the trapezoidal rule is used to estimate I. — 1

(c) $I + J = \int_1^2 x^x(1+\ln x)dx$ — 1A

$= \left[x^x\right]_1^2$ by (a)

$= 3$ — 1

(d) (i)

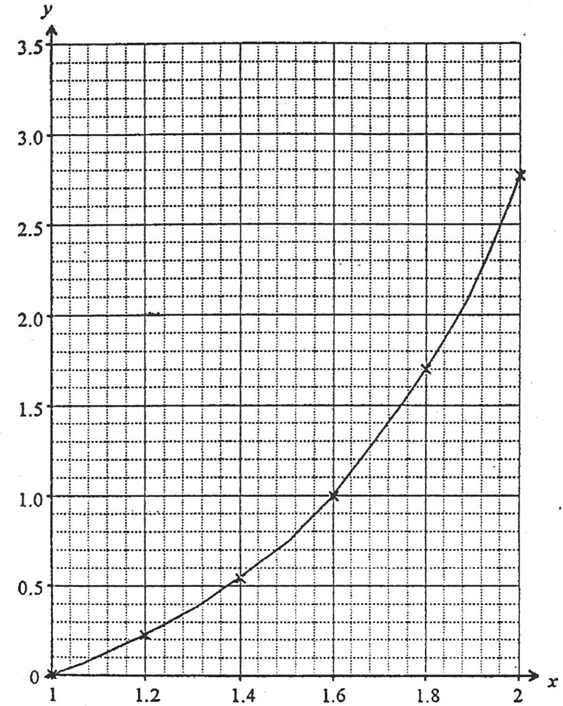
x	1	1.2	1.4	1.6	1.8	2
$x^x\ln x$	0	0.22691	0.53893	0.99700	1.69321	2.77259

1A

$J_0 \approx \frac{0.2}{2}[2.77259 + 2(0.22691 + 0.53893 + 0.99700 + 1.69321)]$ — 1M

≈ 0.9685 — 1A

(ii)



1A+1M

From the plotted graph, $y = x^x \ln x$ is concave upward (or convex) for $1 \le x \le 2$.

$\therefore$ J_0 is an overestimate of J. — 1M

(iii) The estimation can be improved by increasing the number of sub-intervals. — 1

(iv) J_0 is an underestimate of I because the value 3 for $I + J$ is exact and J_0 is an overestimate of J. — 1

50. (1996 ASL-M&S Q9)

(a)

t	0	1	2	3	4	5	6
$e^{\frac{t^2}{10}}$	1	1.10517	1.49182	2.45960	4.95303	12.18249	36.59823

1A

$\therefore \int_0^6 e^{\frac{t^2}{10}}\,dt \approx \frac{1}{2}(1+36.59823)+(1.10517+1.49182+2.45960+4.95303+12.18249)$ 1M

≈ 40.9912 1A

$\because P|_{t=6}-P|_{t=0}=\int_0^6 (5e^{\frac{t^2}{10}}-2t)\,dt$

$\therefore P|_{t=6}=\int_0^6 (5e^{\frac{t^2}{10}}-2t)\,dt+10$ 2A

$=5\int_0^6 e^{\frac{t^2}{10}}\,dt-\left[t^2\right]_0^6+10$ 1A

$\approx 5\times 40.9912-36+10$

≈ 179 1A

(b) (i) Put $t=6$ and $P=179$ into $P=kte^{-0.04t}-50$.

$179=6ke^{-0.24}-50$ 1M

$k=48.5$ 1A

(ii) $P=48.5te^{-0.04t}-50$

$P'=48.5(-0.04te^{-0.04t}+e^{-0.04t})$ 1M

$=48.5(1-0.04t)e^{-0.04t}$

$\because e^{-0.04t}>0$ for all t

$\therefore P'=0$ only when $t=25$ 1A

and $P'\begin{cases}>0 & \text{for } t<25\\ <0 & \text{for } t>25\end{cases}$ 1M

Hence the population size will attain its max. when $t=25$.

The maximum population size $=48.5\cdot 25\cdot e^{-0.04\cdot 25}-50$

≈ 396 1A

(iii) Substitute $y=e^{0.04t}$ into $48.5te^{-0.04t}-50=0$, we have $y=0.97t$. 1M

The graphs $y=e^{0.04t}$ and $y=0.97t$ intersect at $t\approx 1$ or 119

$\because t\geq 6$,

$\therefore$ The species of reptiles becomes extinct ($48.5te^{-0.04t}-50=0$) when $t\approx 119$. 1A Accept 118 - 120

51. (1996 ASL-M&S Q10)

(a) (i) $\ln[C(t)+1]=\ln ae^{bt}$

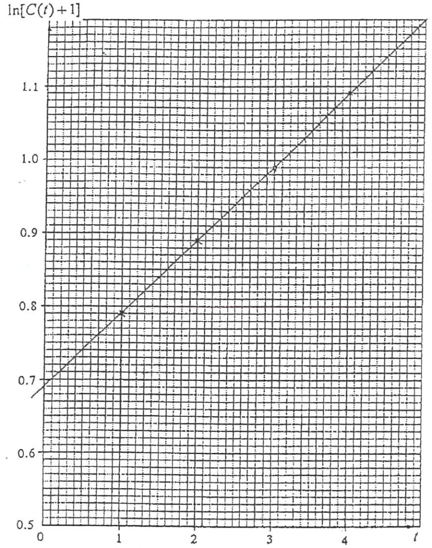
$=\ln a+\ln e^{bt}$ 1M

$=\ln a+bt$ 1A

(ii)

t	1	2	3	4
$\ln[C(t)+1]$	0.79	0.89	0.99	1.09

1A



1A+1A For the points & line

From the graph,

$\ln a\approx 0.69$, $a\approx 2.0$ 1A

$b\approx\frac{1.09-0.79}{4-1}=0.1$ 1A

(iii) $C(t)=2.0e^{0.1t}-1$

$C(36)\approx 72.1965$

When $t=36$, the monthly cost is 72.1965 thousand dollars. 1A

(b) (i) Solve $2.0e^{0.1t}-1=439-e^{0.2t}$ 1M

$e^{0.2t}+2.0e^{0.1t}-440=0$

$\left(e^{0.1t}\right)^2+2.0\left(e^{0.1t}\right)-440=0$ 1M

$e^{0.1t}=20$ or -22 (rej.) 1A

$t\approx 30$ 1A

(ii) $\int_0^{30}[(439-e^{0.2t})-(2.0e^{0.1t}-1)]\,dt$ 1M

$=\int_0^{30}(440-e^{0.2t}-2.0e^{0.1t})\,dt$

$=\left[440t-5e^{0.2t}-20e^{0.1t}\right]_0^{30}$ 1A

≈ 10806

$\therefore$ The total profit is 10806 thousand dollars. 1A

52. (1995 ASL-M&S Q7)

(a) (i)

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	1	1.00504	1.02062	1.04828	1.09109	1.15470

1A

$I_1 = 0.1\left[\frac{1}{2}(1+1.15470) + (1.00504+1.02062+1.04828+1.09109)\right]$ 1M Using correct formula

$= 0.5242$ 1A

(ii) $f'(x) = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$ 1A

$f''(x) = \dfrac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}$ 1A must be simplified

(iii) By (a)(ii), $f''(x) > 0$ for $0 \le x \le \frac{1}{2}$,

$\therefore$ $f(x)$ is concave upward (or convex) on $[0, \frac{1}{2}]$ 1 argument for convexity

Hence I_1 is an over-estimate of I. 1A

(b) (i) $f(x) = (1-x^2)^{-\frac{1}{2}}$

$= 1 + (-\frac{1}{2})(-x^2) + \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-x^2)^2 + \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(-x^2)^3 + \ldots$ 1M + 1A 1M for binomial series

$= 1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\ldots$ for $0 \le x \le \frac{1}{2}$.

$\therefore p(x) = 1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6$ 1A

$I_2 = \int_0^{\frac{1}{2}} (1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6)\,dx$

$= \left[x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7\right]_0^{\frac{1}{2}}$ 1A

$= \frac{1}{2}+\frac{1}{48}+\frac{3}{1280}+\frac{5}{14336}$

≈ 0.5235 1A

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(ii) $\because f(x) = p(x) + \sum_{r=4}^{\infty} \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-r+1)}{r!}(-x^2)^r$

$= p(x) + \sum_{r=4}^{\infty} \dfrac{(\frac{1}{2})(\frac{1}{2}+1)\ldots(\frac{1}{2}+r-1)}{r!}x^{2r}$ 1A

$> p(x)$ for $0<x\le\frac{1}{2}$. 1

Hence $I > I_2$

i.e. I_2 is an under-estimate of I. 1A

Note:

1. 1 mark for the following argument in b(ii)
   $\because$ Sum to infinity
   $\therefore$ p(x) is just a truncation
   Hence underestimate

2. Withhold 1 mark <u>once</u> for incorrect degree of accuracy.

53. (1994 ASL-M&S Q10)

(a) Amount of pollutant

$= \int_0^8 r(t)\,dt$

$\approx \frac{2}{2}[0+2(11+32+59)+90]$ — 1M

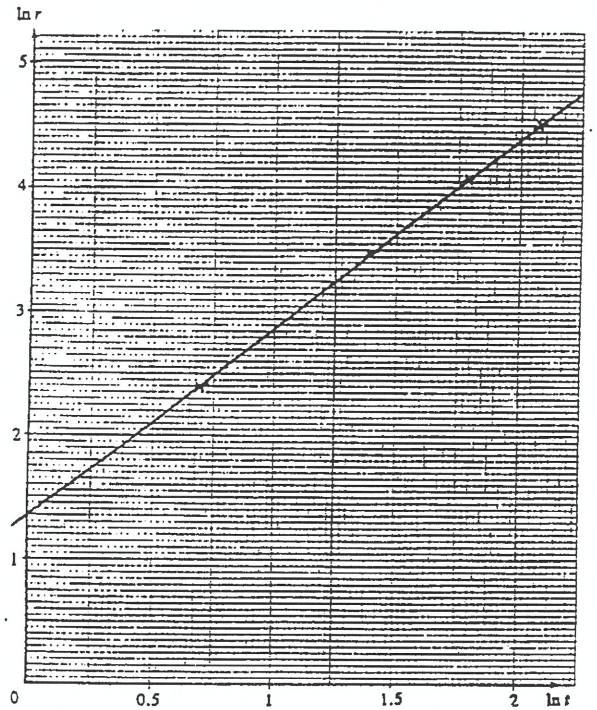
$= 294$ (units) — 1A

(b) (i) $r = at^b$

$\ln r = \ln a + b\ln t$ — 1M — For taking logarithm

$\ln t$	0.69	1.39	1.79	2.08
$\ln r$	2.40	3.47	4.08	4.50

1A — For entries in the table



1 — For the graph

From the graph,

$\ln a \approx 1.35$

$a = 3.9$ — 1A — Accept 1.3 - 1.4 or 3.7 - 4.1 respectively

$b \approx \frac{4.50-2.40}{2.08-0.69} \approx 1.5$ — 1A — Accept 1.4 - 1.6

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(b) (ii) Amount of pollutant

$= \int_0^8 3.9t^{1.5}\,dt$ — 1M

$= \left[\frac{3.9}{2.5}t^{2.5}\right]_0^8$ — 1M

$= \frac{3.9}{2.5}\times 8^{2.5}$

≈ 282.4 (units) — 1A

In general, accept

Amount of pollutant

$= \int_0^8 at^b\,dt$

$= \left[\frac{a}{b+1}t^{b+1}\right]_0^8$

$= \frac{a}{b+1}8^{b+1}$ where $a \in (3.7, 4.1)$

$b \in (1.4, 1.6)$

$\in (226.7, 351.4)$

(c) Amount of pollutant after x months

$= \int_0^x 3.9t^{1.5}\,dt$ — 1M

$= \left[\frac{3.9}{2.5}t^{2.5}\right]_0^x$ — 1M

$= \frac{3.9}{2.5}x^{2.5}$ — 1A

In general, accept

...

$= \frac{a}{b+1}x^{b+1}$ where $a \in (3.7, 4.1)$

$b \in (1.4, 1.6)$

The lake will "die" after x months if

$\frac{3.9}{2.5}x^{2.5} = 1000$ (or $\frac{3.9}{2.5}x^{2.5} \geq 1000$) — 1M

$x = \left(\frac{2.5\times1000}{3.9}\right)^{\frac{1}{2.5}}$

≈ 13 — 1A

In general, accept

...

$x = \left(\frac{1000(b+1)}{a}\right)^{\frac{1}{b+1}}$ where $a \in (3.7, 4.1)$

$b \in (1.4, 1.6)$

$x \in (12, 15)$