6. Definite Integrals

Learning Unit	Learning Objectives
Calculus Area	
Integration with Its Applications	
8. Definite integrals and their applications	8.1 recognise the concept of definite integration 8.2 recognise the Fundamental Theorem of Calculus and understand the properties of definite integrals 8.3 find the definite integrals of algebraic functions and exponential functions 8.4 use integration by substitution to find definite integrals 8.5 use definite integration to find the areas of plane figures 8.6 use definite integration to solve problems
9. Approximation of definite integrals using the trapezoidal rule	9.1 understand the trapezoidal rule and use it to estimate the values of definite integrals

Section A

1. Let m be a non-zero constant.

(a) By considering $\frac{d}{dx}(xe^{mx})$, find $\int xe^{mx}dx$.

(b) If the area of the region bounded by the curve $y = xe^{mx}$, the x-axis and the straight line $x = 1$ is $\frac{1}{m}$, find m.

(7 marks) (2020 DSE-MATH-M1 Q8)

2. Define $f(x) = \frac{6-x}{x+3}$ for all $x > -3$.

(a) Prove that $f(x)$ is decreasing.

(b) Find $\lim_{x\to\infty} f(x)$.

(c) Find the exact value of the area of the region bounded by the graph of $y = f(x)$, the x-axis and the y-axis.

(6 marks) (2019 DSE-MATH-M1 Q5)

3. (a) Express $7^{\frac{-1}{\ln 7}}$ in terms of e.

(b) By considering $\frac{d}{dx}(x7^{-x})$, find $\int x7^{-x}dx$.

(c) Define $h(x) = x7^{-x}$ for all real numbers x. It is given that the equation $h'(x) = 0$ has only one real root α. Find α. Also express $\int_0^{\alpha} h(x)dx$ in terms of e.

(7 marks) (2019 DSE-MATH-M1 Q8)

4. (a) By considering $\frac{d}{dx}(x\ln x)$, find $\int \ln x dx$.

(b) Find $\int \frac{\ln x}{x}dx$.

(c) Let C be the curve $y = \frac{(x-1)(\ln x - 1)}{x}$, where $x > 0$. Express, in terms of e, the area of the region bounded by C and the x-axis.

(7 marks) (2018 DSE-MATH-M1 Q8)

5. Define $g(x) = \frac{1}{x}\ln\left(\frac{e}{x}\right)$ for all $x > 0$.

(a) Using integration by substitution, find $\int g(x)dx$.

(b) Denote the curve $y = g(x)$ by Γ.

(i) Write down the x-intercept(s) of Γ.

(ii) Find the area of the region bounded by Γ, the x-axis and the straight lines $x = 1$ and $x = e^2$.

(7 marks) (2017 DSE-MATH-M1 Q8)

6. Let $f(x) = 3^{2x} - 10(3^x) + 9$.

(a) Find $\int f(x)dx$.

(b) The equation of the curve C is $y = f(x)$. Find

(i) the two x-intercepts of C,

(ii) the exact value of the area of the region bounded by C and the x-axis.

(6 marks) (2016 DSE-MATH-M1 Q6)

7. Define $f(x) = \frac{(\ln x)^2}{x}$ for all $x > 0$. Let α and β be the two roots of the equation $f'(x) = 0$, where $\alpha > \beta$.

(a) Express α in terms of e. Also find β.

(b) Using integration by substitution, evaluate $\int_{\beta}^{\alpha} f(x)dx$.

(7 marks) (2016 DSE-MATH-M1 Q8)

8. Consider the curves C_1: $y = e^{2x} + e^4$ and C_2: $y = e^{x+3} + e^{x+1}$.

(a) Find the x-coordinates or the two points of intersection of C_1 and C_2.

(b) Express, in terms of e, the area of the region bounded by C_1 and C_2.

(Part b is out of Syllabus) (6 marks) (2015 DSE-MATH-M1 Q6)

9. Evaluate the following definite integrals:

(a) $\int_1^3 \frac{t+2}{t^2+4t+11}dt$,

(b) $\int_1^3 \frac{t^2+3t+9}{t^2+4t+11}dt$.

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(6 marks) (2014 DSE-MATH-M1 Q4)

10. (a) Find $\frac{d}{dx}(x\ln x)$.

(b) Use (a) to evaluate $\int_1^e \ln x dx$.

(4 marks) (2013 DSE-MATH-M1 Q5)

11. The slope of the tangent to a curve S at any point (x, y) on S is given by $\frac{dy}{dx} = e^{2x}$. Let L be the tangent to S at the point $A(0,1)$ on S.
    (a) Find the equation of S.
    (b) Find the equation of L.
    (c) Find the area of the region bounded by S, L and the line $x = 1$.

(Part c is out of Syllabus) (7 marks) (2012 DSE-MATH-M1 Q5)

12. Consider the curve C: $y = x(x-2)^{\frac{1}{3}}$ and the straight line L that passes through the origin and is parallel to the tangent to C at $x = 3$.
    (a) Find the equation of L.
    (b) Find the x-coordinates of the two intersecting points of C and L.
    (c) Find the area bounded by C and L.

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(Part c is out of Syllabus) (8 marks) (2013 DSE-MATH-M1 Q3)

13. Consider the two curves C_1: $y = 1 - \frac{e}{e^x}$ and C_2: $y = e^x - e$.
    (a) Find the x-coordinates of all the points of intersection of C_1 and C_2.
    (b) Find the area of the region bounded by C_1 and C_2.

(Part b is out of Syllabus) (5 marks) (PP DSE-MATH-M1 Q5)

14. L is the tangent to the curve C: $y = x^3 + 7$ at $x = 2$.
    (a) Find the equation of the tangent L.
    (b) Using the result of (a), find the area bounded by the y-axis, the tangent L and the curve C.

(Part b is out of Syllabus) (7 marks) (SAMPLE DSE-MATH-M1 Q9)

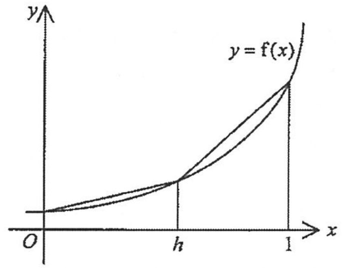
15. The value $R(t)$, in thousand dollars, of a machine can be modelled by

$$R(t) = Ae^{-0.5t} + B ,$$

where t (≥ 0) is the time, in years, since the machine has been purchased. At $t = 0$, its value is 500 thousand dollars and in the long run, its value is 10 thousand dollars.
    (a) Find the values of A and B.
    (b) The machine can generate revenue at a rate of $P'(t) = 600e^{-0.3t}$ thousand dollars per year, where t is the number since the machine has been purchased. Richard purchased the machine for his factory and used it for 5 years before he sold it. How much did he gain in this process? Correct your answer to the nearest thousand dollars.

(6 marks) (2013 ASL-M&S Q3)

16. Let $f(x) = e^{2x}$.
    (a) Use trapezoidal rule with 2 intervals of equal width to find the approximate value of $\int_0^1 f(x)dx$.
    (b) Evaluate the exact value of $\int_0^1 f(x)dx$.
    (c) A student uses trapezoidal rule with 2 trapeziums of unequal widths to approximate $\int_0^1 f(x)dx$. The first trapezium has width h $(0 < h < 1)$ and the second trapezium has width $1-h$ as shown below. Let A be the total area of the two trapeziums.



    (i) Show that $A = \frac{e^{2h} + (1-e^2)h + e^2}{2}$.
    (ii) Find the minimum value of A.

(8 marks) (2010 ASL-M&S Q2)

17. The rate of change of the amount of water in litres flowing into a tanks can be modelled by

$$f(t)=\frac{500}{(t+2)^2e^t},$$

where $t(\geq 0)$ is the time measured in minutes.

(a) Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of water flowing into the tank from $t=1$ to $t=11$.

(b) Find $\dfrac{d^2 f(t)}{dt^2}$.

(c) Determine whether the estimate in (a) is an over-estimate or under-estimate.

(7 marks) (2006 ASL-M&S Q3)

18. (a) Using the trapezoidal rule with 4 sub-intervals, estimate $\int_0^8 te^{\frac{t}{5}}\,dt$.

(b) A researcher modelled the rate of change of the number of certain insects under controlled conditions by

$$\frac{dx}{dt}=4te^{\frac{t}{5}}+\frac{200}{t+1},$$

where x is the number of insects and $t\,(\geq 0)$ is the time measured in weeks. It is known that $x=100$ when $t=0$.

Using the result of (a), estimate the number of insects when $t=8$.

Give your answer correct to 2 significant figures.

(7 marks) (2005 ASL-M&S Q2)

19. Suppose the rate of change of the accumulated bonus, R thousand dollars per month, for a group of salesman can be modeled by

$$R=\frac{1200}{t^2+150} \qquad (0\leq t\leq 6),$$

(a) Use the trapezoidal rule with 4 sub-intervals to estimate the total bonus for the first 6 months in 2001 .

(b) Find $\dfrac{d^2R}{dt^2}$.

Hence or otherwise, state with reasons whether the approximation in (a) is an overestimate or an underestimate.

(6 marks) (2001 ASL-M&S Q5)

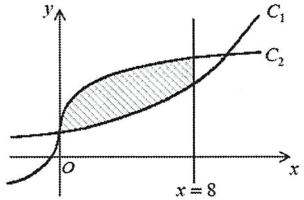
20. The figure shows the graph of the two curves

$C_1: y=e^{\frac{x}{8}}$ and

$C_2: y=1+x^{\frac{1}{3}}$.

Find the area of the shaded region.

(Out of Syllabus) (5 marks) (2000 ASL-M&S Q3)

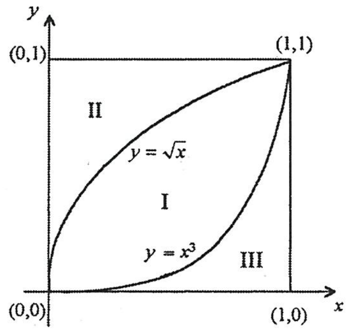


21. The figure shows a unit square target for shooting on the rectangular coordinate plane. The target is divided into three regions I , II and III by the curves $y=\sqrt{x}$ and $y=x^3$. The scores for hitting the regions I , II and III are 10 , 20 and 30 points respectively.

(a) Find the areas of the three regions.

(b) Suppose a child shoots randomly at the target twice and both shots hit the target. Find the probability that he will score 40 points.

(7 marks) (1996 ASL-M&S Q4)



22. The figure shows the graph of the two curves

$C_1: y=2^{2x}+4$ and

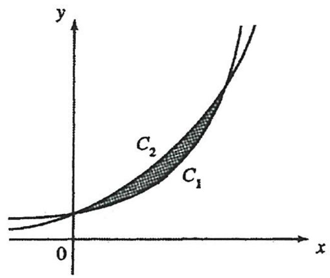
$C_2: y=5(2^x)$.

(a) Find the coordinates of the points of intersection of C_1 and C_2 .

(b) If $2^{2x}=e^{ax}$ for all x, find a.

Hence, or otherwise, find the area of the shaded region in the figure bounded by C_1 and C_2 .

(Part b is out of Syllabus) (8 marks) (1995 ASL-M&S Q6)



23. (a) Use the exponential series to find a polynomial of degree 6 which approximates $e^{\frac{-x^2}{2}}$ for x close to 0 .

Hence estimate the integral $\int_0^1 e^{\frac{-x^2}{2}}\,dx$.

(b) It is known that the area under the standard normal curve between $z=0$ and $z=a$ is $\int_0^a \frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}\,dz$. Use the result of (a) and the normal distribution table to estimate, to 3 decimal places, the value of π .

(7 marks) (1994 ASL-M&S Q6)

Section B

24. Let $f(x)=\dfrac{e^{0.1x}}{x}$. Define $I=\int_{0.5}^{1}f(x)dx$. In order to estimate the value of I, Ada suggests using trapezoidal rule with 5 sub-intervals while Billy suggests replacing $e^{0.1x}$ with $1+0.1x+0.005x^2$ and then evaluating the integral.

(a) Find the estimates of I according to the suggestions of Ada and Billy respectively.

(5 marks)

(b) Determine each of the two estimates in (a) is an over-estimate or an under-estimate. Explain your answer.

(6 marks)

(c) Someone claims that the difference of I and 0.746 is less than 0.002. Do you agree? Explain your answer.

(2 marks)

(2017 DSE-MATH-M1 Q11)

25. An investment consultant, Albert, predicts the total profit made by a factory in the coming year. He models the rate of change of profit (in million dollars per month) made by the factory by

$$\mathrm{A}(t)=\ln\left(t^2-8t+95\right),$$

where t ($0\le t\le 12$) is the number of months elapsed since the prediction begins. Let P_1 million dollars be the total profit made by the factory in the coming year under Albert's model.

(a) (i) Using the trapezoidal rule with 4 sub-intervals, estimate P_1.

(ii) $\dfrac{\mathrm{d}^2\,\mathrm{A}(t)}{\mathrm{d}t^2}$.

(4 marks)

(b) The factory manager, Christine, models the rate of change of profit (in million dollars per month) made by the factory in the coming year by

$$\mathrm{B}(t)=\frac{t+8}{\sqrt{t+3}},$$

where t ($0\le t\le 12$) is the number of months elapsed since the prediction begins. Let P_2 million dollars be the total profit made by the factory in the coming year under Christine's model.

(i) Find P_2.

(ii) Albert claims that the difference between P_1 and P_2 does not exceed 2. Do you agree? Explain your answer.

(9 marks)

(2016 DSE-MATH-M1 Q11)

26. An engineer models the rates of change of the amount of oil produced (in hundred barrels per day) by oil companies X and Y respectively by

$$f(t)=\ln\left(e^t-t\right) \text{ and } g(t)=\frac{8t}{1+t},$$

where t $(2\le t\le 12)$ is the time measured in days.

(a) Using the trapezoidal rule with 5 subintervals, estimate the total amount of oil produced by oil company X from $t=2$ to $t=12$.

(3 marks)

(b) Determine whether the estimate in (a) is an over-estimate or an under-estimate. Explain your answer.

(3 marks)

(c) Find $\int\dfrac{t}{1+t}dt$.

(3 marks)

(d) The engineer claims that the total amount of oil produced by oil company X from $t=2$ to $t=12$ is less than that of oil company Y. Do you agree? Explain your answer.

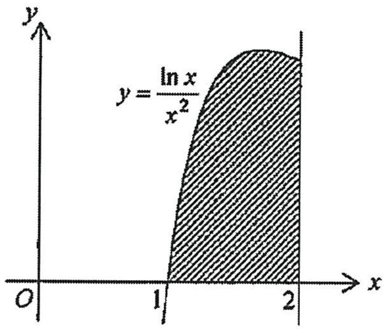
(3 marks) (2015 DSE-MATH-M1 Q11)

27. (a) (i) Find $\frac{d}{dv}\left(ve^{-v}\right)$.

(ii) Using (a)(i), or otherwise, show that $\int ve^{-v}dv = -e^{-v}(1+v)+C$, where C is a constant.

(3 marks)

(b)



The figure shows a shaded region bounded by the curve $y=\frac{\ln x}{x^2}$, the line $x=2$ and the x-axis. Using a suitable substitution and the result of (a), show that the area of the shaded region is $\frac{1-\ln 2}{2}$.

(5 marks)

(c) (i) Find $\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right)$.

(ii) Using (b) and (c)(i), show that

$$\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\frac{\ln 1.3}{1.3^2}+\cdots+\frac{\ln 1.9}{1.9^2}<5-\frac{41}{8}\ln 2 .$$

(6 marks)

(2014 DSE-MATH-M1 Q10)

28. (a) Consider the function $f(x)=\ln(x^2+16)-\ln(3x+20)$ for $x>\frac{-20}{3}$.

(i) Find the range of values of x such that $f(x)<0$.

(ii) Consider the integral $I=\int_0^4 f(x)dx$.

(1) Using the trapezoidal rule with 4 subintervals, find an estimate for I .

(2) Determine whether the estimate in (1) is an over-estimate or under-estimate. Justify your answer.

(8 marks)

(b) A certain species of insects lives in a certain environment. Let $N(t)$ (in thousand) be the number of the insects at time t (in months). Assume that $N(t)$ can be treated as a differentiable function when $N(t)>0$. The birth rate and death rate of the insects at time t are equal to $10\ln(t^2+16)$ and $10\ln(3t+20)$ respectively when $N(t)>0$. It is given that $N(0)=8$.

(i) Express $N'(t)$ in terms of t when $N(t)>0$.

(ii) Jane claims that the species will not die out until $t=4$. Do you agree? Justify your answer.

(4 marks)

(2013 DSE-MATH-M1 Q10)

29. Let $P(t)$ and $C(t)$ (in suitable units) be the electric energy produced and consumed respectively in a city during the time period $[0,t]$, where t is in years and $t\geq 0$. It is known that $P'(t)=4\left(4-e^{\frac{-t}{5}}\right)$ and $C'(t)=9\left(2-e^{\frac{-t}{10}}\right)$. The redundant electric energy being generated during the time period $[0,t]$ is $R(t)$, where $R(t)=P(t)-C(t)$ and $t\geq 0$.

(a) Find t such that $R'(t)=0$.

(3 marks)

(b) Show that $R'(t)$ decreases with t .

(3 marks)

(c) Find the total redundant electric energy generated during the period when $R'(t)>0$.

(3 marks)

(d) The electric energy production is improved at $t=5$. Let $Q(t)$ be the electric energy produced during the period $[5,t]$, where $t\geq 5$, and

$$Q'(t)=\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}+9 .$$

Find the total electric energy produced for the first 3 years after the improvement.

(5 marks)

[Note: For definite integrals, answers obtained by using numerical integration functions in calculators are not accepted.]

(2013 DSE-MATH-M1 Q11)

30. Let $I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,dt$.

(a) (i) Use the trapezoidal rule with 6 sub-intervals to estimate I .

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Justify your answer.

(7 marks)

(b) Using a suitable substitution, show that $I=2\int_1^2 e^{\frac{-x^2}{2}}\,dx$.

(3 marks)

(c) Using the above results and the Standard Normal Distribution Table, show that $\pi<3.25$.

(3 marks)

(2012 DSE-MATH-M1 Q10)

31. An engineer models the rates of the production of an alloy in the first 10 weeks by two new machines A and B respectively by

$$\frac{dx}{dt}=\frac{61t}{(t+1)^{\frac{5}{2}}} \quad \text{and} \quad \frac{dy}{dt}=\frac{15\ln\left(t^2+100\right)}{16} \quad \text{for } 0\le t\le 10,$$

where x (in million kg) and y (in million kg) are the amount of the alloy produced by machines A and B respectively, and t (in weeks) is the time elapsed since the beginning of the production.

(a) Using the substitution $u=t+1$, find the amount of the alloy produced by machine A in the first 10 weeks.

(4 marks)

(b) Using the trapezoidal rule with 5 sub-intervals, estimate the amount of the alloy produced by machine B in the first 10 weeks.

(2 marks)

(c) The engineer uses the results of (a) and (b) to claim that machine B is more productive than machine A in the first 10 weeks. Do you agree? Explain your answer.

(4 marks)

(PP DSE-MATH-M1 Q10)

32. (a) Let $f(t)$ be a function defined for all $t\ge 0$. It is given that

$$f'(t)=e^{2bt}+ae^{bt}+8,$$

where a and b are negative constants and $f(0)=0$, $f'(0)=3$ and $f'(1)=4.73$.

(i) Find the values of a and b .

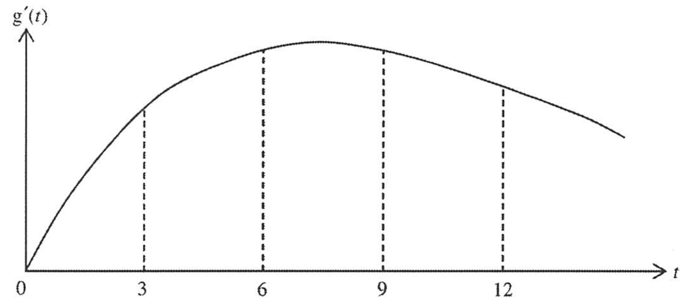
(ii) By taking $b=-0.5$, find $f(12)$.

(5 marks)

(b) Let $g(t)$ be another function defined for all $t\ge 0$. It is given that

$$g'(t)=\frac{33}{10}te^{-kt},$$

where k is a positive constant. Figure 1 shows a sketch of the graph of $g'(t)$ against t. It is given that $g'(t)$ attains the greatest value at $t=7.5$ and $g(0)=0$.



(i) Find the value of k.

(ii) Use the trapezoidal rule with four sub-intervals to estimate $g(12)$.

(6 marks)

(c) From the estimated value obtained in (b)(ii) and Figure 1, Jenny claims that $g(12)>f(12)$. Do you agree? Explain your answer.

(2 marks)

(SAMPLE DSE-MATH-M1 Q12)

33. In a certain country, the daily rate of change of the amount of oil production P , in million barrels per day, can be modelled by

$$\frac{dP}{dt}=\frac{k-3t}{1+ae^{-bt}}$$

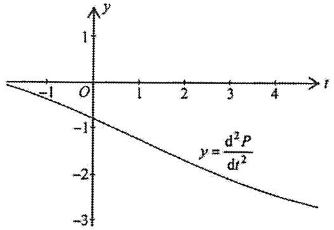
where $t\ (\geq 0)$ is the time measured in days. When $\ln\left(\dfrac{k-3t}{\frac{dP}{dt}}-1\right)$ is plotted against t , the graph is a straight line with slope -0.3 and the intercept on the horizontal axis 0.32 . Moreover, P attains its maximum when $t=3$.

(a) Find the values of a , b and k .

(5 marks)

(b) (i) Using trapezoidal rule with 6 subintervals, estimate the total amount of oil production from $t=0$ to $t=3$.

(ii)



The figure shows the graph of $y=\dfrac{d^2P}{dt^2}$. Using the graph, determine whether the estimation in (i) is an under-estimate or an over-estimate.

(4 marks)

(c) The daily rate of change of the demand for oil D , in million barrels per day, can be modelled by

$$\frac{dD}{dt}=1.63^{2-0.1t}$$

where $t\ (\geq 0)$ is the time measured in days.

(i) Let $y=\alpha^{\beta x}$, where α , β ($\alpha>0$, $\alpha\neq 1$ and $\beta\neq 0$) are constants. Find $\dfrac{dy}{dx}$ in terms of x .

(ii) Find the demand of oil from $t=0$ to $t=3$.

(iii) Does the overall oil production meet the overall demand of oil from $t=0$ to $t=3$? Explain your answer.

(6 marks)

(part (c)(i) is out of syllabus) (2013 ASL-M&S Q8)

34. The population size N (in trillion) of a culture of bacteria increases at the rate of $\dfrac{dN}{dt}=t\ln(2t+1)$, where $t\ (\geq 0)$ is the time measured in days.

It is given that $N=21$ when $t=0$.

(a) (i) Find $\displaystyle\int\frac{t^2}{2t+1}\,dt$.

(ii) Find $\dfrac{d}{dt}[t^2\ln(2t+1)]$.

(iii) Find the population of the culture of bacteria at $t=5$. Correct your answer to the nearest trillion.

(8 marks)

(b) A certain kind of drug is then added to the culture of bacteria at $t=5$. A researcher estimates that the population size M (in trillion) of the bacteria can then be modelled by

$$M=40e^{-2\lambda(t-5)}-20e^{-\lambda(t-5)}+K \qquad (5\leq t\leq 18) ,$$

where t is the time measured in days. K and λ are constants. It is given that $M=27$ when $t=11$.

(i) Using (a), find the value of K correct to the nearest integer. Hence, find the value of λ correct to 1 decimal place.

(ii) By using the value of K correct to the nearest integer and the value of λ correct to 1 decimal place, determine whether M is always decreasing in this model. Hence, explain whether the population of the bacteria will drop to 23 trillion.

(7 marks)

(2013 ASL-M&S Q9)

35. A textile factory has bought two new dyeing machines P and Q. The two machines start to operate at the same time and will emit sewage into a lake near the factory. The manager of the factory estimates the amount of sewage emitted (in tonnes) by the two machines and finds that the rates of emission of sewage by the two machines P and Q can be respectively modelled by

$$p'(t) = 4.5 + 2t(1+6t)^{\frac{-2}{3}} \text{ and}$$

$$q'(t) = 3 + \ln(2t+1) \ ,$$

where $t\ (\geq 0)$ is the number of months that the machines have been in operation.

(a) By using a suitable substitution, find the total amount of sewage emitted by machine P in the first year of operation.

(4 marks)

(b) (i) By using the trapezoidal rule with 5 sub-intervals, estimate the total amount of sewage emitted by machine Q in the first year of operation.

(ii) The manager thinks that the amount of sewage emitted by machine Q will be less than that emitted by machine P in the first year of operation. Do you agree? Explain your answer.

(5 marks)

(c) The manager studies the relationship between the environmental protection tax R (in million dollars) paid by the factory and the amount of sewage x (in tonnes) emitted by the factory. He uses the following model:

$$R = 16 - ae^{-bx} \ ,$$

where a and b are constants.

(i) Express $\ln(16-R)$ as a linear function of x.

(ii) Given that the graph of the linear function in (c)(i) passes through the point $(-10, 1)$ and the x-intercept of the graph is 90, find the values of a and b.

(iii) In addition to the sewage emitted by the machines P and Q, the other operations of the factory emit 80 tonnes of sewage annually. Using the model suggested by the manager and the values of a and b found in (c)(ii), estimate the tax paid by the factory in the first year of the operation of machines P and Q.

(6 marks)

(2012 ASL-M&S Q8)

DSE Mathematics Module 1 **6. Definite Integrals**

36. The current rate of selling of a certain kind of handbags is 30 thousand per day. The sales manager decides to raise the price of the handbags. After the price of the handbags has been raised for t days, the rate of selling of handbags $r(t)$ (in thousand per day) can be modelled by

$$r(t) = 20 - 40e^{-at} + be^{-2at} \qquad (t \geq 0),$$

where a and b are positive constants. From past experience, it is known that after the increase in the price of the handbags, the rate of selling of handbags will decrease for 9 days.

(a) Find the value of b.

(1 mark)

(b) Find the value of a correct to 1 decimal place.

(3 marks)

(c) The sales manager will start to advertise when the rate of change of the rate of selling of handbags reaches a maximum. Use the results obtained in (a) and (b) to find the rate of selling of handbags when the sales manager starts to advertise.

(4 marks)

(d) When the rate of selling of handbags drops below 18 thousand per day, the sales manager will give a 'sales warning' to his team. Use the results obtained in (a) and (b) to find

(i) the duration of the 'sales warning' period correct to the nearest day,

(ii) the average number of handbags sold per day during the 'sales warning' period correct to the nearest thousand.

(7 marks)

(2012 ASL-M&S Q9)

37. An oil tanker leaks out oil for half a day at the rate of

$$\frac{dV}{dt}=\frac{1}{25}e^{t^2+t+2}$$

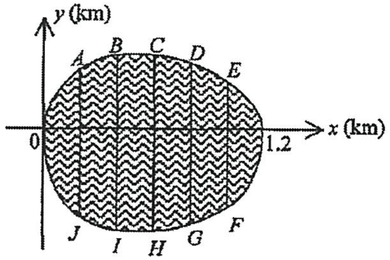
where V is the volume of the oil (in hundred thousand m^3) leaked out and t $(0\le t\le 0.5)$ is the number of days elapsed since the leakage begins.

(a) By finding a polynomial in t of degree 3 which approximates e^{t^2+t}, estimate the volume of the oil leaked out.

Is this an over-estimate or under-estimate? Explain your answer.

(6 marks)

(b) After half a day, the surface area of the ocean affected by the oil spread is as shown by the shaded region in the figure:



$A\,(0.2,1.8)$ $\quad B\,(0.4,2)$ $\quad C\,(0.6,2.2)$ $\quad D\,(0.8,2.1)$ $\quad E\,(1,1.6)$

$F\,(1,-1.8)$ $\quad G\,(0.8,-2)$ $\quad H\,(0.6,-2.1)$ $\quad I\,(0.4,-2.2)$ $\quad J\,(0.2,-2)$

(i) Using the trapezoidal rule, estimate the surface area of the ocean affected.

Is this an over-estimate or under-estimate? Explain your answer.

(ii) Assuming that the thickness of the oil spread is uniform, estimate the thickness of the oil spread.

Is this an over-estimate or under-estimate? Explain your answer.

(5 marks)

(c) Subsequently, the oil company uses a new technology to clean up the oil spread. The rate of cleaning up the oil spread can be modelled by

$$\frac{dW}{dt}=\frac{-(W+1)^{\frac{1}{3}}}{40}$$

where W is the volume of the oil spread (in hundred thousand m^3) remained and t is the number of days elapsed since the beginning of the cleaning up.

How long will it take for all the oil spread to be cleaned up?

(4 marks)

(part (c) is out of syllabus) (2011 ASL-M&S Q8)

38. A company launches a campaign to increase the sales of a product. The monthly increase in sales (in thousand dollars) t months after the launch can be modelled by the function

$$f(t)=-250e^{2at}+300e^{at}-50$$

where a is a non-zero constant.

It is known that the monthly increase in sales attains the maximum 5 months after the launch.

(a) Find the value of a.

(3 marks)

(b) After at least T_1 months, the campaign will not increase the sales.

(i) Find the value of T_1.

(ii) Estimate the total amount of sales increased T_1 months after the launch.

(5 marks)

(c) The start up cost of the campaign is 100 thousand dollars and the running cost at time t is $\frac{100}{t+9}$ thousand dollars. The campaign will be terminated after T_2 months when the total expenditure reaches 200 thousand dollars.

(i) Express the total expenditure E (in thousand dollars) in terms of t.

(ii) Find the value of T_2.

(iii) During the period of the campaign, the manager of the company suggests replacing the campaign by a less costly plan. The monthly increase in sales (in thousand dollars) due to the plan can be modelled by the function

$$g(t)=-(t-\alpha)(t-2\alpha)\ ,\quad \alpha\le t\le 2\alpha$$

where α $(0\le\alpha\le T_2)$ is the time, in months after the launching of the original campaign, of starting the plan.

In order to achieve the maximum total amount of sales increased by the plan, when should it be started? Explain your answer.

(7 marks)

(2010 ASL-M&S Q8)

39. A shop owner wants to launch two promotion plans A and B to raise the revenue. Let R and Q (in million dollars) be the respective cumulative weekly revenues of the shop after the launching of the promotion plans A and B. It is known that R and Q can be modelled by

$$\frac{dR}{dt}=\begin{cases}\ln(2t+1) & \text{when } 0\le t\le 6\\ 0 & \text{when } t>6\end{cases},$$

and

$$\frac{dQ}{dt}=\begin{cases}45t(1-t)+\dfrac{1.58}{t+1} & \text{when } 0\le t\le 1\\ \dfrac{30e^{-t}}{(3+2e^{-t})^2} & \text{when } t>1\end{cases}$$

respectively, where t is the number of weeks elapsed since the launching of a promotion plan.

(a) Suppose plan A is adopted.

(i) Using the trapezoidal rule with 6 sub-intervals, estimate the total amount of revenue in the first 6 weeks since the start of the plan.

(ii) Is the estimate in (a)(i) an over-estimate or under-estimate? Explain your answer briefly.

(4 marks)

(b) Suppose plan B is adopted.

(i) Find the total amount of revenue in the first week since the start of the plan.

(ii) Using the substitution $u=3+2e^{-t}$, or otherwise, find the total amount of revenue in the first n weeks, where $n>1$, since the start of the plan. Express your answer in terms of n.

(6 marks)

(c) Which of the plans will produce more revenue in the long run? Explain your answer briefly.

(5 marks)

(2009 ASL-M&S Q9)

40. The rate of change of yearly average temperature of a city is predicted to be

$$\frac{dx}{dt}=\frac{1}{40}\sqrt{1+t^2}\qquad (t\ge 0),$$

where x is the temperature measured in °C and t is the time measured in years. It is given that $x=22$ and $t=0$.

(a) (i) Using the trapezoidal rule with 4 sub-intervals, estimate the increase of temperature from $t=0$ to $t=10$.

(ii) Determine whether this estimate is an over-estimate or an under-estimate.

(4 marks)

(b) It is known that the electricity consumption $W(x)$, in appropriate units, depends on the yearly average temperature x and is given by

$$W(x)=100(\ln x)^2-630\ln x+1960\qquad (x\ge 22).$$

(i) If $W(x_0)=968$, find all possible value(s) of x_0.

(ii) Find the range of values of x while $W'(x)<0$.

(iii) Find the rate of change of electricity consumption at $t=0$.

(iv) Using (a), estimate the electricity consumption at $t=10$. Determine and explain whether the actual electricity consumption is larger than or smaller than this estimate.

(11 marks)

(2008 ASL-M&S Q9)

41. A financial analyst, Mary, models the rates of change of profit (in billion dollars) made by companies A and B respectively by

$$f(t)=\ln\left(e^t+2\right)+3 \quad \text{and} \quad g(t)=\frac{8e^t}{40-t^2},$$

where t is the time measures in months.

Assume that the two models are valid for $0\le t\le 6$.

(a) (i) Using the trapezoidal rule with 6 sub-intervals, estimate the total profit made by company A from $t=0$ to $t=6$.

(ii) Find $\frac{d^2f(t)}{dt^2}$ and hence determine whether the estimate in (a)(i) is an over-estimate or an under-estimate.

(7 marks)

(b) (i) Expand $\frac{1}{40-t^2}$ in ascending powers of t as far as the term in t^4 .

(ii) Using the result of (b)(i), find the expansion of $\frac{8e^t}{40-t^2}$ in ascending powers of t as far as the term in t^4 .

(iii) Using the result of (b)(ii), estimate the total profit made by company B from $t=0$ to $t=6$.

(6 marks)

(c) Mary claims that the total profit made by company A from $t=0$ to $t=6$ is less than that of company B . Do you agree? Explain your answer.

(2 marks)

(part (b) is out of Syllabus) (2007 ASL-M&S Q8)

42. An engineer. designed a driving test to compare fuel consumption when different driving tactics are used. The rates of change of fuel consumption in litres when using driving tactics A and B can be modelled respectively by

$$f(t)=\frac{1}{4}t(15-t)e^{\frac{-t}{4}} \quad \text{and}$$

$$g(t)=\frac{1}{145}t(15-t)^2$$

where $t(\ge 0)$ is the time measured in minutes from the start of the test.

(a) Use the trapezoidal rule with 5 sub-intervals to estimate the total fuel consumption from $t=0$ to $t=15$ when using driving tactic A .

(3 marks)

(b) Use integration to find the total fuel consumption from $t=0$ to $t=15$ when using driving tactic B .

(3 marks)

(c) Find the greatest value of $f(t)$, where $0\le t\le 15$.

(5 marks)

(d) (i) Find $\frac{d^2f(t)}{dt^2}$.

(ii) By considering $\frac{d^2f(t)}{dt^2}$, can you determine whether the total fuel consumption from $t=0$ to $t=15$ when using driving tactic A will be less than that of using driving tactic B ? Explain your answer.

(4 marks)

(2004 ASL-M&S Q8)

43. According to the past production record, an oil company manager modelled the rate of change of the amount of oil production in thousand barrels by

$$f(t) = 5 + 2^{-kt+h} \ ,$$

where h and k are positive constants and $t(\geq 0)$ is the time measured in months.

(a) Express $\ln(f(t)-5)$ as a linear function of t .

(1 marks)

(b) Given that the slope and the intercept on the vertical axis of the graph of the linear function in (a) are -0.35 and 1.39 respectively, find the values of h and k correct to 1 decimal place.

(2 marks)

(c) The manager decides to start a production improvement plan and predicts the rate of change of the amount of oil production in thousand barrels by

$$g(t) = 5 + \ln(t+1) + 2^{-kt+h} \ ,$$

where h and k are the values obtained in (b) correct to 1 decimal place, and $t(\geq 0)$ is the time measured in months from the start of the plan.
Using the trapezoidal rule with 5 sub-intervals, estimate the total amount of oil production in thousand barrels from $t=2$ to $t=12$.

(2 marks)

(d) It is known that $g(t)$ in (c) satisfies

$$\frac{d^2 g(t)}{dt^2} = p(t) - q(t), \text{ where } q(t) = \frac{1}{(t+1)^2} \ .$$

(i) If $2^t = e^{at}$ for all $t \geq 0$, find a .
(ii) Find $p(t)$.
(iii) It is known that there is no intersection between the curve $y = p(t)$ and the curve $y = q(t)$, where $2 \leq t \leq 12$. Determine whether the estimate in (c) is an over-estimate or under -estimate.

(10 marks)
(2003 ASL-M&S Q8)

44. Lactic acid in large amounts is usually formed during vigorous physical exercise, which leads to fatigue. The amount of lactic acid, M , in muscles is measured in m mol/L. A student modelled the rate of change of the amount of lactic acid in his muscles during vigorous physical exercise by

$$\frac{dM}{dt} = \frac{12e^{\frac{2}{3}t}}{3+t} \qquad (0 \leq t \leq 4) \ ,$$

where t is the time measured in minutes from the start of the exercise.

(a) The student used the trapezoidal rule with 5 sub-intervals to estimate the amount of lactic acid formed after the first 2.5 minutes of exercise.
(i) Find his estimate.
(ii) Find $\frac{d^2}{dt^2}\left[\frac{12e^{\frac{2}{3}t}}{3+t}\right]$ and hence determine whether his estimate is an over-estimate or an under-estimate.

(5 marks)

(b) The student re-estimated the amount of lactic acid formed by expanding $\frac{12e^{\frac{2}{3}t}}{3+t}$ as a series in ascending powers of t .
(i) Expand $\frac{1}{3+t}$ and hence find the expansion of $\frac{12e^{\frac{2}{3}t}}{3+t}$ in ascending powers of t as far as the term in t^3 .
(ii) By integrating the expansion of $\frac{12e^{\frac{2}{3}t}}{3+t}$ in (i), re-estimate the amount of lactic acid formed after the first 2.5 minutes of exercise.

(7 marks)

(c) The student wanted to predict the amount of lactic acid formed in his muscles after the first 4 minutes of exercise. He decided to use the method in (b) to estimate the amount of lactic acid formed. Briefly explain whether his method was valid.

(3 marks)
(part (b) is out of Syllabus) (2002 ASL-M&S Q9)

45. A department store has two promotion plans, F and G, designed to increase its profit, from which only one will be chosen. A marketing agent forecasts that if x hundred thousand dollars is spent on a promotion plan, the respective rates of change of its profit with respect to x can be modelled by

$$f(x)=16+4xe^{-0.25x} \text{ and } g(x)=16+\frac{6x}{\sqrt{1+8x}} .$$

(a) Suppose that promotion plan F is adopted.

(i) Show that $f(x)\le f(4)$ for $x>0$.

(ii) If six hundred thousand dollars is spent on the plan, use the trapezoidal rule with 6 sub-intervals to estimate the expected increase in profit to the nearest hundred thousand dollars.

(6 marks)

(b) Suppose that promotion plan G is adopted.

(i) Show that $g(x)$ is strictly increasing for $x>0$.
As x tends to infinity, what value would $g(x)$ tend to?

(ii) If six hundred thousand dollars is spent on the plan, use the substitution $u=\sqrt{1+8x}$, or otherwise, to find the expected increase in profit to the nearest hundred thousand dollars.

(7 marks)

(c) The manager of the department store notices that if six hundred thousand dollars is spent on promotion, plan F will result in a bigger profit than G . Determine which plan will eventually result in a bigger profit if the amount spent on promotion increases indefinitely. Explain your answer briefly.

(2 marks)

(2000 ASL-M&S Q9)

46. In a 100 m race, the speeds, S_A m/s and S_B m/s , of two athletes A and B respectively can be modelled by the functions

$$S_A=\frac{256}{9625}\left(\frac{1}{3}t^3-\frac{47}{4}t^2+120t\right)$$

$$\text{and } S_B=\frac{183}{50}te^{-kt} ,$$

where k is a positive constant and t is the time measured from the start in seconds.
It is known that A finishes the race in 12.5 seconds and during the race, A and B attain their respective top speeds at the same time.

(a) Find the top speed of A during the race.

(3 marks)

(b) Find the value of k .

(3 marks)

(c) Suppose the model for B is valid for $0\le t\le 12.5$. Use the trapezoidal rule with 5 sub-intervals to estimate the distance covered by B in 12.5 seconds.

(3 marks)

(d) Find $\frac{d^2S_B}{dt^2}$. Hence or otherwise, state with reasons whether B finishes the race ahead of A or not.

(3 marks)

(e) In the same race, the speed, S_C m/s , of another athlete C is modelled by

$$S_C=\frac{50[\ln(t+2)-\ln 2]}{t+2} .$$

Determine whether or not C is the last one to finish the race among the three athletes.

(3 marks)

(1999 ASL-M&S Q8)

47. Mr. Lee has a fish farm in Sai Kung. Last week, the fish in his farm were affected by a certain disease. An expert told Mr. Lee that the number N of fish in his farm could be modelled by the function

$$N = \frac{5000e^{\lambda t}}{t} \qquad (0 < t < 120) \ ,$$

where λ is a constant and t is the number of days elapsed since the disease began to spread.

(a) Suppose that the numbers of fish will be the same when $t = 15$ and $t = 95$.

(i) Find the value of λ .

(ii) How many days after the start of the spread of the disease will the number of fish decrease to the minimum?

(8 marks)

(b) The day that the number of fish decreased to the minimum is called the *Recovery Day*. It is suggested that from the *Recovery Day*, the fish will begin to gain weight according to the model

$$\frac{dW}{ds} = \frac{3}{5}(e^{-\frac{s}{20}} - e^{-\frac{s}{10}}) \qquad (0 < s < 60) \ .$$

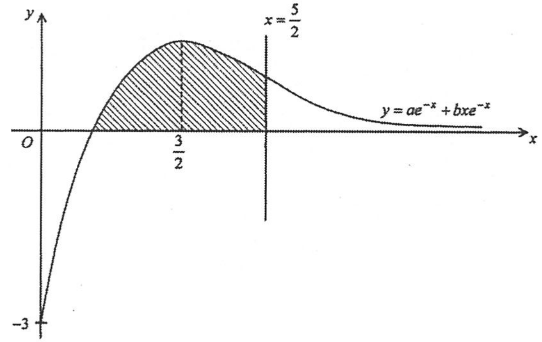
where s is the number of days elapsed since the *Recovery Day* and W is the mean weight of the fish in kg. Find the increase in mean weight of the fish in the first 15 days from the *Recovery Day*. How long will it take for the mean weight of the fish to increase 0.5 kg from the *Recovery Day*?

(7 marks)

(1998 ASL-M&S Q8)

48. The curve in the figure represents the graph of $y = ae^{-x} + bxe^{-x}$ for $x \ge 0$, where a and b are constants. The y-intercept of the curve is -3 and y attains its maximum when $x = \frac{3}{2}$.

Define $I = \int_{\frac{1}{2}}^{\frac{5}{2}} e^{-x}dx$ and $J = \int_{\frac{1}{2}}^{\frac{5}{2}} xe^{-x}dx$.



(a) Evaluate I .

(2 marks)

(b) Find the values of a and b .

(4 marks)

(c) Find the x-intercept and the coordinates of the point(s) of inflection of the curve.

(4 marks)

(d) Let A be the area of the shaded region in Figure 1 bounded by the curve, the x-axis and the line $x = \frac{5}{2}$. Let J_0 be an estimate of J obtained by using the trapezoidal rule with 4 sub-intervals.

A student uses $A_0 = aI + bJ_0$ to estimate A .

(i) Find A_0 .

(ii) The student made the following argument:

> Since $\frac{d^2y}{dx^2} < 0$ for $\frac{1}{2} < x < \frac{5}{2}$,
> the curve is concave downward in the interval,
> therefore J_0 is an underestimate of J ,
> and hence A_0 is an underestimate of A .

Determine whether the student's argument is correct or not. Explain your answer briefly.

(5 marks)

(part (c) is out of Syllabus) (1998 ASL-M&S Q9)

49. Let $y = x^x$, $I = \int_1^2 x^x dx$ and $J = \int_1^2 x^x \ln x dx$.

(a) Using logarithmic differentiation, find $\frac{dy}{dx}$.

(2 marks)

(b) By finding $\frac{d^2y}{dx^2}$, state whether I would be overestimated or underestimated if the trapezoidal rule is used to estimate I . Explain your answer briefly.

(3 marks)

(c) Using (a) or otherwise, show that $I + J = 3$.

(2 marks)

(d) Let J_0 be an estimate of J obtained by using the trapezoidal rule with 5 sub-intervals.

(i) Find J_0 .

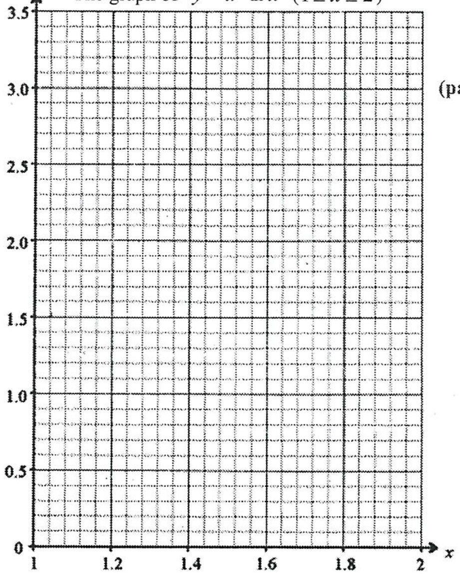
(ii) Plot the graph of $y = x^x \ln x$ on the graph paper. Hence state whether J_0 is an overestimate or underestimate of J . Explain your answer briefly.

(iii) How can the estimation be improved if the trapezoidal rule is applied again to estimate J ?

(iv) Let $I_0 = 3 - J_0$. State whether I_0 is an overestimate or underestimate of I . Explain your answer briefly.

(8 marks)

The graph of $y = x^x \ln x$ $(1 \le x \le 2)$



(1997 ASL-M&S Q10)

(part (a) is out of syllabus)

50. The population size P of a species of reptiles living in a jungle increases at a rate of

$$\frac{dP}{dt} = 5e^{\frac{t^2}{10}} - 2t \qquad (t \ge 0),$$

where t is the time in month. It is known that $P = 10$ when $t = 0$.

(a) Use the trapezoidal rule with 6 sub-intervals to estimate $\int_0^6 e^{\frac{t^2}{10}} dt$.

Hence estimate P , to the nearest integer, at $t = 6$.

(7 marks)

(b) A chemical plant was recently built near the jungle. Pollution from the plant affects the growth of the population of the reptiles from $t = 6$ onwards. An ecologist suggests that the population size of the species of reptiles can then be approximated by

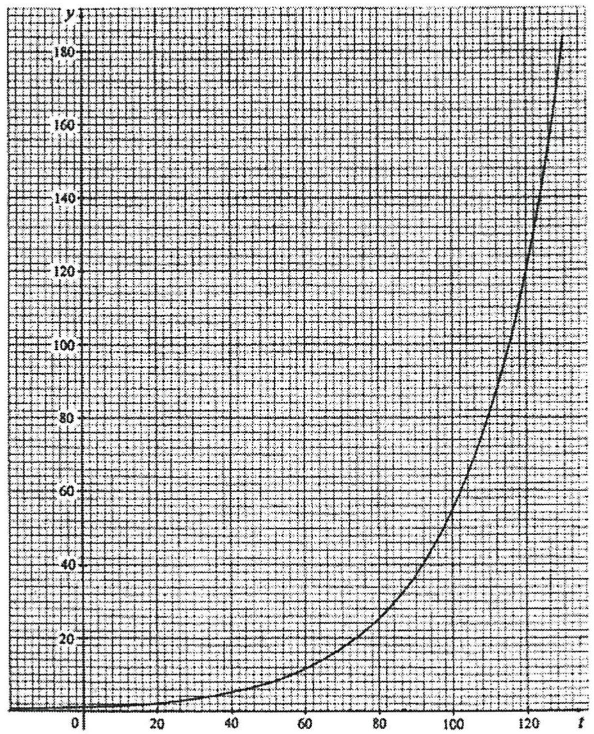
$$P = kte^{-0.04t} - 50 \qquad (t \ge 6).$$

(i) Using (a), find the value of k correct to 1 decimal place.

(ii) Determine the time at which the population size will attain its maximum. Hence find the maximum population size correct to the nearest integer.

(iii) Use the graph in the figure to find the value of t , correct to the nearest integer, when the species of reptiles becomes extinct due to pollution.

The graph of $y = e^{0.04t}$



(8 marks)

(1996 ASL-M&S Q9)

51. The monthly cost $C(t)$ at time t of operating a certain machine in a factory can be modelled by

$$C(t) = ae^{bt} - 1 \qquad (0 < t \leq 36),$$

where t is in month and $C(t)$ is in thousand dollars.

Table 2 shows the values of $C(t)$ when $t = 1, 2, 3, 4$.

Table

t	1	2	3	4
$C(t)$	1.21	1.44	1.70	1.98

(a) (i) Express $\ln[C(t)+1]$ as a linear function of t.

(ii) Use the table and a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

(iii) Using the values of a and b found in (a)(ii), estimate the monthly cost of operating this machine when $t = 36$.

(8 marks)

(b) The monthly income $P(t)$ generated by this machine at time t can be modelled by

$$P(t) = 439 - e^{0.2t} \qquad (0 < t \leq 36),$$

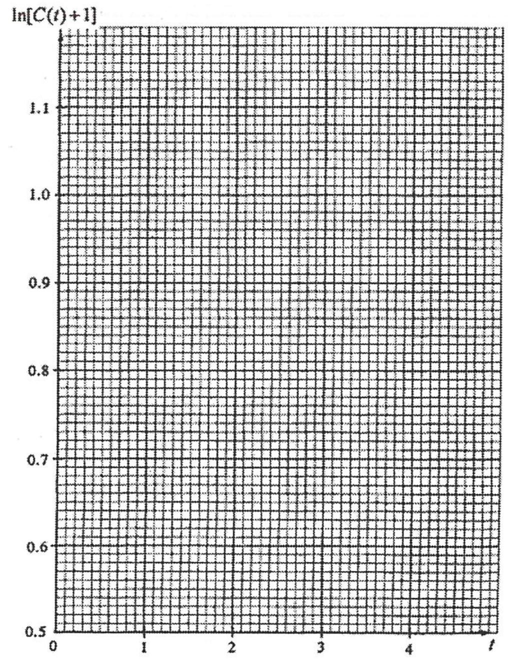
where t is in month and $P(t)$ is in thousand dollars.

The factory will stop using this machine when the monthly cost of operation exceeds the monthly income.

(i) Find the value of t when the factory stops using this machine. Give the answer correct to the nearest integer.

(ii) What is the total profit generated by this machine? Give the answer correct to the nearest thousand dollars.

(7 marks)



(1996 ASL-M&S Q10)

52. Let $f(x)=\dfrac{1}{\sqrt{1-x^2}}$ where $0\le x\le\dfrac{1}{2}$, and $I=\int_0^{\frac{1}{2}}f(x)dx$.

(a) (i) Find the estimate I_1 of I using the trapezoidal rule with 5 sub-intervals.

(ii) Find $f'(x)$ and $f''(x)$.

(iii) Using (a)(ii) or otherwise, state whether in (a)(i) is an over-estimate or under-estimate of I . Explain your answer briefly.

(7 marks)

(b) (i) Using the binomial expansion to find a polynomial $p(x)$ of degree 6 which approximates $f(x)$ for $0\le x\le\dfrac{1}{2}$.

Let $I_2=\int_0^{\frac{1}{2}}p(x)dx$. Find I_2 .

(ii) State whether I_2 in (b)(i) is an over-estimate or under-estimate of I . Explain your answer briefly.

(8 marks)

(part (b) is out of Syllabus) (1995 ASL-M&S Q7)

53. A chemical plant discharges pollutant to be a lake at an unknown rate of $r(t)$ units per month, where t is the number of months that the plant has been in operation.

Suppose that $r(0)=0$.

The government measured $r(t)$ once every two months and reported the following figures:

t	2	4	6	8
$r(t)$	11	32	59	90

(a) Use the trapezoidal rule to estimate the total amount of pollutant which entered the lake in the first 8 months of the plant's operation.

(2 mark)

(b) An environmental scientist suggests that

$$r(t)=at^b,$$

where a and b are constants.

(i) Use a graph paper to estimate graphically the values of a and b correct to 1 decimal place.

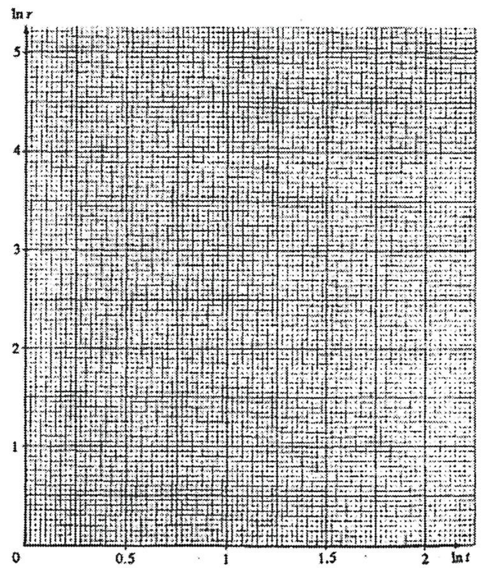
(ii) Based on this scientist's model, estimate the total amount of pollutant, correct to 1 decimal place, which entered the lake in the first 8 months of the plant's operation.

(8 mark)

(c) It is known that no life can survive when 1000 units of pollutant have entered the lake. Adopting the scientist's model in (b), how long does it take for the pollutant from the plant to destroy all life in the lake? Give your answer correct to the nearest month.

(5 mark)

(1994 ASL-M&S Q10)



2021 DSE Q11

Let $f(x)=\left(\frac{x}{2-x}\right)^{\frac{1}{2}}$, where $0 \le x \le 1$.

(a) Find $f'(x)$ and $f''(x)$. (3 marks)

(b) Define $J=\int_0^{0.5} f(x)\,dx$ and $K=\int_{0.5}^1 f(x)\,dx$.

(i) Using the trapezoidal rule with 5 sub-intervals, estimate J.

(ii) Using the fact that $\int_0^1 f(x)\,dx=\frac{\pi-2}{2}$ and the result of (b)(i), estimate K.

(iii) Someone claims that $\frac{J}{K}<0.44$. Do you agree? Explain your answer.

(8 marks)

6. Definite Integrals

Section A

1. (2020 DSE-MATH-M1 Q8)

8. (a) $\frac{d}{dx}(xe^{mx})$		
$= mxe^{mx}+e^{mx}$	1A	
So, we have $xe^{mx}=\frac{1}{m}\left(\frac{d}{dx}(xe^{mx})-e^{mx}\right)$.		
$\int xe^{mx}\,dx$		
$=\frac{1}{m}\left(xe^{mx}-\int e^{mx}\,dx\right)$	1M	
$=\frac{xe^{mx}}{m}-\frac{e^{mx}}{m^2}+\text{constant}$	1A	
(b) Note that the x-intercept of the curve $y=xe^{mx}$ is 0.		
$\int_0^1 xe^{mx}\,dx=\frac{1}{m}$	1M	
$\left[\frac{xe^{mx}}{m}-\frac{e^{mx}}{m^2}\right]_0^1=\frac{1}{m}$	1M	for using the result of (a)
$\frac{e^m}{m}-\frac{e^m}{m^2}+\frac{1}{m^2}=\frac{1}{m}$		
$me^m-e^m-m+1=0$		
$(m-1)(e^m-1)=0$	1M	
$m=1$ or $m=0$ (rejected)		
Thus, we have $m=1$.	1A	
	----------(7)	

2. (2019 DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) For all $x > -3$, $f'(x) = \frac{(x+3)(-1)-(6-x)(1)}{(x+3)^2} = \frac{-9}{(x+3)^2} < 0$. Thus, $f(x)$ is decreasing.	1	
Note that $f(x) = \frac{9}{x+3} - 1$ for all $x > -3$. Thus, $f(x)$ is decreasing.	1	
(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{6}{x}-1}{1+\frac{3}{x}} = -1$	1A	
$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{9}{x+3}-1\right) = -1$	1A	
(c) For $y=0$, we have $x=6$. The required area		
$= \int_0^6 f(x)\,dx$	1M	
$= \int_0^6 \frac{6-x}{x+3}\,dx$		
$= \int_0^6 \left(\frac{9}{x+3}-1\right)dx$	1M	
$= [9\ln(x+3)-x]_0^6$	1M	
$= 9\ln 3 - 6$	1A	
For $y=0$, we have $x=6$. The required area		
$= \int_0^6 f(x)\,dx$	1M	
$= \int_0^6 \frac{6-x}{x+3}\,dx$		
$= \int_3^9 \frac{6-(u-3)}{u}\,du$ (by letting $u = x+3$)	1M	
$= \int_3^9 \left(\frac{9}{u}-1\right)du$		
$= [9\ln u - u]_3^9$	1M	
$= 9\ln 3 - 6$	1A	
	----------(6)	

(a)	Good. Some candidates were unable to show that $f'(x) < 0$ to complete the proof.
(b)	Good. Some candidates were unable to consider $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{6}{x}-1}{1+\frac{3}{x}}$ to obtain the required limit.
(c)	Good. Many candidates were able to use integration to obtain the required area, but some candidates were unable to give the answer in exact value.

3. (2019 DSE-MATH-M1 Q8)

Solution	Marks	Remarks
(a) $\ln 7^{\frac{-1}{\ln 7}} = \frac{-1}{\ln 7}(\ln 7) = -1$		
$7^{\frac{-1}{\ln 7}} = e^{-1} = \frac{1}{e}$	1A	
(b) $\frac{d}{dx}(x7^{-x}) = 7^{-x} - x(7^{-x}\ln 7)$	1M	for $\frac{d}{dx}(7^{-x}) = -7^{-x}\ln 7$
So, we have $x7^{-x} = \frac{1}{\ln 7}\left(7^{-x} - \frac{d}{dx}(x7^{-x})\right)$.		
$\int x7^{-x}dx = \frac{1}{\ln 7}\left(\int 7^{-x}dx - x7^{-x}\right)$	1M	
$= \frac{1}{\ln 7}\left(\frac{-7^{-x}}{\ln 7} - x7^{-x}\right) + \text{constant}$	1A	
$= \frac{-1}{\ln 7}\left(\frac{1}{\ln 7} + x\right)7^{-x} + \text{constant}$		
(c) For $h'(x) = 0$, we have $7^{-x}(1 - x\ln 7) = 0$. So, we have $\alpha = \frac{1}{\ln 7}$.	1A	r.t. 0.5139
$\int_0^{\alpha} h(x)\,dx = \left[\frac{-1}{\ln 7}\left(\frac{1}{\ln 7}+x\right)7^{-x}\right]_0^{\frac{1}{\ln 7}}$	1M	
$= \frac{-1}{\ln 7}\left(\frac{2(7^{\frac{-1}{\ln 7}})}{\ln 7} - \frac{1}{\ln 7}\right)$		
$= \frac{1}{(\ln 7)^2}\left(1 - \frac{2}{e}\right)$ (by (a))	1A	
$= \frac{e-2}{e(\ln 7)^2}$		
	----------(7)	

(a)	Good. Many candidates were able to obtain the required answer by taking logarithms.
(b)	Good. Many candidates were able to obtain the required indefinite integral by considering $\frac{d}{dx}(x7^{-x})$.
(c)	Fair. Some candidates were able to obtain the required definite integral by using the result of (b).

4. (2018 DSE-MATH-M1 Q8)

Note that $3x^2-24x+49=3(x-4)^2+1\neq 0$.

(a) $f'(x)=0$

$\dfrac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

x	$(-\infty,4)$	4	$(4,\infty)$
$f'(x)$	$-$	0	+

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

$f'(x)=0$

$\dfrac{12x-48}{(3x^2-24x+49)^2}=0$ 1M

$x=4$

$f''(x)$

$=\dfrac{-108x^2+864x-1716}{(3x^2-24x+49)^3}$

$f''(4)$

$=12$

>0

So, $f(x)$ attains its minimum value at $x=4$.

Thus, we have $\alpha=4$. 1A

(b) (i) Let $v=3x^2-24x+49$. Then, we have $\dfrac{dv}{dx}=6x-24$.

$f(x)$

$=\displaystyle\int\frac{12x-48}{(3x^2-24x+49)^2}dx$

$=\displaystyle\int\frac{2}{v^2}dv$ 1M

$=\dfrac{-2}{v}+C$

$=\dfrac{-2}{3x^2-24x+49}+C$

Since $f(x)$ has only one extreme value, we have $f(4)=5$. 1M

$\dfrac{-2}{3(4)^2-24(4)+49}+C=5$

$C=7$

Thus, we have $f(x)=\dfrac{-2}{3x^2-24x+49}+7$. 1A

(ii) $\displaystyle\lim_{x\to\infty}f(x)$

$=7$ 1A

----------(6)

(a)	Very good. Over 85% of the candidates were able to find the value of α .
(b) (i)	Good. Many candidates were able to find $f(x)$ by indefinite integral but some candidates were unable to use a suitable substitution.
(ii)	Fair. Only some candidates were able to find the constant of integration in (b)(i), and thus the required limit.

5. (2017 DSE-MATH-M1 Q8)

(a) Let $u=\ln x$. 1M

So, we have $\dfrac{du}{dx}=\dfrac{1}{x}$.

$\displaystyle\int g(x)dx$

$=\displaystyle\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$

$=\displaystyle\int\left(\frac{1}{x}(1-\ln x)\right)dx$

$=\displaystyle\int(1-u)du$ 1M

$=u-\dfrac{1}{2}u^2+\text{constant}$

$=\ln x-\dfrac{1}{2}(\ln x)^2+\text{constant}$ 1A

Let $u=\ln\left(\dfrac{e}{x}\right)$. 1M

Then, we have $\dfrac{du}{dx}=\dfrac{-1}{x}$.

$\displaystyle\int g(x)dx$

$=\displaystyle\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$

$=\displaystyle\int -u\,du$ 1M

$=\dfrac{-1}{2}u^2+\text{constant}$

$=\dfrac{-1}{2}\left(\ln\left(\dfrac{e}{x}\right)\right)^2+\text{constant}$ 1A

(b) (i) e 1A

(ii) The required area

$=\displaystyle\int_1^e g(x)dx+\int_e^{e^2}-g(x)dx$ 1M

$=\left[\ln x-\dfrac{1}{2}(\ln x)^2\right]_1^e+\left[-\ln x+\dfrac{1}{2}(\ln x)^2\right]_e^{e^2}$ (by (a)) 1M for using (a)

$=\dfrac{1}{2}+\dfrac{1}{2}$

$=1$ 1A

The required area

$=\displaystyle\int_1^e g(x)dx+\int_e^{e^2}-g(x)dx$ 1M

$=\left[\dfrac{-1}{2}\left(\ln\left(\dfrac{e}{x}\right)\right)^2\right]_1^e+\left[\dfrac{1}{2}\left(\ln\left(\dfrac{e}{x}\right)\right)^2\right]_e^{e^2}$ (by (a)) 1M for using (a)

$=\dfrac{1}{2}+\dfrac{1}{2}$

$=1$ 1A

----------(7)

(a)	Very good. Most candidates were able to use a correct substitution in finding $\displaystyle\int\left(\frac{1}{x}\ln\left(\frac{e}{x}\right)\right)dx$.
(b) (i)	Very good. Many candidates were able to write down the x-intercept of Γ . However, some candidates wrongly gave $(e,0)$ instead of e as the answer.
(ii)	Fair. Many candidates were unable to note that part of Γ lies above the x-axis while part of Γ lies below the x-axis.

6. (2016 DSE-MATH-M1 Q6)

Solution	Marks	Remarks
(a) $\int f(x)\,dx$		
$=\int (3^{2x}-10(3^x)+9)\,dx$		
$=\dfrac{3^{2x}}{2\ln 3}-\dfrac{10(3^x)}{\ln 3}+9x+\text{constant}$	1M+1A	1M for $\int a^x dx=\dfrac{a^x}{\ln a}+\text{constant}$
(b) (i) $3^{2x}-10(3^x)+9=0$		
$(3^x)^2-10(3^x)+9=0$	1M	
$3^x=1$ or $3^x=9$		
$x=0$ or $x=2$	1A	for both
Thus, the x-intercepts are 0 and 2.		
(ii) The area of the region bounded by C and the x-axis		
$=-\int_0^2 f(x)\,dx$	1M	
$=-\left[\dfrac{3^{2x}}{2\ln 3}-\dfrac{10(3^x)}{\ln 3}+9x\right]_0^2$ (by (a))		
$=-\left(\dfrac{81}{2\ln 3}-\dfrac{90}{\ln 3}+18\right)+\left(\dfrac{1}{2\ln 3}-\dfrac{10}{\ln 3}\right)$		
$=\dfrac{40}{\ln 3}-18$	1A	
	----------(6)	

(a)	Fair. Some candidates wrongly evaluated the indefinite integral $\int 3^x dx$ as $\ln 3(3^x)+\text{constant}$ instead of $\dfrac{3^x}{\ln 3}+\text{constant}$.
(b) (i)	Very good. More than 70% of the candidates were able to find the two x-intercepts of C , while a small number of candidates were unable to write a quadratic equation in 3^x .
(ii)	Fair. Although many candidates were able to use the results of (a) and (b)(i) to find the area of the required region, they were unable to give the answer in exact value. $\dfrac{e^6}{2}-2e^4-\dfrac{e^2}{2}$.

7. (2016 DSE-MATH-M1 Q8)

Solution	Marks	Remarks
(a) $f'(x)$		
$=\dfrac{x\left(2(\ln x)\dfrac{1}{x}\right)-(\ln x)^2}{x^2}$	1M	for quotient rule
$=\dfrac{2\ln x-(\ln x)^2}{x^2}$		
$=\dfrac{(2-\ln x)(\ln x)}{x^2}$		
$f'(x)=0$		
$\ln x=2$ or $\ln x=0$		
$x=e^2$ or $x=1$		
$\alpha=e^2$ and $\beta=1$	1A+1A	
(b) Let $u=\ln x$.	1M	
Then, we have $\dfrac{du}{dx}=\dfrac{1}{x}$.		
$\int_\beta^\alpha f(x)\,dx$		
$=\int_1^{e^2}\dfrac{(\ln x)^2}{x}\,dx$		
$=\int_0^2 u^2\,du$	1A	
$=\left[\dfrac{u^3}{3}\right]_0^2$	1M	
$=\dfrac{8}{3}$	1A	r.t. 2.6667
≈ 2.666666667		
≈ 2.6667		
	----------(7)	

(a)	Very good. More than 60% of the candidates were able to apply quotient rule or product rule to find $f'(x)$ and hence find the values of α and β by solving the equation $f'(x)=0$, while some candidates wrongly wrote the value of β as 0 instead of 1 .
(b)	Good. Many candidates employed a suitable substitution in evaluating the definite integral $\int_1^{e^2}\dfrac{(\ln x)^2}{x}dx$.

8. (2015 DSE-MATH-M1 Q6)

(a) $e^{2x}+e^4=e^{x+3}+e^{x+1}$

$(e^x)^2-(e^3+e)e^x+e^4=0$ — 1M

$(e^x-e)(e^x-e^3)=0$

$e^x=e$ or $e^x=e^3$

$x=1$ or $x=3$

Thus, the x-coordinates are 1 and 3. — 1A

(b) The area of the region bounded by C_1 and C_2

$=\int_1^3 (e^{x+3}+e^{x+1}-(e^{2x}+e^4))\,dx$ — 1M+1A

$=\left[e^{x+3}+e^{x+1}-\frac{e^{2x}}{2}-e^4x\right]_1^3$ — 1M

$=\frac{e^6}{2}-2e^4-\frac{e^2}{2}$ — 1A

----------(6)

(a)	Good. Many candidates were able to find the x-coordinates of the two points of intersection of C_1 and C_2, while some candidates failed to write a quadratic equation in e^x.
(b)	Good. Some candidates failed to give a simplified answer and left an absolute value sign in the answer, and some candidates got a wrong answer $-\frac{e^6}{2}+2e^4+\frac{e^2}{2}$ instead of $\frac{e^6}{2}-2e^4-\frac{e^2}{2}$.

9. (2014 DSE-MATH-M1 Q4)

(a) Let $u=t^2+4t+11$.

$du=(2t+4)dt$ } 1A

When $t=1, u=16$; when $t=3, u=32$.

$\int_1^3 \frac{t+2}{t^2+4t+11}dt=\int_{16}^{32}\frac{1}{u}\frac{du}{2}$ — 1M — OR $\frac{1}{2}\int_{t=1}^3\frac{d(t^2+4t+11)}{t^2+4t+11}$

$=\frac{1}{2}[\ln|u|]_{16}^{32}$ — 1A

$=\frac{\ln 32-\ln 16}{2}$ — 1A

$=\frac{\ln 2}{2}$ — OR 0.3466

(b) $\int_1^3\frac{t^2+3t+9}{t^2+4t+11}dt=\int_1^3\left(1-\frac{t+2}{t^2+4t+11}\right)dt$ — 1M

$=[t]_1^3-\int_1^3\frac{t+2}{t^2+4t+11}dt$

$=2-\frac{\ln 2}{2}$ — 1A — OR 1.6534

(6)

(a)	Very good.
(b)	Poor. Many candidates seemed to have no idea about how to solve the problem.

10. (2013 DSE-MATH-M1 Q5)

(a) $\frac{d}{dx}(x\ln x)=(1)\ln x+x\left(\frac{1}{x}\right)$

$=\ln x+1$ — 1A

(b) $\ln x=\frac{d}{dx}(x\ln x)-1$

$\int_1^e \ln x\,dx=[x\ln x]_1^e-\int_1^e 1\,dx$ — 1M

$=e\ln e-\ln 1-[x]_1^e$ — 1A — For x

$=1$ — 1A

(4)

(a)	Excellent.
(b)	Satisfactory. Some candidates failed to use the result of (a), while some others wrote $x\ln x$ instead of $[x\ln x]_1^e$.

11. (2012 DSE-MATH-M1 Q5)

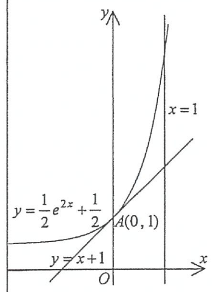
(a) $\frac{dy}{dx}=e^{2x}$

$y=\frac{1}{2}e^{2x}+C$ — 1A

Since $A(0,1)$ lies on S, we have $1=\frac{1}{2}e^{2(0)}+C$. — 1M

i.e. $C=\frac{1}{2}$

Hence the equation of S is $y=\frac{1}{2}e^{2x}+\frac{1}{2}$. — 1A



(b) At $A(0,1)$, $\frac{dy}{dx}=e^{2(0)}=1$.

Hence the equation of L is $y-1=1(x-0)$. — 1M

i.e. $y=x+1$ — 1A

(c) The area of the region bounded by S, L and the line $x=1$

$=\int_0^1\left[\left(\frac{1}{2}e^{2x}+\frac{1}{2}\right)-(x+1)\right]dx$ — 1M — 1M for $A=\int_0^1(y_1-y_2)dx$

$=\left[\frac{1}{4}e^{2x}-\frac{1}{2}x^2-\frac{1}{2}x\right]_0^1$

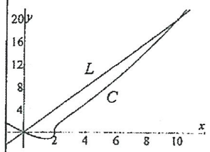
$=\frac{e^2-5}{4}$ — 1A — OR 0.5973

(7)

(a)	Satisfactory. Some candidates omitted the constant of integration or wrote $\int e^{2x}dx=2e^{2x}+C$ while others mixed S with L.
(b)	Satisfactory. Some candidates treated e^{2x} as the slope of L and wrote $y=e^{2x}x+1$ as the equation of L.
(c)	Poor. Some candidates regarded $y=e^{2x}$ as the equation of S.

12. (2013 DSE-MATH-M1 Q3)

Solution	Marks	Remarks
(a) $y=x(x-2)^{\frac{1}{3}}$		
$\frac{dy}{dx}=(x-2)^{\frac{1}{3}}+\frac{1}{3}(x-2)^{\frac{-2}{3}}x$	1M	For product rule
When $x=3$, $\frac{dy}{dx}=2$.		
Hence the equation of L is $y=2x$.	1A	
(b) Solving C and L :		
$x(x-2)^{\frac{1}{3}}=2x$	1M	
$x\left[(x-2)^{\frac{1}{3}}-2\right]=0$		
$x=0$ or 10	1A	
(c) The area bounded by L and C		
$=\int_0^{10}\left[2x-x(x-2)^{\frac{1}{3}}\right]dx$	1M	
$=\int_0^{10}2x\,dx-\int_0^{10}x(x-2)^{\frac{1}{3}}dx$		
Let $u=x-2$ and so $du=dx$.	1M	
When $x=0$, $u=-2$; when $x=10$, $u=8$.		
$\therefore$ the area bounded by L and C		
$=\int_0^{10}2x\,dx-\int_{-2}^{8}(u+2)u^{\frac{1}{3}}du$		
$=[x^2]_0^{10}-\int_{-2}^{8}\left(u^{\frac{4}{3}}+2u^{\frac{1}{3}}\right)du$		
$=100-\left[\frac{3}{7}u^{\frac{7}{3}}+\frac{3}{2}u^{\frac{4}{3}}\right]_{-2}^{8}$	1M	For the primitive function
$=\frac{148+9\sqrt[3]{2}}{7}$	1A	OR 22.7628
	(8)	



(a)	Good. Some candidates found the equation of the tangent to C at $x=3$ instead of the equation of L .
(b)	Good. Some candidates did not know how to solve equations with fraction exponents or missed out the root $x=0$ by dividing both sides of an equation by x .
(c)	Fair. Most candidates made mistakes in finding correct primitive functions or calculating the final answer.

DSE Mathematics Module 1 6. Definite Integrals

13. (PP DSE-MATH-M1 Q5)

Solution	Marks	Remarks
(a) $1-\frac{e}{e^x}=e^x-e$		
$(e^x)^2-(e+1)e^x+e=0$	1A	
$e^x=1$ or e		
$x=0$ or 1	1A	
(b) The area of the region bounded by C_1 and C_2		
$=\int_0^1\left[1-\frac{e}{e^x}-(e^x-e)\right]dx$	1M	For lower and upper limits
$=\left[x+e\cdot e^{-x}-e^x+ex\right]_0^1$	1M	Accept $\left[e^x-ex-x-e\cdot e^{-x}\right]_0^1$
$=1+1-e+e-e+1$		
$=3-e$	1A	
	(5)	

(a)	平平。部分學生不懂分解 $e^{2x}-(e+1)e^x+e=0$。
(b)	平平。部分學生沒有說明積分法正確的上下限。

14. (SAMPLE DSE-MATH-M1 Q9)

Solution	Marks	Remarks
(a) $y=x^3+7$		
$\frac{dy}{dx}=3x^2$	1A	
$\left.\frac{dy}{dx}\right\vert_{x=2}=12$		
Hence L is $y-15=12(x-2)$	1M	For point-slope form
i.e. $y=12x-9$	1A	
(b) The area $=\int_0^2\left(x^3+7-12x+9\right)dx$	1M+1M	1M for $A=\int_a^b(y_1-y_2)dx$ 1M for using (a)
$=\left[\frac{x^4}{4}-6x^2+16x\right]_0^2$	1A	For $\frac{x^4}{4}-6x^2+16x$
$=12$	1A	
	(7)	

15. (2013 ASL-M&S Q3)

(a) $R(t) = Ae^{-0.5t} + B$

$R(t) \to 10$ when $t \to \infty$ — 1M

$\therefore\ B = 10$ — 1A

$R(0) = 500$

$500 = A + B$

$\therefore\ A = 490$ — 1A

(b) $\int_0^5 P'(t)\,dt + R(5) - R(0)$ — 1M

$= \int_0^5 600e^{-0.3t}\,dt + \left[490e^{-0.5(5)} + 10\right] - 500$

$= [-2000e^{-0.3t}]_0^5 + 490e^{-2.5} - 490$ — 1A — For $[-2000e^{-0.3t}]_0^5$

$= -2000e^{-1.5} + 490e^{-2.5} + 1510$

≈ 1104 — 1A

Hence Richard gains 1104 thousand dollars in the process.

(6)

Good.
In (b), some candidates did not consider the depreciation of the value of the machine in five years.

16. (2010 ASL-M&S Q2)

(a) $\int_0^1 f(x)\,dx \approx \frac{0.5}{2}(1 + e^2 + 2e)$ — 1M

$= \frac{(e+1)^2}{4}$ — 1A — OR $\frac{e^2+2e+1}{4}$ OR 3.4564

(b) $\int_0^1 f(x)\,dx = \left[\frac{e^{2x}}{2}\right]_0^1$

$= \frac{e^2 - 1}{2}$ — 1A

(c) (i) $A = \frac{(1+e^{2h})h}{2} + \frac{(e^{2h}+e^2)(1-h)}{2}$

$= \frac{e^{2h} + (1-e^2)h + e^2}{2}$ — 1 — Follow through

(ii) $\frac{dA}{dh} = \frac{2e^{2h} + 1 - e^2}{2}$ — 1A

$\frac{dA}{dh} = 0$ when $h = \frac{1}{2}\ln\frac{e^2-1}{2}$ — 1A — OR 0.5807

$\frac{d^2A}{dh^2} = 2e^{2h} > 0$ — 1M — OR by using sign test

Hence A is minimum when $h = \frac{1}{2}\ln\frac{e^2-1}{2}$.

The minimum value of A is $\frac{3e^2-1}{4} + \frac{1-e^2}{4}\ln\frac{e^2-1}{2}$. — 1A — OR 3.4367

(8)

Very good. Candidates knew the trapezoidal rule very well. Nevertheless, many candidates ignored the requirement for exact value in (b) and some candidates were not able to make use of the formula of area of trapezium in solving (c).

DSE Mathematics Module 1 6. Definite Integrals

17. (2006 ASL-M&S Q3)

(a) The total amount

$= \int_1^{11} f(t)\,dt$ — 1A can be absorbed

$\approx \frac{11-1}{10}\left(f(1) + f(11) + 2(f(3) + f(5) + f(7) + f(9))\right)$ — 1M for trapezoidal rule

≈ 22.57906572

≈ 22.5791 litres — 1A $a-1$ for r.t. 22.579

(b) $f(t) = \frac{500}{(t+2)^2 e^t}$

$\frac{df(t)}{dt} = \frac{-500\left(2(t+2)e^t + (t+2)^2 e^t\right)}{(t+2)^4 e^{2t}}$ — 1M for quotient rule

$= \frac{-500(t+4)}{(t+2)^3 e^t}$

$\frac{d^2f(t)}{dt^2} = -500\left(\frac{(t+2)^3 e^t - (t+4)\left(3(t+2)^2 e^t + (t+2)^3 e^t\right)}{(t+2)^6 e^{2t}}\right)$

$= -500\left(\frac{t+2-(t+4)(t+5)}{(t+2)^4 e^t}\right)$

$= 500\left(\frac{t^2+8t+18}{(t+2)^4 e^t}\right)$ — 1A or equivalent

$f(t) = 500(t+2)^{-2}e^{-t}$

$\frac{df(t)}{dt} = 500(-2)(t+2)^{-3}e^{-t} + 500(t+2)^{-2}(-1)e^{-t}$ — 1M for product rule

$= -1000(t+2)^{-3}e^{-t} - 500(t+2)^{-2}e^{-t}$

$\frac{d^2f(t)}{dt^2} = 3000(t+2)^{-4}e^{-t} + 1000(t+2)^{-3}e^{-t} + 1000(t+2)^{-3}e^{-t} + 500(t+2)^{-2}e^{-t}$

$= 3000(t+2)^{-4}e^{-t} + 2000(t+2)^{-3}e^{-t} + 500(t+2)^{-2}e^{-t}$ — 1A or equivalent

(c) Note that $\frac{d^2f(t)}{dt^2} > 0$ for all $1 \le t \le 11$. — 1M for considering the sign of $\frac{d^2f(t)}{dt^2}$

So, $f(t)$ is concave upward on $[1, 11]$.

Thus, the estimate in (a) is an over-estimate. — 1A f.t.

----------(7)

Fair. Some candidates could not find the second derivative. Some candidates could not make use of the second derivative to determine whether the trapezoidal rule gives an over-estimate or under-estimate.

18. (2005 ASL-M&S Q2)

(a) $\int_0^8 te^{\frac{t}{5}}\,dt$

$\approx \frac{8-0}{2(4)}\left(0+8e^{\frac{8}{5}}+2(2e^{\frac{2}{5}}+4e^{\frac{4}{5}}+6e^{\frac{6}{5}})\right)$ — 1M for trapezoidal rule

≈ 103.2372887

≈ 103.2373 — 1A $a-1$ for r.t. 103.237

(b) $\int_0^8 \frac{dx}{dt}dt = \int_0^8 \left(4te^{\frac{t}{5}}+\frac{200}{t+1}\right)dt$ — 1M for considering $\int_0^8 \frac{dx}{dt}dt$

$x(8)-x(0) = \int_0^8 \left(4te^{\frac{t}{5}}+\frac{200}{t+1}\right)dt$ — 1A

$x(8)-x(0) = 4\int_0^8 te^{\frac{t}{5}}dt + 200\int_0^8 \frac{dt}{t+1}$

$x(8)-100 \approx 4(103.2372887)+200\int_0^8 \frac{dt}{t+1}$ (by (a)) — 1M for using (a)

Note that

$\int_0^8 \frac{dt}{t+1}$

$=[\ln(t+1)]_0^8$ — 1A for $\int\frac{dt}{t+1}=\ln(t+1)+C$

$=\ln 9$

So, we have $x(8)\approx 952.3940702 \approx 950$ (correct to 2 significant figures). — 1A

Thus, the required number is 950.

----------(7)

Fair. Some candidates were still confusing definite integrals with indefinite integrals.

19. (2001 ASL-M&S Q5)

(a)

t	0	1.5	3	4.5	6
R	8	7.88177	7.54717	7.04846	6.45161

1M correct to 4 d.p.

$\int_0^6 R\,dt \approx \frac{1.5}{2}[8+6.45161+2(7.88177+7.54717+7.04846)]$ — 1M

≈ 44.5548 — 1A

$\therefore$ The total bonus for the first 6 months is 44.5548 thousand dollars.

(b) $\frac{dR}{dt}=\frac{-2400t}{(t^2+150)^2}$ — 1A

$\frac{d^2R}{dt^2}=\frac{7200(t^2-50)}{(t^2+150)^3}$ — 1A

<0 for $0<t<6$

$\therefore$ The graph of R is concave downward in the interval $0\le t\le 6$.

The approximation in (a) is an underestimate. — 1M

----------(6)

20. (2000 ASL-M&S Q3)

Area of the shaded region $=\int_0^8 (1+x^{\frac{1}{3}}-e^{\frac{x}{8}})\,dx$ — 1A integrand (accept $e^{\frac{x}{8}}-1-x^{\frac{1}{3}}$); 1A limits

($pp-1$ for missing dx)

$=\left[x+\frac{3}{4}x^{\frac{4}{3}}-8e^{\frac{x}{8}}\right]_0^8$ — 1A for $x+\frac{3}{4}x^{\frac{4}{3}}$; 1A for $-8e^{\frac{x}{8}}$

≈ 6.2537 (or $28-8e$) — 1A $a-1$ for r.t. 6.254

-----(5)

21. (1996 ASL-M&S Q4)

(a) Area of regions I & III $=\int_0^1\sqrt{x}\,dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$ $\left(=\frac{2}{3}\right)$ — 1A — Or 0.6667

Area of region III $=\int_0^1 x^3dx=\left[\frac{1}{4}x^4\right]_0^1=\frac{1}{4}$ — 1A — Or 0.25

Area of region II $=1-\frac{2}{3}=\frac{1}{3}$ — 1A — Or 0.3333

Area of region I $=\frac{2}{3}-\frac{1}{4}=\frac{5}{12}$ — 1A — Or 0.4167

(b) Probability of scoring 40 points $=2\times\frac{5}{12}\times\frac{1}{4}+(\frac{1}{3})^2$ — 1M+1M — 1M for $2\times\frac{5}{12}\times\frac{1}{4}+p$; 1M for $p+(\frac{1}{3})^2$

$=\frac{23}{72}$ (or 0.3194) — 1A

(7)

22. (1995 ASL-M&S Q6)

(a) $2^{2x}+4=5(2^x)$

$(2^x)^2-5(2^x)+4=0$

$(2^x-4)(2^x-1)=0$ — 1M

$2^x=4$ or 1

$x=2$ or 0 — 1A

$\therefore$ The intersection points are $(0,5)$ and $(2,20)$ — 1A

(b) If $2^x=e^{ax}$ for all values of x,

then $a=\ln 2$. — 1A

Area $=\int_0^2 [5(2^x)-2^{2x}-4]\,dx$ — 1A

$=\int_0^2 [5e^{x\ln 2}-e^{x\ln 4}-4]\,dx$ — 1M

$=\frac{5}{\ln 2}[e^{x\ln 2}]_0^2-\frac{1}{\ln 4}[e^{x\ln 4}]_0^2-4[x]_0^2$ — 1M

$=15\left(\frac{1}{\ln 2}-\frac{1}{\ln 4}\right)-8$

$=\frac{15}{2\ln 2}-8$ (or 2.8202) — 1A

(8)

23. (1994 ASL-M&S Q6)

Solution	Marks	Remarks
(a) Since $e^x \approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$ for $x=0$,		
$e^{-\frac{x^2}{2}} = 1+(-\frac{x^2}{2})+\frac{1}{2}(-\frac{x^2}{2})^2+\frac{1}{6}(-\frac{x^2}{2})^3$	1M	
$= 1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}$ for $x=0$.	1A	
$\int_0^1 e^{-\frac{x^2}{2}}dx = \int_0^1 (1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48})\,dx$		
$= \left[x-\frac{x^3}{6}+\frac{x^5}{40}-\frac{x^7}{336}\right]_0^1$	1M	
$= 1-\frac{1}{6}+\frac{1}{40}-\frac{1}{336}$		
$= 0.8554$	1A	
(b) From the normal distribution table,		
$\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-\frac{x^2}{2}}dx = 0.3413$	1A	
Hence $\frac{1}{\sqrt{2\pi}} \times 0.8554 = 0.3413$	1M	
$\therefore \quad \pi = \frac{0.8554^2}{2\times 0.3413^2} = 3.141$	1A	3.140 for using exact value of (a)
	7	

DSE Mathematics Module 1 6. Definite Integrals

Section B

24. (2017 DSE-MATH-M1 Q11)

Solution	Marks	Remarks
(a) According to the suggestion by Ada,		
I		
$\approx \frac{1}{2}\left(\frac{1-0.5}{5}\right)(f(0.5)+f(1)+2(f(0.6)+f(0.7)+f(0.8)+f(0.9)))$	1M	
[redacted]		
≈ 0.7476	1A	r.t. 0.7476
According to the suggestion by Billy,		
I		
$\approx \int_{0.5}^1 \left(\frac{1}{x}+0.1+0.005x\right)dx$	1M	
$= \left[\ln x+0.1x+0.0025x^2\right]_{0.5}^1$	1M	
$= \ln 2+0.051875$	1A	r.t. 0.7450
[redacted]		
≈ 0.7450		
	----------(5)	
(b) $f(x)$		
$= \frac{e^{0.1x}}{x}$		
$f'(x)$		
$= \frac{0.1e^{0.1x}}{x^2}(x-10)$	1M	
$f''(x)$		
$= \frac{0.01e^{0.1x}}{x^3}(x^2-20x+200)$	1A	
$= \frac{0.01e^{0.1x}}{x^3}((x-10)^2+100)$	1M	
>0 for $0.5 \le x \le 1$		
Thus, the estimate suggested by Ada is an over-estimate.	1A	f.t.
$e^{0.1x} = 1+0.1x+\frac{(0.1x)^2}{2!}+\frac{(0.1x)^3}{3!}+\cdots$	1M	
$e^{0.1x} > 1+0.1x+0.005x^2$ for $0.5 \le x \le 1$		
$I > \int_{0.5}^1 \left(\frac{1}{x}+0.1+0.005x\right)dx$		
Thus, the estimate suggested by Billy is an under-estimate.	1A	f.t.
	----------(6)	
(c) $0.7450 < I < 0.7476$	1M	
$-0.0010 < I-0.746 < 0.0016$		
So, we have $-0.002 < I-0.746 < 0.002$.		
Thus, the claim is agreed.	1A	f.t.
$I-0.746 < 0.7476-0.746 = 0.0016$	1M	
$0.746-I < 0.746-0.7450 = 0.0010$		
So, the difference of I and 0.746 is less than 0.002 .		
Thus, the claim is agreed.	1A	f.t.
	----------(2)	

(a)	Very good. Most candidates were able to use correct sub-intervals when applying the trapezoidal rule to find an estimate of I.
(b)	Fair. Many candidates were unable to find $\frac{d^2f(t)}{dt^2}$ correctly, hence they were unable to determine the nature of the estimate according to the suggestion of Ada in (a).
(c)	Poor. Most candidates did not prove that one of the estimates in (a) is an over-estimate while the other is an under-estimate, hence they were unable to finish the argument.

25. (2016 DSE-MATH-M1 Q11)

Solution	Marks	Remarks
(a) (i) P_1		
$=\int_0^{12} A(t)\,dt$		
$\approx \frac{1}{2}\left(\frac{12-0}{4}\right)(A(0)+A(12)+2(A(3)+A(6)+A(9)))$	1M	
≈ 54.61085671		
≈ 54.6109	1A	r.t. 54.6109
(ii) $\frac{dA(t)}{dt}$		
$=\frac{2t-8}{t^2-8t+95}$	1A	
$\frac{d^2A(t)}{dt^2}$		
$=\frac{2(t^2-8t+95)-(2t-8)^2}{(t^2-8t+95)^2}$		
$=\frac{-2t^2+16t+126}{(t^2-8t+95)^2}$	1A	
$=\frac{-2(t^2-8t-63)}{(t^2-8t+95)^2}$		
	----------(4)	
(b) (i) Let $u=t+3$.	1M	
Then, we have $\frac{du}{dt}=1$.		
P_2		
$=\int_0^{12} B(t)\,dt$	1M	
$=\int_0^{12}\frac{t+8}{\sqrt{t+3}}\,dt$		
$=\int_3^{15}\frac{u-3+8}{\sqrt{u}}\,du$	1A	
$=\int_3^{15}(u^{\frac{1}{2}}+5u^{\frac{-1}{2}})\,du$		
$=\left[\frac{2}{3}u^{\frac{3}{2}}+10u^{\frac{1}{2}}\right]_3^{15}$	1M	
$=20\sqrt{15}-12\sqrt{3}$	1A	r.t. 56.6751
≈ 56.67505723		
≈ 56.6751		

Solution	Marks	Remarks
(b) (ii) $\frac{d^2A(t)}{dt^2}=\frac{-2[t-(4-\sqrt{79})][t-(4+\sqrt{79})]}{(t^2-8t+95)^2}$	1M	1M for considering $\frac{d^2A(t)}{dt^2}$
Note that $4-\sqrt{79}<0$ and $4+\sqrt{79}>12$.		
Therefore, we have $\frac{(t-(4-\sqrt{79}))(t-(4+\sqrt{79}))}{(t^2-8t+95)^2}<0$		
for $0\le t\le 12$.		
Hence, we have $\frac{d^2A(t)}{dt^2}>0$ for $0\le t\le 12$.	1A	f.t.
So, the estimate of P_1 is an over-estimate. $P_1<54.61085671$.		
P_2-P_1		
$=20\sqrt{15}-12\sqrt{3}-P_1$		
$>20\sqrt{15}-12\sqrt{3}-54.61085671$	1M	
≈ 2.064200523		
>2		
Thus, the claim is disagreed.	1A	f.t.
	----------(9)	

(a) (i)	Very good. More than 60% of the candidates were able to find the correct answer using trapezoidal rule. However, a small number of candidates were unable to use the correct sub-intervals when applying the trapezoidal rule.
(ii)	Good. Many candidates were able to find $\frac{dA(t)}{dt}$ by quotient rule, but some candidates were unable to simplify $\frac{d^2A(t)}{dt^2}$.
(b) (i)	Very good. Most candidates were able to formulate and evaluate the definite integral $\int_0^{12}\frac{t+8}{\sqrt{t+3}}dt$ by using a suitable substitution.
(ii)	Poor. Most candidates just mentioned $\frac{d^2A(t)}{dt^2}>0$ without proof. They showed difficulties in using inequality to express the relation between P_1 and its over-estimate, hence unable to complete the argument.

26. (2015 DSE-MATH-M1 Q11)

(a) The total amount of oil produced by oil company X

$=\int_2^{12} f(t)\,dt$ 1M

$\approx \frac{1}{2}\left(\frac{12-2}{5}\right)(f(2)+f(12)+2(f(4)+f(6)+f(8)+f(10)))$ 1M

≈ 69.49587529

≈ 69.4959 hundred barrels 1A r.t. 69.4959

----------(3)

(b) $\frac{df(t)}{dt}$

$=\frac{e^t-1}{e^t-t}$ 1A

$\frac{d^2f(t)}{dt^2}$

$=\frac{(e^t-t)e^t-(e^t-1)(e^t-1)}{(e^t-t)^2}$ 1A

$=\frac{e^t(2-t)-1}{(e^t-t)^2}$

<0 (since $2\le t\le 12$)

Thus, the estimate in (a) is an under-estimate. 1A f.t.

----------(3)

(c) Let $u=1+t$. Then, we have $\frac{du}{dt}=1$. 1M

$\int \frac{t}{1+t}dt$

$=\int \frac{u-1}{u}du$

$=\int\left(1-\frac{1}{u}\right)du$ 1A

$=u-\ln u+\text{constant}$

$=t-\ln(1+t)+\text{constant}$ 1A

Note that $\frac{t}{1+t}=1-\frac{1}{1+t}$. 1A

$\int \frac{t}{1+t}dt$

$=\int\left(1-\frac{1}{1+t}\right)dt$ 1M

$=t-\ln(1+t)+\text{constant}$ 1A

----------(3)

(d) The total amount of oil produced by oil company Y

$=8\int_2^{12}\frac{t}{1+t}dt$

$=8\left[t-\ln(1+t)\right]_2^{12}$ (by (c)) 1M for using the result of (c)

≈ 68.26930345 1A

<69.49587529

By (b), the claim is disagreed. 1A f.t.

----------(3)

(a)	Good. Some candidates did not formulate the required amount as a definite integral, and some candidates did not use the correct number of sub-intervals when applying the trapezoidal rule.
(b)	Fair. Many candidates failed to find $\frac{d^2f(t)}{dt^2}$ correctly, as a result, many candidates failed to determine the nature of the estimate in (a).
(c)	Very good. Most candidates were able to find the indefinite integral by using the method of substitution.
(d)	Good. Many candidates were able to find the correct total amount of oil produced by company Y for comparison. However, some candidates failed to show that the estimate in (a) is an under-estimate and hence could not complete the argument.

27. (2014 DSE-MATH-M1 Q10)

(a) (i) $\frac{d}{dv}(ve^{-v})=e^{-v}-ve^{-v}$ 1A

(ii) $ve^{-v}=e^{-v}-\frac{d}{dv}ve^{-v}$

$\int ve^{-v}dv=\int e^{-v}dv-ve^{-v}$ 1M

$=-e^{-v}-ve^{-v}+C$

$=-e^{-v}(1+v)+C$ 1

(3)

(b) The area of the shaded region $=\int_1^2\frac{\ln x}{x^2}dx$ 1A

Let $x=e^u$. OR $u=\ln x$

$dx=e^u du$

When $x=1$, $u=0$; when $x=2$, $u=\ln 2$ 1A

$\therefore$ the area $=\int_0^{\ln 2}\frac{u}{e^{2u}}\cdot e^u du$ 1M

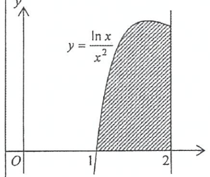
$=\int_0^{\ln 2}ue^{-u}du$

$=[-e^{-u}(1+u)]_0^{\ln 2}$ by (a) 1M

$=\frac{-1}{2}(1+\ln 2)+1$

$=\frac{1-\ln 2}{2}$ 1

(5)



(c) (i) $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right)=\frac{x^2\cdot\frac{1}{x}-\ln x\cdot 2x}{(x^2)^2}$ 1M

$=\frac{1-2\ln x}{x^3}$

$\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right)=\frac{x^3\cdot\frac{-2}{x}-(1-2\ln x)3x^2}{x^6}$

$=\frac{6\ln x-5}{x^4}$ 1A

(ii) $\frac{d^2}{dx^2}\left(\frac{\ln x}{x^2}\right)<0$ when $x<e^{\frac{5}{6}}\approx 2.30098$ 1A OR when $1\le x\le 2$

Hence the trapezoidal rule will underestimate $\int_1^2\frac{\ln x}{x^2}dx$. 1A

Consider the trapezoidal rule with 10 intervals.

$\therefore\ \frac{1}{2}\cdot\frac{1}{10}\left[\frac{\ln 1}{1^2}+2\left(\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\cdots+\frac{\ln 1.9}{1.9^2}\right)+\frac{\ln 2}{2^2}\right]<\frac{1-\ln 2}{2}$ 1M For L.H.S.

$0+2\left(\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\cdots+\frac{\ln 1.9}{1.9^2}\right)+\frac{\ln 2}{4}<10-10\ln 2$

$\frac{\ln 1.1}{1.1^2}+\frac{\ln 1.2}{1.2^2}+\frac{\ln 1.3}{1.3^2}+\cdots+\frac{\ln 1.9}{1.9^2}<5-\frac{41}{8}\ln 2$ 1

(6)

(a)	(i)	Excellent.
	(ii)	Fair. Some candidates wrote nothing but only the expression provided in the question.
(b)		Fair. Some candidates failed to write the correct integral for the area. Some omitted details showing why $[-e^{-u}(1+u)]_0^{\ln 2}=\frac{1-\ln 2}{2}$.
(c)	(i)	Satisfactory. Careless mistakes prevented some candidates from obtaining the correct answer.
	(ii)	Very poor. Most candidates did not attempt this part.

28. (2013 DSE-MATH-M1 Q10)

(a) (i) $\ln(x^2+16)-\ln(3x+20)<0$

$\ln(x^2+16)<\ln(3x+20)$

$x^2+16<3x+20$ 1A

$x^2-3x-4<0$

$-1<x<4$ 1A

(ii) (1) $I=\int_0^4[\ln(x^2+16)-\ln(3x+20)]dx$

$\approx\frac{1}{2}[-0.223143551+0+2(-0.302280871-0.262364264-0.148420005)]$ 1M OR $\frac{1}{2}\{(\ln16-\ln20)+0+2[(\ln17-\ln23)+(\ln20-\ln26)+(\ln25-\ln29)]\}$

≈ -0.824636917

≈ -0.8246 1A

(2) $f(x)=\ln(x^2+16)-\ln(3x+20)$

$f'(x)=\frac{2x}{x^2+16}-\frac{3}{3x+20}$ 1M+1A OR $\frac{3x^2+40x-48}{(x^2+16)(3x+20)}$

$f''(x)=2\cdot\frac{(x^2+16)-x(2x)}{(x^2+16)^2}-3\cdot\frac{(-1)\cdot 3}{(3x+20)^2}$ 1M

$=\frac{2(4+x)(4-x)}{(x^2+16)^2}+\frac{9}{(3x+20)^2}$

>0 for $0\le x\le 4$

Hence the estimate in (1) is an over-estimate. 1A Follow through

(8)

(b) (i) $N'(t)=10\ln(t^2+16)-10\ln(3t+20)$ 1A

(ii) Assume that Jane's claim is true: the species will not die out until $t=4$, i.e. $N(t)>0$ for $0\le t\le 4$.

$N(4)-N(0)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$ 1A

$N(4)-8<-8.24636917$ (since the estimate is an over-estimate) 1M

$N(4)<0$

Hence Jane's claim is false and cannot be agreed with. 1A Follow through

(4)

(a)	(i)		Satisfactory. Some candidates were not able to solve the inequality $\ln(x^2+16)-\ln(3x+20)<0$. Some considered $[\ln(x^2+16)-\ln(3x+20)]'<0$ instead.
	(ii)	(1)	Satisfactory. Some candidates did not apply the formula for trapezoidal rule correctly. Some found the absolute value of I instead.
		(2)	Satisfactory. After obtaining $f'(x)$, many candidates got a point $x_0\in[0,4]$ such that $f(x)$ would decrease on $[0,x_0)$ and increase on $(x_0,4]$, and then claimed immediately that the estimate in (1) was an over-estimate. Among those who were able to find $f''(x)$, only few showed that $f''(x)>0$ for all $x\in[0,4]$ correctly.
(b)	(i)		Poor. A common mistake was to write $N(t)=10\ln(t^2+16)-10\ln(3t+20)$ and then to differentiate both sides of it.
	(ii)		Very poor. Common mistakes included writing $N(t)=\int_0^4[10\ln(t^2+16)-10\ln(3t+20)]\,dt$ and failing to apply the result of (a)(ii)(2) in additional to that of (a)(ii)(1).

29. (2013 DSE-MATH-M1 Q11)

(a) $R'(t)=0$

$P'(t)-C'(t)=0$

$4(4-e^{\frac{-t}{5}})-9(2-e^{\frac{-t}{10}})=0$ 1A

$-4\left(e^{\frac{-t}{10}}\right)^2+9e^{\frac{-t}{10}}-2=0$ 1M For $e^{\frac{-t}{5}}=\left(e^{\frac{-t}{10}}\right)^2$

$e^{\frac{-t}{10}}=0.25$ or 2

$t=20\ln2$ or $-10\ln2$ (rejected as $t\ge0$) 1A OR $t\approx13.8629$

(3)

(b) $R'(t)=-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2$

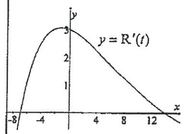
$R''(t)=\frac{4}{5}e^{\frac{-t}{5}}-\frac{9}{10}e^{\frac{-t}{10}}$ 1A

$=\frac{1}{10}e^{\frac{-t}{10}}\left(8e^{\frac{-t}{10}}-9\right)$ 1M

<0 for $t\ge0$ (since $e^{\frac{-t}{10}}\le1$ for $t\ge0$)

Therefore $R'(t)$ decreases with t. 1

(3)



(c) By (a) and (b), $R'(t)>0$ when $0\le t<20\ln2$.

The total redundant electric energy generated during the period when $R'(t)>0$

$=\int_0^{20\ln2}\left(-4e^{\frac{-t}{5}}+9e^{\frac{-t}{10}}-2\right)dt$ 1M For lower and upper limits

$=\left[20e^{\frac{-t}{5}}-90e^{\frac{-t}{10}}-2t\right]_0^{20\ln2}$ 1A For primitive function

$=48.75-40\ln2$ 1A OR 21.0241

(3)

(d) Consider $\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt$

Let $u=\ln(t^2+2t+3)$. 1M

$du=\frac{2t+2}{t^2+2t+3}dt$ 1A

When $t=5$, $u=\ln38$; when $t=8$, $u=\ln83$.

$\therefore \int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt=\int_{\ln38}^{\ln83}u^3\frac{du}{2}$ 1A For $\frac{u^3}{2}$

$=\frac{1}{8}\left[u^4\right]_{\ln38}^{\ln83}$

$=\frac{1}{8}[(\ln83)^4-(\ln38)^4]$

Hence the total electric energy produced for the first 3 years after the improvement

$=\int_5^8\left[\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}+9\right]dt$ 1A

$=\int_5^8\frac{(t+1)[\ln(t^2+2t+3)]^3}{t^2+2t+3}dt+\int_5^8 9\,dt$

$=\frac{1}{8}[(\ln83)^4-(\ln38)^4]+[9t]_5^8$

$=\frac{1}{8}[(\ln83)^4-(\ln38)^4]+27$ 1A OR 52.7730

(5)

(a)	Fair. Some candidates confused $R(t)$ with $R'(t)$, or found $R(t)=P(t)-C(t)$ by integration first and then obtained the expression for $R'(t)=P'(t)-C'(t)$ by differentiation. Many candidates failed to make use of knowledge about quadratic equations to solve for t. Some got wrong answers such as '$e^{\frac{-t}{5}}=0.25$ or 2' or did not reject $t=-10\ln 2$.
(b)	Very poor. Many candidates failed to find $R''(t)$ correctly. Among those who were able to find $R''(t)$, only few provided sufficient reasons to conclude that '$R'(t)$ decreases with t'.
(c)	Very poor. Common mistakes included putting wrong values as limits of the definite integral involved and getting wrong primitives of its integrand.
(d)	Poor. Few candidates were able to use correctly the method of substitution for integration. Among them, some put wrong values as limits of definite integrals, while some others missed out the term $\int_5^8 9\,dt$ or wrote $\int_{\ln 38}^{\ln 83} 9\,dt$.

30. (2012 DSE-MATH-M1 Q10)

Solution	Marks	Remarks
(a) (i) $I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,dt$		
$=\frac{1}{2}\cdot\frac{4-1}{6}\left[\frac{1}{\sqrt{1}}e^{\frac{-1}{2}}+\frac{1}{\sqrt{4}}e^{\frac{-4}{2}}+2\left(\frac{1}{\sqrt{1.5}}e^{\frac{-1.5}{2}}+\frac{1}{\sqrt{2}}e^{\frac{-2}{2}}+\frac{1}{\sqrt{2.5}}e^{\frac{-2.5}{2}}+\frac{1}{\sqrt{3}}e^{\frac{-3}{2}}+\frac{1}{\sqrt{3.5}}e^{\frac{-3.5}{2}}\right)\right]$	1M	
≈ 0.692913377		
≈ 0.6929	1A	
(ii) $\frac{d}{dt}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)=\frac{-1}{2}t^{\frac{-3}{2}}e^{\frac{-t}{2}}+t^{\frac{-1}{2}}\cdot\frac{-1}{2}e^{\frac{-t}{2}}$	1M+1A	
$=\frac{-1}{2}e^{\frac{-t}{2}}\left(t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)$		
$\frac{d^2}{dt^2}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)=\frac{-1}{2}\left[e^{\frac{-t}{2}}\left(\frac{-3}{2}t^{\frac{-5}{2}}+\frac{-1}{2}t^{\frac{-3}{2}}\right)+\frac{-1}{2}e^{\frac{-t}{2}}\left(t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)\right]$	1M+1A	
$=\frac{1}{4}e^{\frac{-t}{2}}\left(3t^{\frac{-5}{2}}+2t^{\frac{-3}{2}}+t^{\frac{-1}{2}}\right)$		
>0 for $1\le t\le 4$.		
Hence the estimation in (i) is an over-estimate.	1	
	(7)	
(b) Let $t=x^2$.	1M	
$dt=2x\,dx$	1A	
When $t=1$, $x=1$; when $t=4$, $x=2$.		
$I=\int_1^4 \frac{1}{\sqrt{t}}e^{\frac{-t}{2}}\,dt$		
$=\int_1^2 \frac{1}{x}e^{\frac{-x^2}{2}}\,2x\,dx$		
$=2\int_1^2 e^{\frac{-x^2}{2}}\,dx$	1	
	(3)	
(c) $2\int_1^2 e^{\frac{-x^2}{2}}\,dx<0.692913377$	1M	
$2\sqrt{2\pi}\int_1^2 \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\,dx<0.692913377$		
$2\sqrt{2\pi}(0.4772-0.3413)<0.692913377$	1A	For 0.4772 and 0.3413
$\pi<3.249593152$		
$\therefore\ \pi<3.25$	1	
	(3)	

(a) (i)	Good. Many candidates applied the trapezoidal rule correctly.
(ii)	Poor. Many candidates used $\frac{d}{dt}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)$ instead of $\frac{d^2}{dt^2}\left(t^{\frac{-1}{2}}e^{\frac{-t}{2}}\right)$ to determine whether the estimate in (i) is an over-estimate or under-estimate.
(b)	Fair. Many candidates used wrong substitutions.
(c)	Very poor. Only a few candidates attempted this part. Among them, some wrote $I\approx 0.692913377$ instead of $I<0.692913377$.

31. (PP DSE-MATH-M1 Q10)

(a) $\dfrac{dx}{dt}=\dfrac{61t}{(t+1)^{\frac{5}{2}}}$

Let $u=t+1$ and hence $du=dt$.

The amount of alloy produced by A

$=\displaystyle\int_0^{10}\frac{61t}{(t+1)^{\frac{5}{2}}}\,dt$ 1A

$=\displaystyle\int_1^{11}\frac{61(u-1)}{u^{\frac{5}{2}}}\,du$

$=\displaystyle\int_1^{11}\left(61u^{\frac{-3}{2}}-61u^{\frac{-5}{2}}\right)du$ 1M

$=\left[-122u^{\frac{-1}{2}}+\dfrac{122}{3}u^{\frac{-3}{2}}\right]_1^{11}$ 1A For primitive function

Alternative Solution

$x=\displaystyle\int\frac{61t}{(t+1)^{\frac{5}{2}}}\,dt$

$=\displaystyle\int\frac{61(u-1)}{u^{\frac{5}{2}}}\,du$ 1A

$=\displaystyle\int\left(61u^{\frac{-3}{2}}-61u^{\frac{-5}{2}}\right)du$ 1M

$=-122u^{\frac{-1}{2}}+\dfrac{122}{3}u^{\frac{-3}{2}}+C$

$=-122(t+1)^{\frac{-1}{2}}+\dfrac{122}{3}(t+1)^{\frac{-3}{2}}+C$ 1A

The amount of alloy produced by A

$=\left[-122(10+1)^{\frac{-1}{2}}+\dfrac{122}{3}(10+1)^{\frac{-3}{2}}+C\right]-\left[-122+\dfrac{122}{3}+C\right]$

≈ 45.6636 1A OR $=\dfrac{244}{3}-\dfrac{3904}{33\sqrt{11}}$

(4)

(b) The amount of alloy produced by B

$=\displaystyle\int_0^{10}\frac{15\ln(t^2+100)}{16}\,dt$

$\approx\dfrac{2}{2}\cdot\dfrac{15}{16}\{\ln(0+100)+\ln(10^2+100)+2[\ln(2^2+100)$
$+\ln(4^2+100)+\ln(6^2+100)+\ln(8^2+100)]\}$ 1M

≈ 45.6792 1A

(2)

(c) $\dfrac{d}{dt}\left(\dfrac{dy}{dt}\right)=\dfrac{d}{dt}\dfrac{15\ln(t^2+100)}{16}$

$=\dfrac{15t}{8(t^2+100)}$ 1A

$\dfrac{d^2}{dt^2}\left(\dfrac{dy}{dt}\right)=\dfrac{15}{8}\cdot\dfrac{(t^2+100)-t(2t)}{(t^2+100)^2}$

$=\dfrac{15(100-t^2)}{8(t^2+100)^2}$ 1A

$\therefore\ \dfrac{d^2}{dt^2}\left(\dfrac{dy}{dt}\right)>0$ for $0<t<10$

Thus, 45.6792 is an over-estimate of the amount of alloy produced by B . 1A

Hence it is uncertain whether machine B is more productive than machine A by the results of (a) and (b). The engineer cannot be agreed with. 1A

(4)

(a)	平平。部分學生未能正確運用代換法。
(b)	平平。部分學生未能正確運用梯形法則公式。
(c)	甚差。部分同學對凹凸曲線概念不了解。

32. (SAMPLE DSE-MATH-M1 Q12)

(a) (i) $f'(0)=e^{2b(0)}+ae^{b(0)}+8=3$

$a=-6$ — 1A

$f'(1)=e^{2b(1)}+ae^{b(1)}+8=4.73$

$e^{2b}-6e^{b}+3.27=0$

$e^{b}=0.606258159$ or 5.393741841 — 1M

$b=\ln 0.606258159$ or $\ln 5.393741841$

$=-0.50044937$ or 1.685239363 (rejected)

i.e. $b=-0.5004$ — 1A

(ii) $f(12)-f(0)=\int_0^{12} f'(t)\,dt$

$f(12)-0=\int_0^{12}(e^{-t}-6e^{-0.5t}+8)\,dt$

$=\left[-e^{-t}+12e^{-0.5t}+8t\right]_0^{12}$ — 1M

$f(12)=85.02973888$

$=85.0297$ — 1A

(5)

(b) (i) $g'(t)=\frac{33}{10}te^{-kt}$

$g''(t)=\frac{33}{10}e^{-kt}(1-kt)$ — 1A

Since $g'(t)$ is greatest when $t=7.5$, $g''(7.5)=0$

$\frac{33}{10}e^{-7.5k}(1-7.5k)=0$ — 1M

$k=\frac{2}{15}$ — 1A

(ii)

t	0	3	6	9	12
$g'(t)$	0	6.63617	8.89671	8.94547	7.99510

$g(12)-g(0)=\int_0^{12} g'(t)\,dt$ — 1M

$g(12)-0\approx\frac{3}{2}[0+7.99510+2(6.63617+8.89671+8.94547)]$ — 1M — For RHS

$g(12)\approx 85.427703$

≈ 85.4277 — 1A

(6)

(c) Agree. Since the graph of $g'(t)$ is concave downward for $0\le t\le 12$, the estimated value obtained in b(ii) is under-estimated and the estimate 85.4277 is greater than 85.0297 in a(iii). — 1M — For pointing out that (b)(ii) is under-estimated

Hence $g(12)>f(12)$. — 1

(2)

33. (2013 ASL-M&S Q8)

(a) $\frac{dP}{dt}=\frac{k-3t}{1+ae^{-bt}}$

$\ln\left(\frac{k-3t}{\frac{dP}{dt}}-1\right)=-bt+\ln a$ — 1A

$\because$ slope $=-0.3$

$\therefore\ b=0.3$ — 1A

$\because$ intercept on the horizontal axis $=0.32$

$\therefore\ 0=-(0.3)(0.32)+\ln a$

$a\approx 1.100759064$

≈ 1.1008 — 1A — OR $e^{0.096}$

When $t=3$ P attains maximum and hence $\frac{dP}{dt}=0$. — 1M

$\frac{k-3(3)}{1+(1.100759064)e^{-(0.3)(3)}}=0$

$k=9$ — 1A

(5)

(b) (i) $P=\int_0^3\frac{9-3t}{1+1.100759064e^{-0.3t}}\,dt$

$\approx\frac{0.5}{2}[4.284165735+0+2(3.851225403+3.30494319+2.644142541$
$+1.870196654+0.986866929)]$ — 1M

≈ 7.3997 million barrels — 1A

(ii) From the graph $\frac{d^2P}{dt^2}$ is decreasing for $0<t<3$. — 1A

Thus, $\frac{d^3P}{dt^3}<0$ for $0<t<3$ and hence the estimation is under-estimate. — 1

(4)

(c) (i) $y=\alpha^{\beta x}$

$\ln y=\beta x\ln\alpha$ — 1M

$\frac{1}{y}\cdot\frac{dy}{dx}=\beta\ln\alpha$

$\frac{dy}{dx}=\beta\alpha^{\beta x}\ln\alpha$ — 1A

(ii) $\therefore\ \int\alpha^{\beta x}\,dx=\frac{1}{\beta\ln\alpha}\alpha^{\beta x}+C$ ------------------------------ (*)

$D=\int_0^3 1.63^{2-0.1t}\,dt$

$=1.63^2\left[\frac{1}{-0.1\ln 1.63}1.63^{-0.1t}\right]_0^3$ by (*) — 1M

≈ 7.414075736

≈ 7.4141 (million barrels) — 1A

(iii) The amount of oil production is approximately 7.3997 million barrels which is an underestimate. Compare with (c)(ii), we cannot conclude that whether the overall oil production meets the overall demand of oil. — 1M, 1

(6)

(a)			Good.
(b)	(i)		Good.
	(ii)		Satisfactory. Some candidates were not able to gather relevant information from the graph to support that $\frac{d^2P}{dt^2}$ is decreasing for $0<t<3$.
(c)			Satisfactory.

34. (2013 ASL-M&S Q9)

Solution	Marks
(a) (i) Let $u=2t+1$.	1A
$\therefore\ t=\dfrac{u-1}{2}$	
$dt=\dfrac{1}{2}du$	
$\therefore\ \displaystyle\int\frac{t^2}{2t+1}dt=\int\frac{1}{u}\left(\frac{u-1}{2}\right)^2\frac{1}{2}du$	
$=\dfrac{1}{8}\displaystyle\int\left(u-2+\frac{1}{u}\right)du$	1A
$=\dfrac{u^2}{16}-\dfrac{u}{4}+\dfrac{1}{8}\ln\lvert u\rvert+C$	
$=\dfrac{(2t+1)^2}{16}-\dfrac{2t+1}{4}+\dfrac{1}{8}\ln\lvert 2t+1\rvert+C$	1A
(ii) $\dfrac{d}{dt}[t^2\ln(2t+1)]=2t\ln(2t+1)+\dfrac{2t^2}{2t+1}$	1A
(iii) $\therefore\ t\ln(2t+1)=\dfrac{1}{2}\cdot\dfrac{d}{dt}t^2\ln(2t+1)-\dfrac{t^2}{2t+1}$	
$\displaystyle\int t\ln(2t+1)\,dt=\frac{1}{2}t^2\ln(2t+1)-\int\frac{t^2}{2t+1}dt$	1M
$=\dfrac{1}{2}t^2\ln(2t+1)-\dfrac{(2t+1)^2}{16}+\dfrac{2t+1}{4}-\dfrac{1}{8}\ln\lvert 2t+1\rvert-C$ by (a)(i)	1M
$N\rvert_{t=5}-N\rvert_{t=0}=\displaystyle\int_0^5 t\ln(2t+1)\,dt$	1M
$N\rvert_{t=5}-21=\left[\dfrac{1}{2}t^2\ln(2t+1)-\dfrac{(2t+1)^2}{16}+\dfrac{2t+1}{4}-\dfrac{1}{8}\ln\lvert 2t+1\rvert\right]_0^5$	
$N\rvert_{t=5}\approx 45.673954$	
Hence the population of the culture of bacteria is approximately 46 trillions.	1A
	(8)
(b) (i) By (a)(iii), $45.673954=40e^{-2\lambda(5-5)}-20e^{-\lambda(5-5)}+K$	
$K\approx 25.673954$	
≈ 26	1A
$27=40e^{-2\lambda(11-5)}-20e^{-\lambda(11-5)}+25.673954$	1M
$40e^{-12\lambda}-20e^{-6\lambda}-1.326046=0$	
$e^{-6\lambda}=0.559275201$ or -0.059275201 (rejected)	
$\lambda\approx 0.1$	1A
(ii) $M=40e^{-0.2(t-5)}-20e^{-0.1(t-5)}+26$	
$M'=-8e^{-0.2(t-5)}+2e^{-0.1(t-5)}$	1M
$=-2e^{-0.2(t-5)}[4-e^{0.1(t-5)}]$	
<0 since $e^{0.1(t-5)}\le e^{1.3}<4$ for $t\le 18$	1M
Thus, M is always decreasing for $t\le 18$.	1
Since we have $M\approx 23.5203$ when $t=18$, the population of the bacteria will not drop to 23 trillion.	1
	(7)

(a)	(i)(ii)		Very good.
	(iii)		Satisfactory. Some candidates were not able to make use of the results of (i) and (ii) to integrate $t\ln(2t+1)$.
(b)			Fair.

35. (2012 ASL-M&S Q8)

Solution	Marks	Remarks
(a) Let $u=1+6t$.	1A	
$du=6dt$		
When $t=0$, $u=1$; when $t=12$, $u=73$		
$\displaystyle\int_0^{12}\left[4.5+2t(1+6t)^{\frac{-2}{3}}\right]dt$		
$=\displaystyle\int_1^{73}\left(4.5+\frac{u-1}{3}u^{\frac{-2}{3}}\right)\frac{du}{6}$	1M	For integrand
$=\displaystyle\int_1^{73}\left(\frac{3}{4}+\frac{1}{18}u^{\frac{1}{3}}-\frac{1}{18}u^{\frac{-2}{3}}\right)du$		
$=\left[\dfrac{3u}{4}+\dfrac{1}{24}u^{\frac{4}{3}}-\dfrac{1}{6}u^{\frac{1}{3}}\right]_1^{73}$	1A	For primitive function
≈ 66.14060019		
$\therefore$ the total amount of sewage emitted by machine $P\approx 66.1406$ tonnes.	1A	OR $\dfrac{433+23\sqrt[3]{73}}{8}$ tonnes
	(4)	
(b) (i) $\displaystyle\int_0^{12}[3+\ln(2t+1)]\,dt$		
$=\dfrac{12-0}{2(5)}[3+\ln 1+3+\ln 25+2(3+\ln 5.8+3+\ln 10.6+3+\ln 15.4+3+\ln 20.2)]$	1M	
≈ 63.52367987		
$\therefore$ the total amount of sewage emitted by machine $Q\approx 63.5237$ tonnes.	1A	
(ii) $q''(t)=\dfrac{2}{2t+1}$		
$q'''(t)=\dfrac{-4}{(2t+1)^2}<0$ for all $t\ge 0$	1A	For $\dfrac{-4}{(2t+1)^2}$
Hence the estimate in (b)(i) is an under-estimate.	1M	
Therefore we cannot conclude that the amount of sewage emitted by Q will be less than that by P and so the manager cannot be agreed with.	1	
	(5)	
(c) (i) $R=16-ae^{-bx}$		
$\ln(16-R)=\ln a-bx$	1A	
(ii) $\begin{cases}1=\ln a+10b\\0=\ln a-90b\end{cases}$	1M	
Solving, $a=e^{0.9}$ and $b=0.01$	1A+1A	OR $a\approx 2.4596$
(iii) Total amount of sewage		
$\approx 80+66.14060019+63.52367987$	1M	
$=209.6642801$		
Hence $R=16-e^{0.9}e^{-0.01(209.6642801)}$		
≈ 15.69779292		
i.e. the tax paid is 15.6978 million dollars.	1A	
	(6)	

(a)		Satisfactory. Many candidates were able to use substitution appropriately but some did not change the limits of integration accordingly.
(b) (i)		Good.
(ii)		Fair. Some candidates did not realise that the second derivative of the rate of emission should be q''' rather than q''.
(c) (i)		Very good.
(ii)		Good. Some candidates failed to see that a was not the intercept of the function $\ln(16-R)$ when expressed as a linear function of x.
(iii)		Very poor. Many candidates were not able to comprehend the given situation and failed to state that the total amount of sewage was '80 + result of (a) + result of (b)(i)'.

36. (2012 ASL-M&S Q9)

(a) $r(t)=20-40e^{-at}+be^{-2at}$

$\because\ r(0)=20-40e^{0}+be^{0}=30$

$\therefore\ b=50$ — 1A

(1)

(b) $r'(t)<0$ for 9 days

$40ae^{-at}-100ae^{-2at}<0$ for $t<9$ — 1M

$20ae^{-2at}(2e^{at}-5)<0$

$e^{at}<2.5$

$t<\dfrac{\ln 2.5}{a}$ — 1A

$\therefore\ \dfrac{\ln 2.5}{a}=9$

i.e. $a\approx 0.1$ (correct to 1 decimal place) — 1A

(3)

(c) The rate of change of the rate of selling of handbags is $r'(t)=4e^{-0.1t}-10e^{-0.2t}$.

$\dfrac{d}{dt}r'(t)=-0.4e^{-0.1t}+2e^{-0.2t}$

$\dfrac{d}{dt}r'(t)=0$ when $0.4e^{-0.1t}=2e^{-0.2t}$ — 1M

$e^{0.1t}=5$

$t=10\ln 5$ — 1A — OR 16.0944

$\dfrac{d^2}{dt^2}r'(t)=0.04e^{-0.1t}-0.4e^{-0.2t}$

When $t=10\ln 5$, $\dfrac{d^2}{dt^2}r'(t)=-0.008<0$ — 1M — OR by using sign test

Hence $r'(t)$ is maximum when $t=10\ln 5$

$r(10\ln 5)=20-40e^{-0.1(10\ln 5)}+50e^{-0.2(10\ln 5)}=14$

The rate of selling $=14$ thousand per day — 1A — OR 14000 per day

(4)

(d) (i) $r(t)=20-40e^{-0.1t}+50e^{-0.2t}<18$ — 1M

$25e^{-0.2t}-20e^{-0.1t}+1<0$

$0.053589838<e^{-0.1t}<0.746410161$

$2.924800155<t<29.26395809$ — 1A

$\because\ 29.26395809-2.924800155=26.33915794$

$\therefore$ the 'sales warning' will last for 26 days. — 1A

(ii) Number of handbags sold (in thousand) during the 'sales warning' period

$=\displaystyle\int_{2.92480155}^{29.26395809}(20-40e^{-0.1t}+50e^{-0.2t})\,dt$ — 1M

$=[20t+400e^{-0.1t}-250e^{-0.2t}]_{2.924800155}^{29.26395809}$

≈ 388.2190941 — 1A — Accept 388.2191

$\dfrac{388.2190941}{26.33915794}\approx 14.7392$ — 1M

Hence the average number of handbags sold per day is 15 thousand. — 1A — OR 15000

(7)

(a) (b) (c) (d) (i) (ii)		Very good. Satisfactory. Many candidates used an equation rather than an inequality to solve for the value of a. Fair. Some candidates overlooked that the given condition was for the rate of change of the rate of selling. When consider the maximum rate of change, candidates should set the second derivative $\dfrac{d^2r}{dt^2}$ zero. Poor. Many candidates were not able to handle the quadratic inequality. Fair. Many candidates were not able to get the correct answer due to errors made in the previous parts.

DSE Mathematics Module 1 6. Definite Integrals

37. (2011 ASL-M&S Q8)

(a) $e^{t^2+t}=1+(t^2+t)+\dfrac{(t^2+t)^2}{2}+\dfrac{(t^2+t)^3}{3!}+\cdots$ — 1M

$==1+t^2+t+\dfrac{2t^3+t^2+\cdots}{2}+\dfrac{t^3+\cdots}{6}+\cdots$

$=1+t+\dfrac{3t^2}{2}+\dfrac{7t^3}{6}+\cdots$ — 1A

$V=\displaystyle\int_0^{\frac{1}{2}}\frac{1}{25}e^{t^2+t+2}\,dt$

$\approx\dfrac{e^2}{25}\displaystyle\int_0^{\frac{1}{2}}\left(1+t+\frac{3t^2}{2}+\frac{7t^3}{6}\right)dt$ — 1M

$=\dfrac{e^2}{25}\left[t+\dfrac{t^2}{2}+\dfrac{t^3}{2}+\dfrac{7t^4}{24}\right]_0^{\frac{1}{2}}$

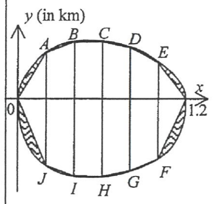
$=\dfrac{271}{9600}e^2$ hundred thousand m^3 — 1A — OR 0.2086 hund. th. m^3 OR $20858.6896\ m^3$

Since for $t>0$, $e^{t^2+t}=1+t+\dfrac{3t^2}{2}+\dfrac{7t^3}{6}+$ positive terms , — 1M

$e^{t^2+t}>1+t+\dfrac{3t^2}{2}+\dfrac{7t^3}{6}$

Hence the estimation is an under-estimate. — 1A

(6)



(b) (i) Area $\approx\dfrac{0.2}{2}[0+2(3.8+4.2+4.3+4.1+3.4)+0]$ — 1M

$=3.96\ km^2$ — 1A

Since the upper half of the curve is concave downwards and the lower half is concave upwards, the estimation is an under-estimate. — 1A

(ii) Thickness $\approx\dfrac{20858.6896}{3.96\times 1000^2}$ m

≈ 0.0053 m — 1A — OR 5.2673×10^{-3} m OR 5.2673 mm

Since both the numerator and denominator are under-estimates, we cannot determine whether the thickness is an over- or under-estimate. — 1A

(5)

(c) $\dfrac{dW}{dt}=\dfrac{-(W+1)^{\frac{1}{3}}}{40}$

$\dfrac{dt}{dW}=-40(W+1)^{\frac{-1}{3}}$ 1M

$t=-40\displaystyle\int(W+1)^{\frac{-1}{3}}\,dW$

$=-60(W+1)^{\frac{2}{3}}+C$ 1A

When $t=0$, $W=\dfrac{271}{9600}e^2$. 1M

$\therefore\ 0=-60\left(\dfrac{271}{9600}e^2+1\right)^{\frac{2}{3}}+C$

$C=68.07743296$

Hence $t=-60(W+1)^{\frac{2}{3}}+68.07743296$

When $W=0$, $t\approx 8.0774$. 1A

Thus all the oil spread will be cleaned up after 8.0774 days.

(4)

(a)		Satisfactory. In order to arrive at a polynomial in t of degree 3, the exponential function had to be expanded to the fourth term which is $\dfrac{(t^2+t)^3}{3!}$. Many candidates only considered three terms and hence did not meet the requirement.
(b) (i)		Fair. Candidates' concept about concave and convex curves was unclear.
(ii)		Fair. Some candidates did not know how to conclude when the denominator and numerator of a fraction were both under-estimates.
(c)		Poor. Many candidates were unable to deal with $\dfrac{dW}{dt}$ when it was expressed as a function of W rather than a function of t .

38. (2010 ASL-M&S Q8)

(a) $f'(t)=-500ae^{2at}+300ae^{at}$ 1A

Since $f(t)$ attains maximum when $t=5$, $f'(5)=0$ 1M

$-500ae^{10a}+300ae^{5a}=0$

$a=0.2\ln 0.6$ 1A OR -0.1022

(3)

(b) (i) $-250e^{0.4T_1\ln 0.6}+300e^{0.2T_1\ln 0.6}-50=0$ 1M

$e^{0.2T_1\ln 0.6}=0.2$ or 1 (rejected as $T_1>0$)

$T_1=\dfrac{5\ln 0.2}{\ln 0.6}$ 1A OR 15.7533

(ii) The total amount of sales increased

$=\displaystyle\int_0^{T_1}(-250e^{2at}+300e^{at}-50)\,dt$ 1M

$=\left[\dfrac{-125e^{0.4t\ln 0.6}}{0.2\ln 0.6}+\dfrac{300e^{0.2t\ln 0.6}}{0.2\ln 0.6}-50t\right]_0^{\frac{5\ln 0.2}{\ln 0.6}}$ 1A

$=\dfrac{-125}{0.2\ln 0.6}(0.2^2-1)+\dfrac{300}{0.2\ln 0.6}(0.2-1)-50\left(\dfrac{5\ln 0.2}{\ln 0.6}\right)$

$=\dfrac{-600+250\ln 5}{\ln 0.6}$ thousand dollars 1A OR 386.9041 thousand dollars OR \$ 386904.0876

(5)

(c) (i) $E=100+\displaystyle\int\frac{100}{t+9}\,dt$

$=100+100\ln(t+9)+C$ 1A

When $t=0, E=100$.

$100=100+100\ln 9+C$ 1M

$C=-100\ln 9$

$\therefore\ E=100\,[\ln(t+9)+1-\ln 9]$ 1A

(ii) $200=100\ln(t+9)+100-100\ln 9$

$T_2=9(e-1)$ 1A OR 15.4645

(iii) Total sales increased

$=\displaystyle\int_{\alpha}^{2\alpha}-(t-\alpha)(t-2\alpha)\,dt$ 1M

$=\displaystyle\int_{\alpha}^{2\alpha}(-t^2+3\alpha t-2\alpha^2)\,dt$

$=\left[\dfrac{-t^3}{3}+\dfrac{3\alpha t^2}{2}-2\alpha^2 t\right]_{\alpha}^{2\alpha}$

$=\dfrac{\alpha^3}{6}$ 1A

Hence the maximum total increase of sales can be achieved when

$\alpha=T_2$ 1M

$=9(e-1)$

Hence the plan should be started $9(e-1)$ months after the launching of the campaign.

(7)

(a)		Very good.
(b) (i)		Good. Some candidates could not make use of the given condition $f(T_1)=0$ to solve for T_1 .
(ii)		Fair. Some candidates had difficulty in performing integration involving exponential functions.
(c) (i)		Poor. Many candidates could not express the total expenditure E , which is the sum of a fixed cost and the integrated total of a variable cost.
(ii)		Poor, since it depends on (c)(i).
(iii)		Very poor. Very few candidates attempted this part.

39. (2009 ASL-M&S Q9)

Solution	Marks	Remarks
(a) (i) $R_6 = \int_0^6 \ln(2t+1)\,dt$		
$\approx \frac{1}{2}\{\ln(2\cdot 0+1)+2[\ln(2\cdot 1+1)+\ln(2\cdot 2+1)+\ln(2\cdot 3+1)+\ln(2\cdot 4+1)$		
$+\ln(2\cdot 5+1)]+\ln(2\cdot 6+1)\}$	1M	
$=10.53155488$		
The total amount of revenue in the first 6 weeks is 10.5316 million dollars.	1A	
(ii) Let $f(t)=\ln(2t+1)$		
$f'(t)=\frac{2}{2t+1}$		
$f''(t)=\frac{-4}{(2t+1)^2}$	1A	
<0 for $0 \le t \le 6$		
$\therefore$ $f(t)$ is concave downward for $0 \le t \le 6$.		
Hence the estimate in (a)(i) is an under-estimate.	1A	Follow through
	(4)	
(b) (i) $Q_1 = \int_0^1 \left[45t(1-t)+\frac{1.58}{t+1}\right] dt$	1A	
$=\left[45\left(\frac{t^2}{2}-\frac{t^3}{3}\right)+1.58\ln\lvert t+1\rvert\right]_0^1$		
$=\frac{15}{2}+1.58\ln 2$		
≈ 8.595172545		
The total amount of revenue in the first week is 8.5952 million dollars.	1A	
(ii) $Q_n = Q_1 + \int_1^n \frac{30e^{-t}}{(3+2e^{-t})^2} dt$	1M	
Let $u=3+2e^{-t}$		
$du=-2e^{-t}dt$		
$\therefore$ $Q_n = Q_1 + \int_{3+2e^{-1}}^{3+2e^{-n}} \frac{-15}{u^2} du$	1M	For $\frac{-15}{u^2}$
$=Q_1+\left[\frac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$	1A	For $\left[\frac{15}{u}\right]_{3+2e^{-1}}^{3+2e^{-n}}$
$=\frac{15}{2}+1.58\ln 2+\frac{15}{3+2e^{-n}}-\frac{15}{3+2e^{-1}}$		
Hence the total amount of revenue in the first n weeks is $\left(\frac{15}{2}+1.58\ln 2+\frac{15}{3+2e^{-n}}-\frac{15}{3+2e^{-1}}\right)$ million dollars, where $n>1$.	1A	Accept $4.5799+\frac{15}{3+2e^{-n}}$
	(6)	
(c) For $n>6$, $R_n = R_6 + \int_6^n 0\,dt \approx 10.5316$ (by (a)(i))	1M	For $\int_6^n 0\,dt = 0$
When $n \to \infty$, $e^{-n} \to 0$ and so $Q_n \to 4.5799+\frac{15}{3+0}=9.5799$	1M	For $e^{-n} \to 0$
Therefore, over a long period of time, plan A produces approximately 10.5316 million dollars and plan B produces 9.5799 million dollars of revenue.	1A	
Moreover, the revenue of plan A is even an under-estimate.	1M	
Hence, plan A will produce more revenue over a long period of time.	1A	Follow through
	(5)	

(a) (i)		Good.
(ii)		Poor. The poor performance was rather unexpected since applying the concept of concave and convex curves should be quite standard.
(b) (i)		Good.
(ii)		Poor. The problem might look unfamiliar. Many candidates did not realize that the lower and upper limits of the integral should be 1 and n , and Q_1 should be added to the integral to get Q_n .
(c)		Very poor. Most candidates got the wrong conclusion due to mistakes made in the previous parts.

40. (2008 ASL-M&S Q9)

Solution	Marks	Remarks
(a) (i) $\int_0^{10} \frac{1}{40}\sqrt{1+t^2}\,dt$		
$\approx \frac{10}{2(4)}\cdot\frac{1}{40}\left(\sqrt{1+0^2}+2\sqrt{1+2.5^2}+2\sqrt{1+5^2}+2\sqrt{1+7.5^2}+\sqrt{1+10^2}\right)$	1M	
≈ 1.305182044		
≈ 1.3052	1A	
So the increase of temperature is about 1.3052 °C .		
(ii) $\frac{d}{dt}\left(\frac{1}{40}\sqrt{1+t^2}\right)=\frac{t}{40\sqrt{1+t^2}}$		
$\frac{d^2}{dt^2}\left(\frac{1}{40}\sqrt{1+t^2}\right)=\frac{1}{40(1+t^2)^{\frac{3}{2}}}$	1A	
>0		
Hence it is an over-estimate.	1A	Follow through
	(4)	
(b) (i) $100(\ln x_0)^2-630\ln x_0+1960=968$		
$50(\ln x_0)^2-315\ln x_0+496=0$	1A	
$\ln x_0=\frac{31}{10}$ or $\frac{16}{5}$		
$x_0 \approx 22.1980$ or 24.5325	1A	
(ii) $W'(x)=\frac{200\ln x}{x}-\frac{630}{x}$	1A	
$\therefore W'(x)<0$ when $200\ln x-630<0$ $(\because x \ge 22)$		
$\therefore \ln x<3.15$		
i.e. $22 \le x < e^{3.15} \approx 23.3361$	1A	Accept $x<23.3361$
(iii) $\frac{dW}{dt}=\frac{dW}{dx}\cdot\frac{dx}{dt}$		
$=\frac{200\ln x-630}{x}\cdot\frac{\sqrt{1+t^2}}{40}$	1M	
When $t=0$, $x=22$.		
$\therefore \left.\frac{dW}{dt}\right\rvert_{t=0}=\frac{200\ln 22-630}{22}\cdot\frac{\sqrt{1+0}}{40}$	1M	
≈ -0.0134	1A	
Hence the rate of change of electricity consumption at $t=0$ is -0.0134 units per year.		
(iv) The electricity consumption at $t=10$ is approximately $W(22+1.305182044)$	1M	
$=100(\ln 23.305182044)^2-630\ln 23.305182044+1960$		
≈ 967.7502 units	1A	
Since the estimate in (a)(i) is an over-estimate, the actual temperature when $t=10$ is $x<23.305182044$. Moreover, $W(x)$ is decreasing for $22 \le x < 23.3361$ by (b)(ii).	1M	
Therefore the actual electricity consumption is larger than this estimate.	1A	Follow through
	(11)	

(a) (i)		Very good.
(ii)		Poor. Many candidates were not aware that the second derivative of the given equation is $\frac{d^3x}{dt^3}$ rather than $\frac{d^2x}{dt^2}$.
(b) (i) (ii)		Good.
(iii)		Poor. Many candidates did not realise that $\frac{dW}{dt}$ should be found and some failed to apply chain rule to find $\frac{dW}{dt}$.
(iv)		Very poor. Most candidates could not interpret their own mathematical findings and hence failed to make use of the results to make judgement.

41. (2007 ASL-M&S Q8)

(a) (i) The total profit made by company A

$= \int_0^6 f(t)\,dt$ — 1A withhold 1A for omitting this step

$\approx \frac{1}{2}\left(f(0)+f(6)+2\left(f(1)+f(2)+f(3)+f(4)+f(5)\right)\right)$ — 1M for trapezoidal rule

≈ 37.48705341

≈ 37.4871 billion dollars — 1A $a-1$ for r.t. 37.487

(ii) $f(t)=\ln(e^t+2)+3$

$\frac{df(t)}{dt}=\frac{e^t}{e^t+2}$ — 1A

$\frac{d^2f(t)}{dt^2}$

$=\frac{(e^t+2)e^t-e^t(e^t)}{(e^t+2)^2}$

$=\frac{2e^t}{(e^t+2)^2}$ — 1A

Since $\frac{d^2f(t)}{dt^2}>0$, $f(t)$ is concave upward for $0\le t\le 6$. — 1M

Thus, the estimate in (a)(i) is an over-estimate. — 1A f.t.

----------(7)

(b) (i) $\frac{1}{40-t^2}=\frac{1}{40}\left(1+\frac{t^2}{40}+\frac{t^4}{1600}+\cdots\right)=\frac{1}{40}+\frac{1}{1600}t^2+\frac{1}{64000}t^4+\cdots$ — 1A pp−1 for omitting ' … '

(ii) Note that $e^t=1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\cdots$. Hence, we have — 1M for any four terms correct

$\frac{8e^t}{40-t^2}$

$=8\left(\frac{1}{40}+\frac{1}{1600}t^2+\frac{1}{64000}t^4+\cdots\right)\left(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\cdots\right)$

$=\frac{1}{5}+\frac{1}{5}t+\frac{21}{200}t^2+\frac{23}{600}t^3+\frac{263}{24000}t^4+\cdots$ — 1A pp−1 for omitting ' … '

(iii) The total profit made by company B

$=\int_0^6 g(t)\,dt$

$\approx\int_0^6\left(\frac{1}{5}+\frac{1}{5}t+\frac{21}{200}t^2+\frac{23}{600}t^3+\frac{263}{24000}t^4\right)dt$ — 1M

$=\left[\frac{1}{5}t+\frac{1}{10}t^2+\frac{7}{200}t^3+\frac{23}{2400}t^4+\frac{263}{120000}t^5\right]_0^6$ — 1A for correct integration

$=41.8224$ billion dollars — 1A $a-1$ for r.t. 41.822

----------(6)

(c) Since the estimate in (b)(iii) is an under-estimate, we have — 1A

$\int_0^6 f(t)\,dt<37.4871<41.8224<\int_0^6 g(t)\,dt$.

Thus, Mary's claim is correct. — 1A f.t.

----------(2)

(a) (i)		Good. Most candidates could apply the trapezoidal rule.
(ii)		Fair.
(b) (i)		Good.
(ii)		Fair. Some candidates could not expand the exponential function.
(iii)		Fair. Failing to get the correct result was mainly due to the performance of the previous parts.
(c)		Poor. Many candidates did not attempt this part and others could not make use of the concept of over- and under-estimate of a mathematical model to explain the underlying meaning.

42. (2004 ASL-M&S Q8)

(a) The total fuel consumption

$=\int_0^{15} f(t)\,dt$ — 1A withhold 1A for omitting this step

$\approx\frac{15-0}{10}\left(f(0)+f(15)+2\left(f(3)+f(6)+f(9)+f(12)\right)\right)$ — 1M for trapezoidal rule

≈ 27.40358785

≈ 27.4036 litres — 1A $a-1$ for r.t. 27.404

----------(3)

(b) The total fuel consumption

$=\int_0^{15} g(t)\,dt$

$=\int_0^{15}\frac{1}{145}t(15-t)^2\,dt$

$=\frac{1}{145}\int_0^{15}(225t-30t^2+t^3)\,dt$ — 1A

$=\frac{1}{145}\left[\frac{225t^2}{2}-10t^3+\frac{t^4}{4}\right]_0^{15}$ — 1A for correct integration

$=\frac{3375}{116}$ litres — 1A

≈ 29.09482759

≈ 29.0948 litres — $a-1$ for r.t. 29.095

----------(3)

(c) $f(t)=\frac{1}{4}t(15-t)e^{\frac{-t}{4}}$

$\frac{df(t)}{dt}=\frac{1}{4}(15-2t)\,e^{\frac{-t}{4}}-\frac{1}{16}t(15-t)\,e^{\frac{-t}{4}}$ — 1M for Product Rule or Chain Rule

$=\frac{1}{16}(t^2-23t+60)\,e^{\frac{-t}{4}}$ — 1A must be simplified

$=\frac{1}{16}(t-3)(t-20)\,e^{\frac{-t}{4}}$

$\frac{df(t)}{dt}\begin{cases}>0 & \text{if } 0\le t<3\\ =0 & \text{if } t=3\\ <0 & \text{if } 3<t\le 15\end{cases}$ — 1M for testing + 1A

So, we have

the greatest value

$=f(3)$

$=9e^{\frac{-3}{4}}$ — 1A provided the testing is correct

≈ 4.251298975

≈ 4.2513 — $a-1$ for r.t. 4.251

$f(t)=\frac{1}{4}t(15-t)e^{\frac{-t}{4}}$

$\frac{df(t)}{dt}=\frac{1}{4}(15-2t)e^{\frac{-t}{4}}-\frac{1}{16}t(15-t)e^{\frac{-t}{4}}$ — 1M for Product Rule or Chain Rule

$=\frac{1}{16}(t^2-23t+60)e^{\frac{-t}{4}}$ — 1A must be simplified

$=\frac{1}{16}(t-3)(t-20)e^{\frac{-t}{4}}$

For $\frac{df(t)}{dt}=0$, we have $t=3$ or $t=20$ (rejected since $0\le t\le 15$).

$\frac{d^2f(t)}{dt^2}=\frac{-1}{64}(t-3)(t-20)e^{\frac{-t}{4}}+\frac{1}{16}(2t-23)e^{\frac{-t}{4}}$

$=\frac{-1}{64}(t^2-31t+152)e^{\frac{-t}{4}}$

$\left.\frac{d^2f(t)}{dt^2}\right|_{t=3}=\frac{-17}{16}e^{\frac{-3}{4}}$ [redacted] <0 — 1M for testing + 1A

So, we have
the greatest value
$=f(3)$
$=9e^{\frac{-3}{4}}$
[redacted]
≈ 4.2513 — 1A provided the testing is correct

$a-1$ for r.t. 4.251

----------(5)

(d) (i) $\frac{d^2f(t)}{dt^2}$

$=\frac{-1}{64}(t-3)(t-20)e^{\frac{-t}{4}}+\frac{1}{16}(2t-23)e^{\frac{-t}{4}}$

$=\frac{-1}{64}(t^2-31t+152)e^{\frac{-t}{4}}$ — 1A must be simplified

(ii) $\left.\frac{d^2f(t)}{dt^2}\right|_{t=0}=\frac{-19}{8}<0$

$\left.\frac{d^2f(t)}{dt^2}\right|_{t=15}=\frac{11}{8}e^{\frac{-15}{4}}>0$

} 1M for testing two values of t in [0, 15] or for factorizing $\frac{d^2f(t)}{dt^2}e^{\frac{t}{4}}$

Therefore, by considering $\frac{d^2f(t)}{dt^2}$, we cannot determine whether the estimate in (a) is an over-estimate or an under-estimate. } 1A

Thus, by considering $\frac{d^2f(t)}{dt^2}$, we cannot determine whether the total fuel consumption from $t=0$ to $t=15$ when using driving tactic A will be less than that of using driving tactic B. } 1M

----------(4)

(a/b/c)		Good.
(d)(i)		Fair.
(ii)		Poor. Very few candidates were able to explain their answers correctly.

43. (2003 ASL-M&S Q8)

(a) $f(t)=5+2^{-kt+h}$

$\ln(f(t)-5)=-(k\ln 2)t+h\ln 2$ — 1A do not accept $-k\ln 2t+h\ln 2$

----------(1)

(b) $-k\ln 2=-0.35$

$k\approx 0.504943264$

$k\approx 0.5$ (correct to 1 decimal place) — 1A

$h\ln 2=1.39$

$h\approx 2.005346107$

$h\approx 2.0$ (correct to 1 decimal place) — 1A

----------(2)

(c) The total amount

$=\int_2^{12}g(t)dt$

$\approx \frac{12-2}{10}\left(g(2)+g(12)+2(g(4)+g(6)+g(8)+g(10))\right)$ — 1M for trapezoidal rule

≈ 75.77699747

≈ 75.7770 thousand barrels — 1A

----------(2)

(d) (i) $2^t=e^{at}$ for all $t\ge 0$

$t\ln 2=at$ for all $t\ge 0$

$a=\ln 2$ — 1A accept $a\approx 0.6931$

(ii) $g(t)=5+\ln(t+1)+2^{\frac{-t}{2}+2}$

$=5+\ln(t+1)+4e^{\left(\frac{-\ln 2}{2}\right)t}$

$\frac{dg(t)}{dt}=\frac{1}{t+1}+(4)\left(\frac{-\ln 2}{2}\right)e^{\left(\frac{-\ln 2}{2}\right)t}$ — 1A for the first term + 1M for Chain Rule

$=\frac{1}{t+1}-2(\ln 2)e^{\left(\frac{-\ln 2}{2}\right)t}$ — 1M for the second term

$\frac{d^2g(t)}{dt^2}=\frac{-1}{(t+1)^2}-(2\ln 2)\left(\frac{-\ln 2}{2}\right)e^{\left(\frac{-\ln 2}{2}\right)t}$

$=(\ln 2)^2e^{\left(\frac{-\ln 2}{2}\right)t}-\frac{1}{(t+1)^2}$ — 1A for all being correct

$=(\ln 2)^2 2^{\frac{-t}{2}}-\frac{1}{(t+1)^2}$

$\therefore\ p(t)=(\ln 2)^2 2^{\frac{-t}{2}}$ — 1A accept $p(t)=(\ln 2)^2e^{\left(\frac{-\ln 2}{2}\right)t}$

(iii) $\because$ $p(2)=(\ln 2)^2 2^{-1} \approx 0.240226506 \approx 0.2402$

$q(2)=\frac{1}{9} \approx 0.1111111111 \approx 0.1111$ } 1M for testing

$\therefore$ $p(2)>q(2)$ 1A

It is known that $y=p(t)$ and $y=q(t)$ have no intersection, where $2 \le t \le 12$.

So, we have $p(t)>q(t)$ for all $2 \le t \le 12$.

$\therefore$ $\frac{d^2 g(t)}{dt^2}>0$ on $[2, 12]$ 1M

Thus, the estimate is an over-estimate of I. 1M

----------(10)

(a/b/c)		Good.
(d)		Fair. Some candidates were unable to perform differentiation involving 'ln' function. Very few candidates were able to explain why the estimate was an over-estimate of I.

44. (2002 ASL-M&S Q9)

(a) (i)

t	0	0.5	1.0	1.5	2	2.5
$\frac{dM}{dt}$	4	4.78496	5.84320	7.24875	9.10480	11.55161

$M=\int_0^{2.5}\frac{12e^{\frac{2}{3}t}}{3+t}\,dt \approx \frac{0.5}{2}[4+11.55161$

$+2(4.78496+5.8432+7.24875+9.1048)]$ 1M

$=17.3788$ (m mol/L) 1A a−1 for r.t. 17.379

(ii) $\because$ $\frac{dM}{dt}=\frac{12e^{\frac{2}{3}t}}{3+t}$,

$\frac{d}{dt}\left(\frac{12e^{\frac{2}{3}t}}{3+t}\right)=12\left[\frac{2}{3}\cdot\frac{e^{\frac{2}{3}t}}{3+t}-\frac{e^{\frac{2}{3}t}}{(3+t)^2}\right]=\frac{4(3+2t)e^{\frac{2}{3}t}}{(3+t)^2}$ 1A need not simplify

and $\frac{d^2}{dt^2}\left(\frac{12e^{\frac{2}{3}t}}{3+t}\right)=\frac{8(9+6t+2t^2)}{3(3+t)^3}e^{\frac{2}{3}t}$ 1A need simplification

$\therefore$ $\frac{d^2}{dt^2}\left(\frac{dM}{dt}\right)>0$ (for $0 \le t \le 2.5$)

So, $\frac{dM}{dt}$ is concave upward on $[0, 2.5]$.

Hence it is over-estimate. 1

----------(5)

(b) (i) $\frac{1}{3+t}=\frac{1}{3}(1-\frac{1}{3}t+\frac{1}{9}t^2-\frac{1}{27}t^3+\cdots)$ 1A

$=\frac{1}{3}-\frac{1}{9}t+\frac{1}{27}t^2-\frac{1}{81}t^3+\cdots$

$e^{\frac{2}{3}t}=1+\frac{2}{3}t+\frac{1}{2!}(\frac{2}{3}t)^2+\frac{1}{3!}(\frac{2}{3}t)^3+\cdots$ 1M any three terms

$=1+\frac{2}{3}t+\frac{2}{9}t^2+\frac{4}{81}t^3+\cdots$ 1A

$\frac{12e^{\frac{2}{3}t}}{3+t}=12(\frac{1}{3}-\frac{1}{9}t+\frac{1}{27}t^2-\frac{1}{81}t^3+\cdots)(1+\frac{2}{3}t+\frac{2}{9}t^2+\frac{4}{81}t^3+\cdots)$

$=4+\frac{4}{3}t+\frac{4}{9}t^2+\frac{4}{81}t^3+\cdots$ 1A for the first three terms or the term t^3

1A for all being correct

(ii) $\int_0^{2.5}\frac{12e^{\frac{2}{3}t}}{3+t}\,dt \approx \int_0^{2.5}(4+\frac{4}{3}t+\frac{4}{9}t^2+\frac{4}{81}t^3)\,dt$

$=\left[4t+\frac{2}{3}t^2+\frac{4}{27}t^3+\frac{1}{81}t^4\right]_0^{2.5}$ 1M

$=16.9637$ (m mol/L) 1A a−1 for r.t. 16.964

----------(7)

(c) The expansion is valid only when

$-1<\frac{t}{3}<1$ 2A

$-3<t<3$

Hence $0 \le t < 3$ (as $t \ge 0$)

$\therefore$ this method is not valid to estimate the amount of lactic acid for $t \ge 3$. 1A

----------(3)

45. (2000 ASL-M&S Q9)

(a) (i) $f(x) = 16 + 4xe^{-0.25x}$

$f'(x) = 4e^{-0.25x}(1 - 0.25x)$ — 1M attempting to find f'; 1A

accept considering $f'(x) = e^{-0.25x}(0.25x - 2)$

$$\begin{cases} > 0 & \text{if } 0 < x < 4 \\ = 0 & \text{if } x = 4 \\ < 0 & \text{if } x > 4 \end{cases}$$

$\therefore$ $f(x) \le f(4)$ for $x > 0$. — 1 follow through

(ii)

x	0	1	2	3	4	5	6
f(x)	16 (16)	19.1152 (19.1)	20.8522 (20.9)	21.6684 (21.7)	21.8861 (21.9)	21.7301 (21.7)	21.3551 (21.4)

1A correct to 1 d.p.

$\int_0^6 f(x)dx$

$\approx \frac{1}{2}[16 + 21.3551 + 2(19.1152 + 20.8523 + 21.6684 + 21.8861 + 21.7301)]$ — 1M

≈ 124

$\therefore$ The expected increase in profit is 124 hundred thousand dollars. — 1A a−1 for r.t. 124; pp−1 for wrong/missing unit

(b) (i) $g(x) = 16 + \dfrac{6x}{\sqrt{1+8x}}$

$$g'(x) = \frac{6\sqrt{1+8x} - \dfrac{6x \cdot 8}{2\sqrt{1+8x}}}{1+8x}$$ — 1A

$$= \frac{6(1+4x)}{(1+8x)^{\frac{3}{2}}}$$

> 0 for $x > 0$. — 1

$\therefore$ $g(x)$ is strictly increasing for $x > 0$.

$\because$ $\lim\limits_{x\to\infty}\left(16 + \dfrac{6x}{\sqrt{1+8x}}\right) = \lim\limits_{x\to\infty}\left(16 + \dfrac{6\sqrt{x}}{\sqrt{\dfrac{1}{x}+8}}\right)$

$\therefore$ $g(x) \to \infty$ as $x \to \infty$ — 1A

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(ii) Let $u = \sqrt{1+8x}$, then $u^2 = 1 + 8x$, $2udu = 8dx$

$\int_0^6 g(x)dx = \int_0^6\left(16 + \dfrac{6x}{\sqrt{1+8x}}\right)dx$ (or $\int_0^6 16dx + \int_0^6 \dfrac{6x}{\sqrt{1+8x}}dx$)

$= \int_1^7\left(16 + \dfrac{6(u^2-1)}{8u}\right)\dfrac{1}{4}udu$ (or $[16x]_0^6 + \int_1^7 \dfrac{6(u^2-1)}{8u}\dfrac{1}{4}udu$) — 1A integrand; 1A limits

$= \int_1^7\left(\dfrac{3}{16}u^2 + 4u - \dfrac{3}{16}\right)du$ (or $96 + \int_1^7\left(\dfrac{3}{16}u^2 - \dfrac{3}{16}\right)du$)

$= \left[\dfrac{1}{16}u^3 + 2u^2 - \dfrac{3}{16}u\right]_1^7$ (or $96 + \left[\dfrac{1}{16}u^3 - \dfrac{3}{16}u\right]_1^7$) — 1A ignore limits

$= 116\dfrac{1}{4}$

≈ 116

$\therefore$ The expected increase in profit is 116 hundred thousand dollars. — 1A a−1 for r.t. 116; pp−1 for wrong/missing unit

(c) From (a)(i), $f(x) \le f(4)$ (≈ 21.8861) for $x > 0$.
i.e. $f(x)$ is bounded above by $f(4)$.
From (b)(i), $g(x)$ increases to infinity as x increases to infinity. — 1M

$\because$ $f(x) > 0$ and $g(x) > 0$ for $x > 0$,
the area under the graph of $g(x)$ will be greater than that of $f(x)$ as x increases indefinitely.

$\therefore$ Plan G will eventually result in a bigger profit. — 1A

46. (1999 ASL-M&S Q8)

(a) $S_A=\frac{256}{9625}\left(\frac{1}{3}t^3-\frac{47}{4}t^2+120t\right)$

$\frac{dS_A}{dt}=\frac{256}{9625}\left(t^2-\frac{47}{2}t+120\right)$ 1A

$=\frac{128}{9625}(t-16)(2t-15)$

$\frac{dS_A}{dt}\begin{cases}>0 & \text{when } 0\le t<\frac{15}{2}\\ =0 & \text{when } t=\frac{15}{2}\\ <0 & \text{when } \frac{15}{2}<t\le 12.5\end{cases}$ 1M

$\therefore$ A attains its top speed at $t=\frac{15}{2}$ (or 7.5)

Top speed of $A=\frac{256}{9625}\left[\frac{1}{3}\left(\frac{15}{2}\right)^3-\frac{47}{4}\left(\frac{15}{2}\right)^2+120\left(\frac{15}{2}\right)\right]$ m/s

≈ 10.0987 m/s 1A $a-1$ for r.t. 10.099; $pp-1$ for missing unit

(b) $S_B=\frac{183}{50}te^{-kt}$

$\frac{dS_B}{dt}=\frac{183}{50}e^{-kt}(1-kt)$ 1A

$\because$ $k>0$

$\therefore$ $\frac{dS_B}{dt}\begin{cases}>0 & \text{when } 0\le t<\frac{1}{k}\\ =0 & \text{when } t=\frac{1}{k}\\ <0 & \text{when } t>\frac{1}{k}\end{cases}$ 1M for solving $\frac{dS_B}{dt}=0$ or sub. $t=\frac{15}{2}$ into $\frac{dS_B}{dt}=0$

B attains its top speed at $t=\frac{1}{k}$.

From (a), $\frac{1}{k}=\frac{15}{2}$

$k=\frac{2}{15}$ (or 0.1333) 1A $a-1$ for r.t. 0.133

(c)

t	0	2.5	5	7.5	10	12.5
S_B	0	6.55626 (6.5563)	9.39553 (9.3955)	10.09829 (10.0983)	9.64766 (9.6477)	8.64106 (8.6411)

1M correct to 4 d.p.

The distance covered by B in 12.5 seconds

$=\int_0^{12.5}S_B\,dt$ m

$\approx\frac{2.5}{2}[0+8.64106+2(6.55626+9.39553+10.09829+9.64766)]$ m 1M

≈ 100.0457 m 1A accept 100.0457 to 100.0699; $a-1$ for r.t. 3 d.p.

(d) $\frac{d^2S_B}{dt^2}=\frac{183}{50}k^2e^{-kt}(t-\frac{2}{k})$ (or $\frac{183}{50}ke^{-kt}(kt-2)$) 1M

$=\frac{122}{1875}e^{-\frac{2t}{15}}(t-15)$ (or $\frac{61}{125}e^{-\frac{2t}{15}}(\frac{2}{15}t-2)$)

<0 for $0\le t\le 12.5$

$\therefore$ The graph of S_B is concave downward for $0\le t\le 12.5$.

i.e., The estimated distance covered by B in (c) is underestimated. 1M

Hence B covers more than 100 m in 12.5 seconds.

B finishes the race ahead of A. 1

(e) $\int_0^{12.5}\frac{50[\ln(t+2)-\ln 2]}{t+2}dt$

$=\int_0^{12.5}\frac{25[\ln\frac{t+2}{2}]}{\frac{t+2}{2}}dt$ (or $50\int_0^{12.5}\left(\frac{\ln(t+2)}{t+2}-\frac{\ln 2}{t+2}\right)dt$)

$=25\left[\left(\ln\frac{t+2}{2}\right)^2\right]_0^{12.5}$ (or $50\left[\frac{(\ln(t+2))^2}{2}-\ln 2\ln(t+2)\right]_0^{12.5}$) 1A

≈ 98.1092 1A

$\therefore$ C covers only 98.1092 m but both A and B finish the race in 12.5 seconds. C is the last one to finish the race among the three athletes. 1

<u>Alternatively,</u>

$\int_0^x\frac{50[\ln(t+2)-\ln 2]}{t+2}dt=25\left[\left(\ln\frac{t+2}{2}\right)^2\right]_0^x$ 1A

If $25\left(\ln\frac{x+2}{2}\right)^2=100$

then $\ln\frac{x+2}{2}=2$

$x\approx 12.78$ 1A

$\therefore$ C needs 12.78 seconds to finish the race but both A and B finish the race within 12.5 seconds. C is the last one to finish the race among the three athletes. 1

47. (1998 ASL-M&S Q8)

(a) (i) If $\dfrac{5000e^{15\lambda}}{15} = \dfrac{5000e^{95\lambda}}{95}$ — 1A

then $e^{80\lambda} = \dfrac{19}{3}$

$\lambda = \dfrac{1}{80}\ln\left(\dfrac{19}{3}\right)$ — 1M+1A

≈ 0.0231

(ii) $N = \dfrac{5000e^{\lambda t}}{t} \approx \dfrac{5000e^{0.0231t}}{t}$

$\dfrac{dN}{dt} = 5000\left(\dfrac{\lambda t e^{\lambda t} - e^{\lambda t}}{t^2}\right)$ — 1M+1A

$= \dfrac{5000e^{\lambda t}(\lambda t - 1)}{t^2}$

$\begin{cases} <0 & \text{when } 0 < t < \dfrac{1}{\lambda} \\ =0 & \text{when } t = \dfrac{1}{\lambda} \quad (\approx 43.3410) \\ >0 & \text{when } \dfrac{1}{\lambda} < t < 120 \end{cases}$ — 1M+1A

$\therefore$ N attains its minimum when $t \approx 43.3410$ — 1A — r.t. 43
(The number of fish decreased to the minimum in about 43 days after the spread of the disease.)

(b) $\displaystyle\int_0^{15} \frac{dW}{ds}\,ds$

$= \displaystyle\int_0^{15} \frac{3}{50}\left(e^{-\frac{s}{20}} - e^{-\frac{s}{10}}\right)ds$ — 1A

$= \dfrac{3}{50}\left[-20e^{-\frac{s}{20}} + 10e^{-\frac{s}{10}}\right]_0^{15}$ — 1A

≈ 0.1670 — 1A

$\therefore$ The increase in the mean weight of fish in the first 15 days is 0.1670 kg.

If $\displaystyle\int_0^{a} \frac{dW}{ds}\,ds = 0.5$,

then $\dfrac{3}{50}\left[-20e^{-\frac{s}{20}} + 10e^{-\frac{s}{10}}\right]_0^{a} = 0.5$ — 1M

$10e^{-\frac{a}{10}} - 20e^{-\frac{a}{20}} = \dfrac{25}{3} - 10$

$\left(e^{-\frac{a}{20}}\right)^2 - 2\left(e^{-\frac{a}{20}}\right) + \dfrac{1}{6} = 0$ — 1M

$e^{-\frac{a}{20}} \approx 0.0871$ or 1.9129 — 1A

$a \approx 48.8073$ or -12.9721 (rej.) — 1A

$\therefore$ It takes about 49 days for the mean weight of the fish to increase 0.5 kg from the *Recovery Day*.

DSE Mathematics Module 1 — 6. Definite Integrals

48. (1998 ASL-M&S Q9)

(a) $I = \displaystyle\int_{0.5}^{2.5} e^{-x}\,dx$

$= \left[-e^{-x}\right]_{0.5}^{2.5}$ — 1A

$= e^{-0.5} - e^{-2.5}$ — 1A

$\approx 0.5244 \quad (0.524446)$

(b) $y = ae^{-x} + bxe^{-x}$

$\because$ y-intercept is -3

$\therefore$ $a = -3$ — 1A

$y' = -ae^{-x} + be^{-x} - bxe^{-x}$ — 1A — neglecting the value of a

$= (-a + b - bx)e^{-x}$

$\because$ y attains its maximum when $x = \dfrac{3}{2}$

$\therefore$ $-a + b - \dfrac{3}{2}b = 0$ — 1M

$3 - \dfrac{1}{2}b = 0$

$b = 6$ — 1A

Hence $y = -3e^{-x} + 6xe^{-x}$

(c) If $y = 0$, $3e^{-x}(2x - 1) = 0$

$x = \dfrac{1}{2}$ — 1A

$\therefore$ The x-intercept of the curve is $\dfrac{1}{2}$.

$y' = 9e^{-x} - 6xe^{-x}$

$y'' = -9e^{-x} - 6e^{-x} + 6xe^{-x}$

$= -15e^{-x} + 6xe^{-x}$ — 1M

$= 3(2x - 5)e^{-x}$

$\therefore$ $y'' \begin{cases} <0 & \text{if } 0 \le x < \dfrac{5}{2} \\ =0 & \text{if } x = \dfrac{5}{2} \\ >0 & \text{if } x > \dfrac{5}{2} \end{cases}$ — 1M

The point of inflection is $\left(\dfrac{5}{2}, 12e^{-\frac{5}{2}}\right)$ [or $\left(\dfrac{5}{2}, 0.9850\right)$] — 1A

(d) (i)

x	0.5	1	1.5	2	2.5
xe^{-x}	0.303265	0.367879	0.334695	0.270671	0.205212

$J_0 \approx \dfrac{0.5}{2}[0.303265 + 0.205212 + 2(0.367879 + 0.334695 + 0.270671)]$ — 1M+1A

$\approx 0.6137 \quad (0.613742)$

$A_0 \approx -3 \times 0.524446 + 6 \times 0.613742$

$\approx 2.1091 \quad (2.109114)$ — 1A

(ii) The argument is not correct because the trapezoidal rule was used to approximate the value of J only.
The convexity of the function xe^{-x} should be considered instead of the function $-3e^{-x} + 6xe^{-x}$. — 1A+1 — 1A for either reason; 1 for both

49. (1997 ASL-M&S Q10)

(a) $y = x^x$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$ 1A

$\frac{dy}{dx} = x^x(1+\ln x)$ 1A

(b) $\frac{d^2y}{dx^2} = x^x \frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}x^x$

$= x^x \cdot \frac{1}{x} + (1+\ln x)x^x(1+\ln x)$ 1A

$= x^{x-1} + x^x(1+\ln x)^2$

> 0 for $1 \le x \le 2$ 1A

y is concave upward (or convex) for $1 \le x \le 2$

$\therefore$ I would be overestimated if the trapezoidal rule is used to estimate I. 1

(c) $I + J = \int_1^2 x^x(1+\ln x)dx$ 1A

$= \left[x^x\right]_1^2$ by (a)

$= 3$ 1

(d) (i)

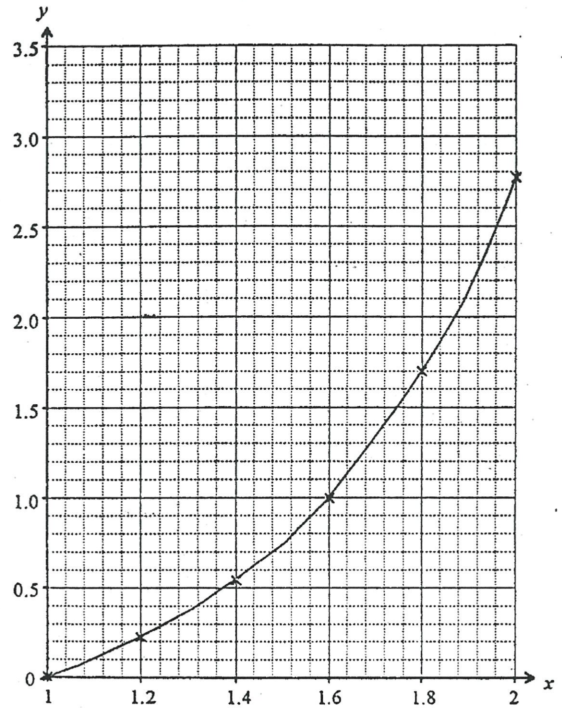
x	1	1.2	1.4	1.6	1.8	2
$x^x \ln x$	0	0.22691	0.53893	0.99700	1.69321	2.77259

1A

$J_0 \approx \approx \frac{0.2}{2}[2.77259 + 2(0.22691 + 0.53893 + 0.99700 + 1.69321)]$ 1M

≈ 0.9685 1A

(ii)



1A+1M

From the plotted graph, $y = x^x \ln x$ is concave upward (or convex) for $1 \le x \le 2$.

$\therefore$ J_0 is an overestimate of J . 1M

(iii) The estimation can be improved by increasing the number of sub-intervals. 1

(iv) J_0 is an underestimate of I because the value 3 for $I + J$ is exact and J_0 is an overestimate of J. 1

50. (1996 ASL-M&S Q9)

(a)

t	0	1	2	3	4	5	6
$e^{\frac{t^2}{10}}$	1	1.10517	1.49182	2.45960	4.95303	12.18249	36.59823

1A

$\therefore \int_0^6 e^{\frac{t^2}{10}}\,dt \approx \frac{1}{2}(1+36.59823)+(1.10517+1.49182+2.45960+4.95303+12.18249)$ 1M

≈ 40.9912 1A

$\because P|_{t=6}-P|_{t=0}=\int_0^6 (5e^{\frac{t^2}{10}}-2t)\,dt$

$\therefore P|_{t=6}=\int_0^6 (5e^{\frac{t^2}{10}}-2t)\,dt+10$ 2A

$=5\int_0^6 e^{\frac{t^2}{10}}\,dt-\left[t^2\right]_0^6+10$ 1A

$\approx 5\times 40.9912-36+10$

≈ 179 1A

(b) (i) Put $t=6$ and $P=179$ into $P=kte^{-0.04t}-50$.

$179=6ke^{-0.24}-50$ 1M

$k=48.5$ 1A

(ii) $P=48.5te^{-0.04t}-50$

$P'=48.5(-0.04te^{-0.04t}+e^{-0.04t})$ 1M

$=48.5(1-0.04t)e^{-0.04t}$

$\because e^{-0.04t}>0$ for all t

$\therefore P'=0$ only when $t=25$ 1A

and $P'\begin{cases}>0 & \text{for } t<25\\ <0 & \text{for } t>25\end{cases}$ 1M

Hence the population size will attain its max. when $t=25$.

The maximum population size $=48.5\cdot 25\cdot e^{-0.04\cdot 25}-50$

≈ 396 1A

(iii) Substitute $y=e^{0.04t}$ into $48.5te^{-0.04t}-50=0$, we have $y=0.97t$. 1M

The graphs $y=e^{0.04t}$ and $y=0.97t$ intersect at $t\approx 1$ or 119

$\because t\geq 6$,

$\therefore$ The species of reptiles becomes extinct ($48.5te^{-0.04t}-50=0$) when $t\approx 119$. 1A Accept 118 - 120

51. (1996 ASL-M&S Q10)

(a) (i) $\ln[C(t)+1]=\ln ae^{bt}$

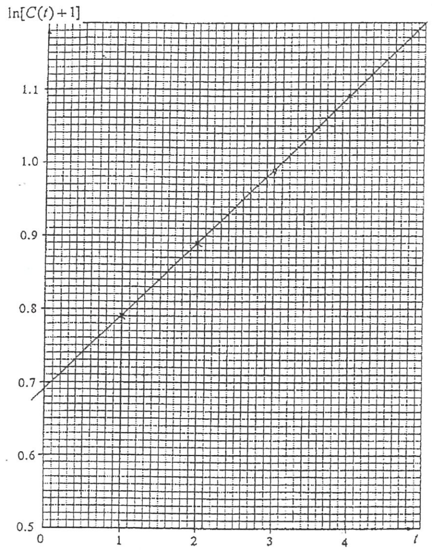
$=\ln a+\ln e^{bt}$ 1M

$=\ln a+bt$ 1A

(ii)

t	1	2	3	4
$\ln[C(t)+1]$	0.79	0.89	0.99	1.09

1A



1A+1A For the points & line

From the graph,

$\ln a\approx 0.69$, $a\approx 2.0$ 1A

$b\approx \frac{1.09-0.79}{4-1}=0.1$ 1A

(iii) $C(t)=2.0e^{0.1t}-1$

$C(36)\approx 72.1965$

When $t=36$, the monthly cost is 72.1965 thousand dollars. 1A

(b) (i) Solve $2.0e^{0.1t}-1=439-e^{0.2t}$ 1M

$e^{0.2t}+2.0e^{0.1t}-440=0$

$\left(e^{0.1t}\right)^2+2.0\left(e^{0.1t}\right)-440=0$ 1M

$e^{0.1t}=20$ or -22 (rej.) 1A

$t\approx 30$ 1A

(ii) $\int_0^{30}[(439-e^{0.2t})-(2.0e^{0.1t}-1)]\,dt$ 1M

$=\int_0^{30}(440-e^{0.2t}-2.0e^{0.1t})\,dt$

$=\left[440t-5e^{0.2t}-20e^{0.1t}\right]_0^{30}$ 1A

≈ 10806

$\therefore$ The total profit is 10806 thousand dollars. 1A

52. (1995 ASL-M&S Q7)

(a) (i)

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	1	1.00504	1.02062	1.04828	1.09109	1.15470

1A

$I_1 = 0.1\left[\frac{1}{2}(1+1.15470)\right.$

$\left.+(1.00504+1.02062+1.04828+1.09109)\right]$ 1M Using correct formula

$= 0.5242$ 1A

(ii) $f'(x) = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$ 1A

$f''(x) = \dfrac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}$ 1A must be simplified

(iii) By (a)(ii), $f''(x) > 0$ for $0 \le x \le \frac{1}{2}$,

$\therefore$ $f(x)$ is concave upward (or convex) on $[0, \frac{1}{2}]$ 1 argument for convexity

Hence I_1 is an over-estimate of I. 1A

(b) (i) $f(x) = (1-x^2)^{-\frac{1}{2}}$

$= 1 + (-\frac{1}{2})(-x^2) + \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-x^2)^2$

$+ \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(-x^2)^3 + \ldots$ 1M + 1A 1M for binomial series

$= 1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\ldots$ for $0 \le x \le \frac{1}{2}$.

$\therefore$ $p(x) = 1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6$ 1A

$I_2 = \int_0^{\frac{1}{2}} (1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6)\,dx$

$= \left[x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7\right]_0^{\frac{1}{2}}$ 1A

$= \frac{1}{2}+\frac{1}{48}+\frac{3}{1280}+\frac{5}{14336}$

≈ 0.5235 1A

(ii) $\because$ $f(x) = p(x) + \displaystyle\sum_{r=4}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-r+1)}{r!}(-x^2)^r$

$= p(x) + \displaystyle\sum_{r=4}^{\infty} \frac{(\frac{1}{2})(\frac{1}{2}+1)\ldots(\frac{1}{2}+r-1)}{r!}x^{2r}$ 1A

$> p(x)$ for $0 < x \le \frac{1}{2}$. 1

Hence $I > I_2$

i.e. I_2 is an under-estimate of I. 1A

<u>Note</u>:

1. 1 mark for the following argument in b(ii)
   $\because$ Sum to infinity
   $\therefore$ $p(x)$ is just a truncation
   Hence underestimate

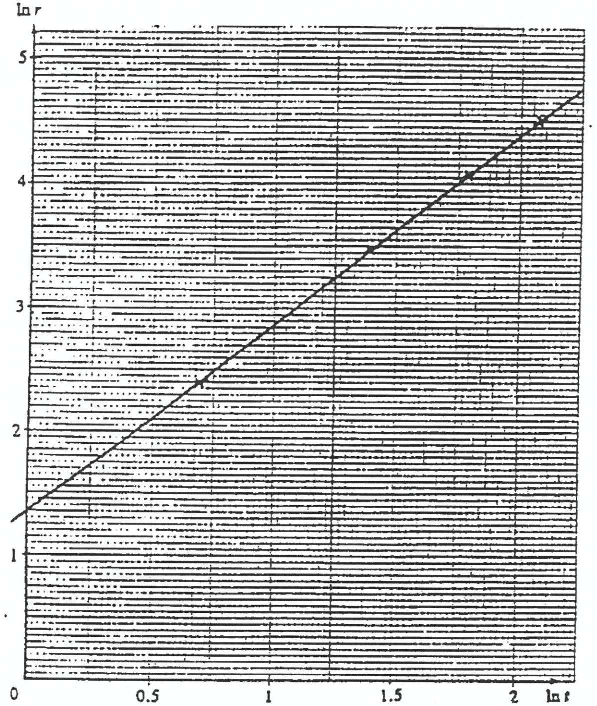
2. Withhold 1 mark <u>once</u> for incorrect degree of accuracy.

53. (1994 ASL-M&S Q10)

(a) Amount of pollutant		
$= \int_0^8 r(t)\,dt$		
$\approx \frac{2}{2}[0+2(11+32+59)+90]$	1M	
$= 294$ (units)	1A	
(b) (i) $r = at^b$		
$\ln r = \ln a + b\ln t$	1M	For taking logarithm

$\ln t$	0.69	1.39	1.79	2.08
$\ln r$	2.40	3.47	4.08	4.50

1A For entries in the table



	1	For the graph
From the graph,		
$\ln a \approx 1.35$		Accept 1.3 - 1.4
$a = 3.9$	1A	or 3.7 - 4.1 respectively
$b \approx \frac{4.50-2.40}{2.08-0.69} \approx 1.5$	1A	Accept 1.4 - 1.6

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(b) (ii) Amount of pollutant	
$= \int_0^8 3.9t^{1.5}\,dt$	1M
$= \left[\frac{3.9}{2.5}t^{2.5}\right]_0^8$	1M
$= \frac{3.9}{2.5}\times 8^{2.5}$	
≈ 282.4 (units)	1A

In general, accept

Amount of pollutant

$= \int_0^8 at^b\,dt$

$= \left[\frac{a}{b+1}t^{b+1}\right]_0^8$

$= \frac{a}{b+1}8^{b+1}$ where $a \in (3.7, 4.1)$, $b \in (1.4, 1.6)$

$\in (226.7, 351.4)$

(c) Amount of pollutant after x months	
$= \int_0^x 3.9t^{1.5}\,dt$	1M
$= \left[\frac{3.9}{2.5}t^{2.5}\right]_0^x$	1M
$= \frac{3.9}{2.5}x^{2.5}$	1A

In general, accept

...

$= \frac{a}{b+1}x^{b+1}$ where $a \in (3.7, 4.1)$, $b \in (1.4, 1.6)$

The lake will "die" after x months if	
$\frac{3.9}{2.5}x^{2.5} = 1000$ (or $\frac{3.9}{2.5}x^{2.5} \ge 1000$)	1M
$x = \left(\frac{2.5\times1000}{3.9}\right)^{\frac{1}{2.5}}$	
≈ 13	1A

In general, accept

...

$x = \left(\frac{1000(b+1)}{a}\right)^{\frac{1}{b+1}}$ where $a \in (3.7, 4.1)$, $b \in (1.4, 1.6)$

$x \in (12, 15)$