7 Functions and Graphs

7A General functions

7A.1 HKCEE MA 1992 – I – 4

(a) Factorize
  (i) x^2-2x,
  (ii) x^2-6x+8.

(b) Simplify $\frac{1}{x^2-2x}+\frac{1}{x^2-6x+8}$.

7A.2 HKCEE MA 1993 – I – 2(a)

Let $f(x)=\frac{x^2+1}{x-1}$. Find $f(3)$.

7A.3 HKCEE MA 2006 – I – 10

Let $f(x)=(x-a)(x-b)(x+1)-3$, where a and b are positive integers with $a<b$. It is given that $f(1)=1$.

(a) (i) Prove that $(a-1)(b-1)=2$.
  (ii) Write down the values of a and b.

(b) Let $g(x)=x^3-6x^2-2x+7$. Using the results of (a)(ii), find $f(x)-g(x)$.
Hence find the exact values of all the roots of the equation $f(x)=g(x)$.

7A.4 HKDSE MA 2016 – I – 3

Simplify $\frac{2}{4x-5}+\frac{3}{1-6x}$.

7A.5 HKDSE MA 2019 – I – 2

Simplify $\frac{3}{7x-6}-\frac{2}{5x-4}$.

7B Quadratic functions and their graphs

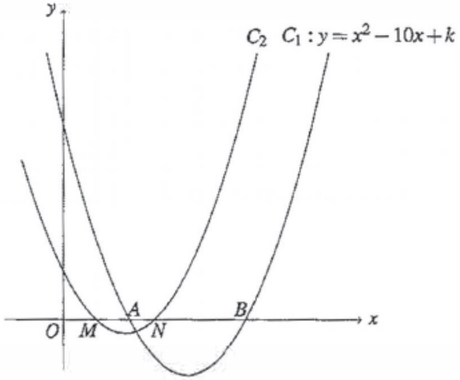
7B.1 HKCEE MA 1982(1/2/3) – I – 11

In the figure, O is the origin. The curve $C_1: y=x^2-10x+k$ (where k is a fixed constant) intersects the x-axis at the points A and B.

(a) By considering the sum and the product of the roots of $x^2-10x+k=0$, or otherwise,
  (i) find $OA+OB$,
  (ii) find $OA\times OB$ in terms of k.

(b) M and N are the mid-points of OA and OB respectively (see the figure).
  (i) Find $OM+ON$.
  (ii) Find $OM\times ON$ in terms of k.

(c) Another curve $C_2: y=x^2+px+r$ (where p and r are fixed constants) passes through the points M and N.
  (i) Using the results in (b) or otherwise, find the value of p and express r in terms of k.
  (ii) If $OM=2$, find k.

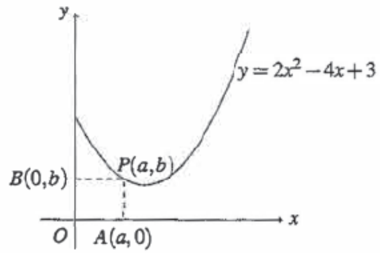


7B.2 HKCEE MA 1992 – I – 9

The figure shows the graph of $y=2x^2-4x+3$, where $x\geq 0$. $P(a,b)$ is a variable point on the graph. A rectangle $OAPB$ is drawn with A and B lying on the x and y axes respectively.

(a) (i) Find the area of rectangle $OAPB$ in terms of a.
  (ii) Find the two values of a for which $OAPB$ is a square.

(b) Suppose the area of $OAPB=\frac{3}{2}$.
  (i) Show that $4a^3-8a^2+6a-3=0$.
  (ii) ***[Out of syllabus]***



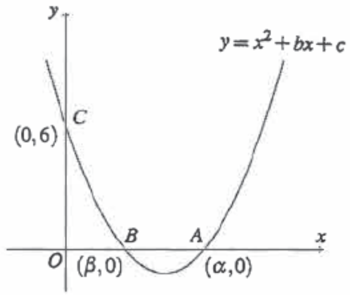
7B.3 HKCEE MA 1994 – I – 8

In the figure, the curve $y=x^2+bx+c$ meets the y-axis at $C(0,6)$ and the x axis at $A(\alpha,0)$ and $B(\beta,0)$, where $\alpha>\beta$.

(a) Find c and hence find the value of $\alpha\beta$.

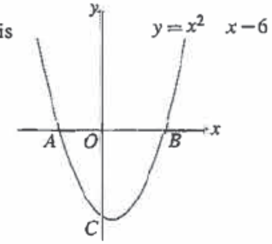
(b) Express $\alpha+\beta$ in terms of b.

(c) Using the results in (a) and (b), express $(\alpha-\beta)^2$ in terms of b. Hence find the area of $\triangle ABC$ in terms of b.



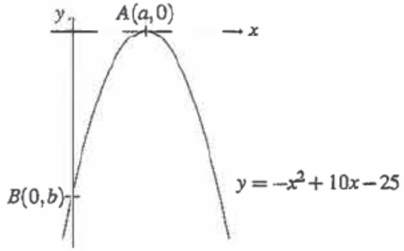
7B.4 HKCEE MA 1999 – I – 7

The graph of $y=x^2-x-6$ cuts the x-axis at $A(a,0)$, $B(b,0)$ and the y-axis at $C(0,c)$ as shown in the figure. Find a, b and c.



7B.5 HKCEE MA 2004 – I – 4

In the figure, the graph of $y=-x^2+10x-25$ touches the x-axis at $A(a,0)$ and cuts the y-axis at $B(0,b)$. Find a and b.



7B.6 HKCEE MA 2008 – I – 11

Consider the function $f(x)=x^2+bx-15$, where b is a constant. It is given that the graph of $y=f(x)$ passes through the point (4, 9).

(a) Find b. Hence, or otherwise, find the two x-intercepts of the graph of $y=f(x)$.

(b) Let k be a constant. If the equation $f(x)=k$ has two distinct real roots, find the range of values of k.

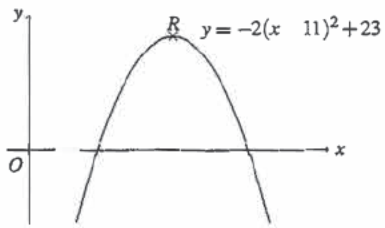
(c) Write down the equation of a straight line which intersects the graph of $y=f(x)$ at only one point.

7B.7 HKCEE MA 2009 – I – 12

In the figure, R is the vertex of the graph of $y=-2(x-11)^2+23$.

(a) Write down
  (i) the equation of the axis of symmetry of the graph,
  (ii) the coordinates of R.

(b) It is given that $P(p,5)$ and $Q(q,5)$ are two distinct points lying on the graph. Find
  (i) the distance between P and Q;
  (ii) the area of the quadrilateral $PQRS$, where S is a point lying on the x axis.



7B.8 HKCEE MA 2010 – I – 16 (To continue as 7E.1.)

Let $f(x)=\frac{1}{2}x-\frac{1}{144}x^2-6$.

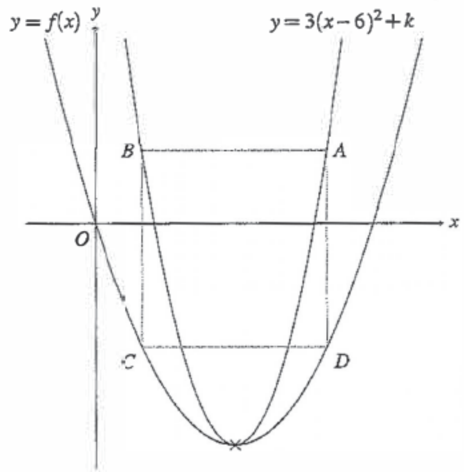
(a) (i) Using the method of completing the square, find the coordinates of the vertex of the graph of $y=f(x)$.

7B.9 HKCEE MA 2011 – I – 11 (Continued from 8C.20.)

It is given that $f(x)$ is the sum of two parts, one part varies as x^2 and the other part varies as x. Suppose that $f(-2)=28$ and $f(6)=-36$.

(a) Find $f(x)$.

(b) The figure shows the graph of $y=3(x-6)^2+k$ and the graph of $y=f(x)$, where k is a constant. The two graphs have the same vertex.
  (i) Find the value of k.
  (ii) It is given that A and B are points lying on the graph of $y=3(x-6)^2+k$ while C and D are points lying on the graph of $y=f(x)$. Also, $ABCD$ is a rectangle and AB is parallel to the x axis. The x coordinate of A is 10. Find the area of the rectangle $ABCD$.



7B.10 HKCEE AM 1988 – I – 10 (To continue as 10C.9.)

Let $f(x)=x^2+2x-1$ and $g(x)=-x^2+2kx-k^2+6$ (where k is a constant.)

(a) Suppose the graph of $y=f(x)$ cuts the x axis at the points P and Q, and the graph of $y=g(x)$ cuts the x-axis at the points R and S.
  (i) Find the lengths of PQ and RS.
  (ii) Find, in terms of k, the x-coordinate of the mid-point of RS.
  If the mid points of PQ and RS coincide with each other, find the value of k.

(b) If the graphs of $y=f(x)$ and $y=g(x)$ intersect at only one point, find the possible values of k; and for each value of k, find the point of intersection.

7B.11 HKCEE AM 1991 – I – 9 (To continue as 10C.11.)

Let $f(x)=x^2+2x-2$ and $g(x)=-2x^2-12x-23$.

(a) Express $g(x)$ in the form $a(x+b)^2+c$, where a, b and c are real constants.
  Hence show that $g(x)<0$ for all real values of x.

(b) Let k_1 and k_2 ($k_1>k_2$) be the two values of k such that the equation $f(x)+kg(x)=0$ has equal roots.
  (i) Find k_1 and k_2.

7B.12 (HKCEE AM 1993 – I – 10)

$C(k)$ is the curve $y=\frac{1}{k+1}[2x^2+(k+7)x+4]$, where k is a real number not equal to -1.

(a) If $C(k)$ cuts the x axis at two points P and Q and $PQ=1$, find the value(s) of k.

(b) Find the range of values of k such that $C(k)$ does not cut the x-axis.

(c) (i) Find the points of intersection of the curves $C(1)$ and $C(-2)$.
  (ii) Show that $C(k)$ passes through the two points in (c)(i) for all values of k.

7B.13 HKCEE AM 1998 – I – 11

Let $f(x) = x^2 - kx$, where k is a real constant, and $g(x) = \quad x$.

(a) Show that the least value of $f(x)$ is $\frac{k^2}{4}$ and find the corresponding value of x.

(b) Find the coordinates of the two intersecting points of curves $y = f(x)$ and $y = g(x)$.

(c) Suppose $k = 3$.

(i) In the same diagram, sketch the graphs of $y = f(x)$ and $y = g(x)$ and label their intersecting points.

(ii) Find the range of values of x such that $f(x) \le g(x)$.
Hence find the least value of $f(x)$ within this range of values of x.

(d) Suppose $k = \frac{3}{2}$. Find the least value of $f(x)$ within the range of values of x such that $f(x) \le g(x)$.

7B.14 HKCEE AM 2000 – I – 12

Consider the function $f(x) = x^2 - 4mx - (5m^2 - 6m + 1)$, where $m > \frac{1}{3}$.

(a) Show that the equation $f(x) = 0$ has distinct real roots.

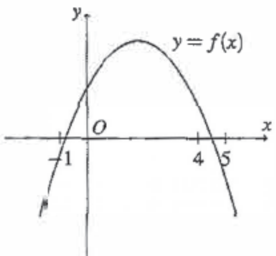
(b) Let α and β be the roots of the equation $f(x) = 0$, where $\alpha < \beta$.

(i) Express α and β in terms of m.

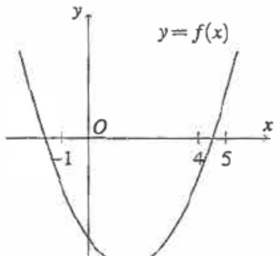
(ii) Furthermore, it is known that $4 < \beta < 5$.

(1) Show that $1 < m < \frac{6}{5}$.

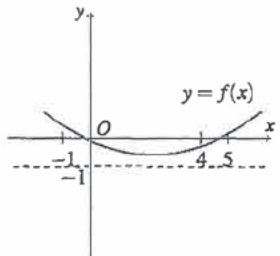
(2) The following figure shows three sketches of the graph of $y = f(x)$ drawn by three students. Their teacher points out that the three sketches are all incorrect. Explain why each of the sketches is incorrect.



Sketch A



Sketch B



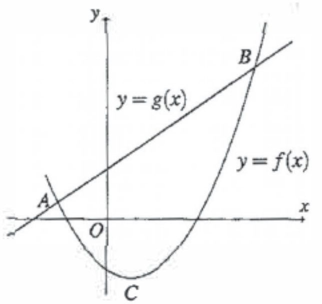
Sketch C

7B.15 HKCEE AM 2002 – 11

Let $f(x) = x^2 - 2x - 6$ and $g(x) = 2x + 6$. The graphs of $y = f(x)$ and $y = g(x)$ intersect at points A and B (see the figure). C is the vertex of the graph of $y = f(x)$.

(a) Find the coordinates of points A, B and C.

(b) Write down the range of values of x such that $f(x) \le g(x)$.
Hence write down the value(s) of k such that the equation $f(x) = k$ has only one real root in this range.



7B.16 HKCEE AM 2003 17

Let $f(x) = \quad (x - a)^2 + b$, where a and b are real. Point P is the vertex of the graph of $y = f(x)$.

(a) Write down the coordinates of point P.

(b) Let $g(x)$ be a quadratic function such that the coefficient of x^2 is 1 and the vertex of the graph of $y = g(x)$ is the point $Q(b, a)$. It is given that the graph of $y = f(x)$ passes through point Q.

(i) Write down $g(x)$ and show that the graph of $y = g(x)$ passes through point P.

(ii) Furthermore, the graph of $y = f(x)$ touches the x-axis. For each of the possible cases, sketch the graphs of $y = f(x)$ and $y = g(x)$ in the same diagram.

7B.17 HKDSE MA 2012 – I 13

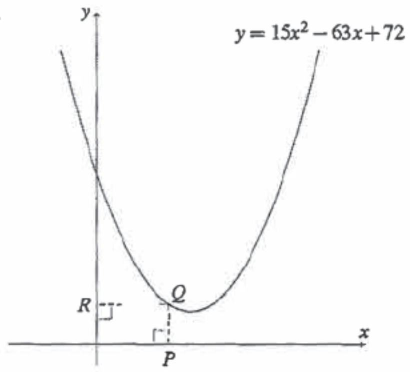
(Continued from 4B.22.)

(a) Find the value of k such that $x - 2$ is a factor of $kx^3 - 21x^2 + 24x - 4$.

(b) The figure shows the graph of $y = 15x^2 - 63x + 72$. Q is a variable point on the graph in the first quadrant. P and R are the feet of the perpendiculars from Q to the x axis and the y axis respectively.

(i) Let $(m, 0)$ be the coordinates of P. Express the area of the rectangle $OPQR$ in terms of m.

(ii) Are there three different positions of Q such that the area of the rectangle $OPQR$ is 12? Explain your answer.



7B.18 HKDSE MA 2015 I 18

(To continue as 7E.2.)

Let $f(x) = 2x^2 - 4kx + 3k^2 + 5$, where k is a real constant.

(a) Does the graph of $y = f(x)$ cut the x axis? Explain your answer.

(b) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y = f(x)$.

7B.19 HKDSE MA 2016 – I – 18

(To continue as 7E.3.)

Let $f(x) = \frac{-1}{3}x^2 + 12x \quad 121$.

(a) Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

7B.20 HKDSE MA 2017 – I 18

The equation of the parabola Γ is $y = 2x^2 - 2kx + 2x - 3k + 8$, where k is a real constant. Denote the straight line $y = 19$ by L.

(a) Prove that L and Γ intersect at two distinct points.

(b) The points of intersection of L and Γ are A and B.

(i) Let a and b be the x coordinates of A and B respectively. Prove that $(a \quad b)^2 = k^2 + 4k + 23$.

(ii) Is it possible that the distance between A and B is less than 4? Explain your answer.

7B.21 HKDSE MA 2018 – I – 18 (Continued from 8C.29 and to continue as 7E.4.)

It is given that $f(x)$ partly varies as x^2 and partly varies as x. Suppose that $f(2) = 60$ and $f(3) = 99$.

(a) Find $f(x)$.

(b) Let Q be the vertex of the graph of $y = f(x)$ and R be the vertex of the graph of $y = 27 - f(x)$.

(i) Using the method of completing the square, find the coordinates of Q.

7B.22 HKDSE MA 2020 – I –

Let $p(x) = 4x^2 + 12x + c$, where c is a constant. The equation $p(x) = 0$ has equal roots. Find

(a) c,

(b) the x-intercept(s) of the graph of $y = p(x) - 169$.

(5 marks)

7B.23 HKDSE MA 2020 – I – 17

Let $g(x) = x^2 - 2kx + 2k^2 + 4$, where k is a real constant.

(a) Using the method of completing the square, express, in terms of k , the coordinates of the vertex of the graph of $y = g(x)$. (2 marks)

(b) On the same rectangular coordinate system, let D and E be the vertex of the graph of $y = g(x+2)$ and the vertex of the graph of $y = -g(x-2)$ respectively. Is there a point F on this rectangular coordinate system such that the coordinates of the circumcentre of ΔDEF are $(0, 3)$? Explain your answer. (4 marks)

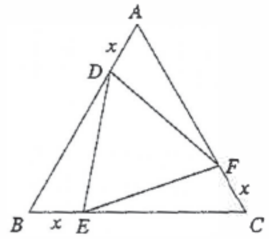
7C Extreme values of quadratic functions

7C.1 HKCEE MA 1985(A/B) – I – 13 (Continued from 14A.3 and to continue as 10C.2.)

In the figure, ABC is an equilateral triangle. $AB = 2$. D, E, F are points on AB, BC, CA respectively such that $AD = BE = CF = x$.

(a) By using the cosine formula or otherwise, express DE^2 in terms of x.

(b) Show that the area of $\triangle DEF = \frac{\sqrt{3}}{4}(3x^2 - 6x + 4)$.
Hence, by using the method of completing the square, find the value of x such that the area of $\triangle DEF$ is smallest.



7C.2 HKCEE MA 1982(1/2) – I – 12 (Continued from 8C.1.)

The price of a certain monthly magazine is x dollars per copy. The total profit on the sale of the magazine is P dollars. It is given that $P = Y + Z$, where Y varies directly as x and Z varies directly as the square of x. When x is 20, P is 80 000; when x is 35, P is 87 500.

(a) Find P when $x = 15$.

(b) Using the method of completing the square, express P in the form $P = a \quad b(x \quad c)^2$ where a, b and c are constants. Find the values of a, b and c.

(c) Hence, or otherwise, find the value of x when P is a maximum.

7C.3 HKCEE MA 1988 – I – 10 (Continued from 8C.5.)

A variable quantity y is the sum of two parts. The first part varies directly as another variable x, while the second part varies directly as x^2. When $x = 1$, $y = -5$; when $x = 2$, $y = -8$.

(a) Express y in terms of x. Hence find the value of y when $x = 6$.

(b) Express y in the form $(x \quad p)^2 - q$, where p and q are constants. Hence find the least possible value of y when x varies.

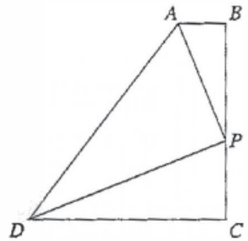
7C.4 HKCEE MA 2011 – I – 12

In the figure, $ABCD$ is a trapezium, where AB is parallel to CD. P is a point lying on BC such that $BP = x$ cm. It is given that $AB = 3$ cm, $BC = 11$ cm, $CD = k$ cm and $\angle ABP = \angle APD = 90°$.

(a) Prove that $\triangle ABP \sim \triangle PCD$.

(b) Prove that $x^2 \quad 11x + 3k = 0$.

(c) If k is an integer, find the greatest value of k.



7C.5 HKCEE AM 1986 I – 3

The maximum value of the function $f(x) = 4k + 18x \quad kx^2$ (k is a positive constant) is 45. Find k.

7C.6 HKCEE AM 1996 – I – 4

Given $x^2 \quad 6x + 11 = (x + a)^2 + b$, where x is real.

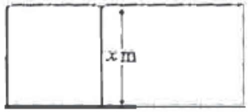
(a) Find the values of a and b. Hence write down the least value of $x^2 - 6x + 11$.

(b) Using (a), or otherwise, write down the range of possible values of $\frac{1}{x^2 - 6x + 11}$.

7C.7 HKDSE MA 2013 – I – 17

(a) Let $f(x) = 36x - x^2$. Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

(b) The length of a piece of string is 108 m. A guard cuts the string into two pieces. One piece is used to enclose a rectangular restricted zone of area A m^2. The other piece of length x m is used to divide this restricted zone into two rectangular regions as shown in the figure.

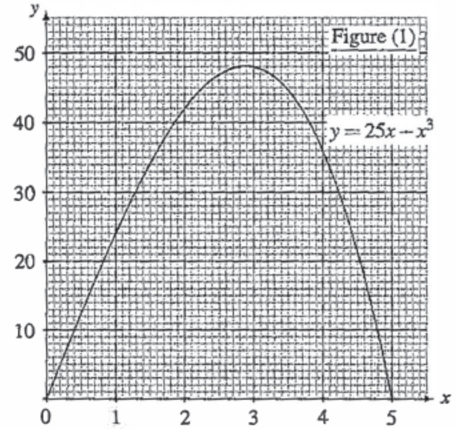


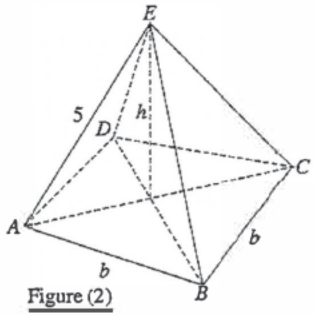
(i) Express A in terms of x.

(ii) The guard claims that the area of this restricted zone can be greater than 500 m^2. Do you agree? Explain your answer.

7D Solving equations using graphs of functions

7D.1 HKCEE MA 1980(3) I 16





(a) Figure (1) shows the graph of $y = 25x - x^3$ for $0 \le x \le 5$. By adding a suitable straight line to the graph, solve the equation $30 = 25x - x^3$, where $0 \le x \le 5$. Give your answers correct to 2 significant figures.

(b) Figure (2) shows a right pyramid with a square base $ABCD$. $AB = b$ units and $AE = 5$ units. The height of the pyramid is h units and its volume is V cubic units.

(i) Express b in terms of h. Hence show that $V = \frac{2}{3}(25h - h^3)$.

(ii) Using (a), find the two values of h such that $V = 20$.
(Your answers should be correct to 2 significant figures.)

(iii) *[Out of syllabus]*

7D.2 HKCEE MA 1981(1) – I – 11

A piece of wire 20 cm long is bent into a rectangle. Let one side of the rectangle be x cm long and the area be y cm^2.

(a) Show that $y = 10x - x^2$.

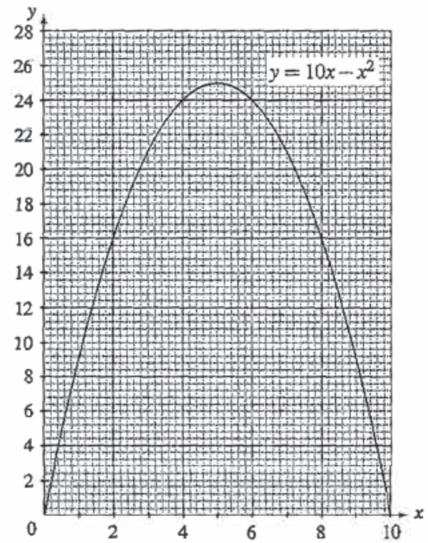
(b) The figure shows the graph of $y = 10x - x^2$ for $0 \le x \le 10$. Using the graph, find

(i) the value of y, correct to 1 decimal place, when $x = 3.4$,

(ii) the values of x, correct to 1 decimal place, when the area of the rectangle is 12 cm^2,

(iii) the greatest area of the rectangle,

(iv) *[Out of syllabus]*



7D.3 <u>HKCEE MA 1983(A) – I – 14</u>

Equal squares each of side k cm are cut from the four corners of a square sheet of paper of side 7 cm (see Figure (1)). The remaining part is folded along the dotted lines to form a rectangular box as shown in Figure (2).

(a) Show that the volume V of the rectangular box, in cm^3, is $V = 4k^3 - 28k^2 + 49k$.

(b) Figure (3) shows the graph of $y = 4x^3 - 28x^2 + 49x$ for $0 \le x \le 5$. Draw a suitable straight line in Figure (3) and use it to find all the possible values of x such that $4x^3 - 28x^2 + 49x - 20 = 0$.
(Give the answers to 1 decimal place.)

(c) Using the results of (a) and (b), deduce the values of k such that the volume of the box is $20 cm^3$.
(Give the answers to 1 decimal place.)

(d) ***[Out of syllabus]***

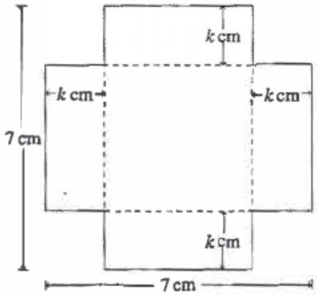


Figure (1)

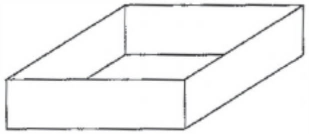


Figure (2)

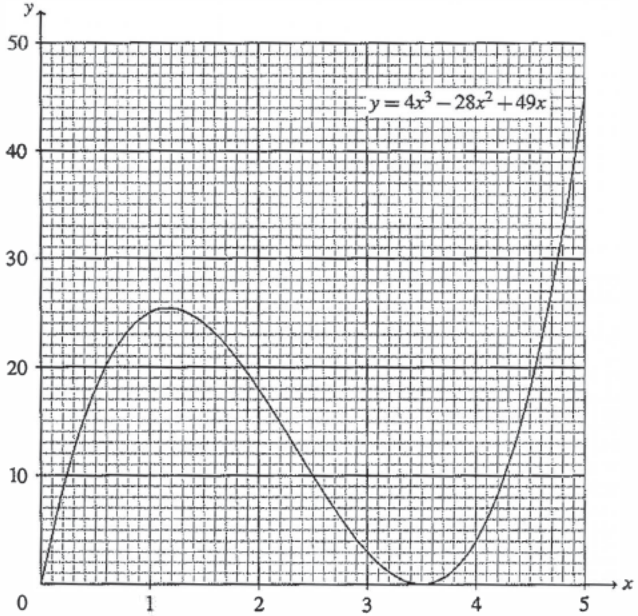


Figure (3)

7D.4 <u>HKCEE MA 1985(A) – I – 12</u>

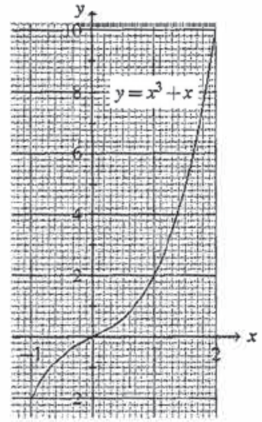
The figure shows the graph of $y = x^3 + x$ for $-1 \le x \le 2$.

(a) (i) Draw a suitable straight line in the figure and hence find, correct to 1 decimal place, the real root of the equation $x^3 + x - 1 = 0$.

(ii) *[Out of syllabus. The result $x = 0.68$ (correct to 2 d.p.) is obtained for the equation in (i).]*

(b) (i) Expand and simplify the expression $(x+1)^4 - (x-1)^4$.

(ii) Using the result in (a)(ii), find, correct to 2 decimal places, the real root of the equation $(x+1)^4 - (x-1)^4 = 8$.



7D.5 <u>HKCEE MA 1985(B) – I – 12</u>

In Figure (1), ABC is an isosceles triangle with $\angle A = 90°$. $PQRS$ is a rectangle inscribed in $\triangle ABC$. $BC = 16$ cm, $BQ = x$ cm.

(a) Show that the area of $PQRS = 2(8x - x^2)$ cm^2.

(b) Figure (2) shows the graph of $y = 8x - x^2$ for $0 \le x \le 8$.
Using the graph,

(i) find the value of x such that the area of $PQRS$ is greatest;

(ii) find the two values of x, correct to 1 decimal place, such that the area of $PQRS$ is $28 cm^2$.

(c) ***[Out of syllabus]***

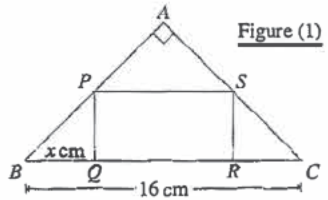


Figure (1)

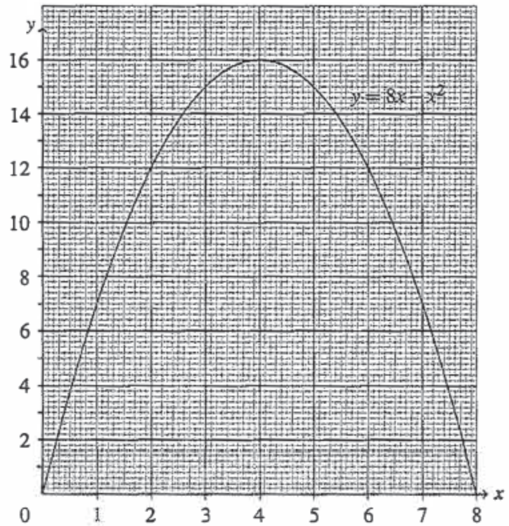


Figure (2)

7D.6 HKCEE MA 1986(B) – I 14

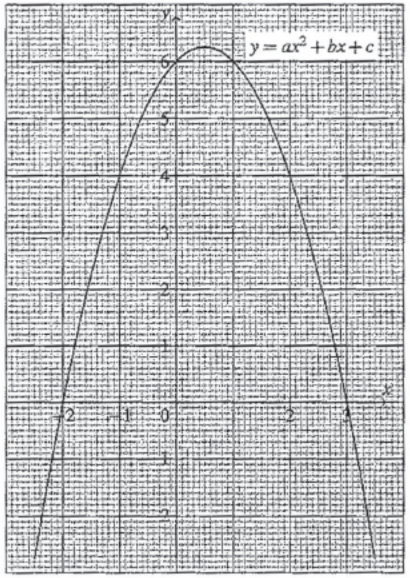
The figure shows the graph of $y = ax^2 + bx + c$.

(a) Find the value of c and hence the values of a and b.

(b) Solve the following equations by adding a suitable straight line to the figure for each case. Give your answers correct to 1 decimal place.

(i) $(x+2)(x-3) = -1$,

(ii) *[Out of syllabus]*



7D.7 HKCEE MA 1987(A) I – 14

The figure shows the graph of $y = x^3 - 6x^2 + 9x$.

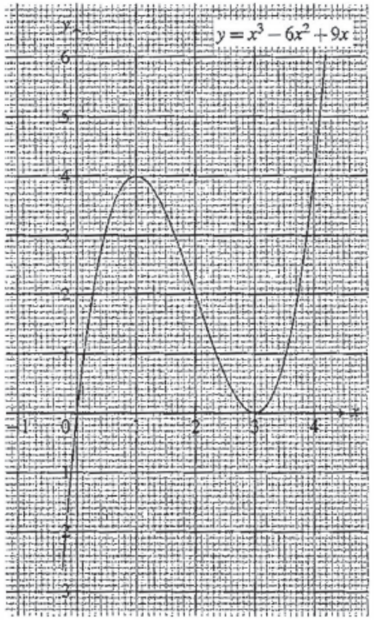
(a) By adding suitable straight lines to the figure, find, correct to 1 decimal place, the real roots of the following equations:

(i) $x^3 - 6x^2 + 9x - 1 = 0$,

(ii) *[Out of syllabus]*

(b) *[Out of syllabus]*

(c) From the figure, find the range of values of k such that the equation $x^3 - 6x^2 + 9x - k = 0$ has three distinct real roots.



7D.8 HKCEE MA 1997 I – 13

Miss Lee makes and sells handmade leather belts and handbags. She finds that if a batch of x belts is made, where $1 \le x \le 11$, the cost per belt \$$B$ is given by $B = x^2 - 20x + 120$. The figure shows the graph of the function $y = x^2 - 20x + 120$.

(a) Use the given graph to write down the number(s) of belts in a batch that will make the cost per belt

(i) a minimum,

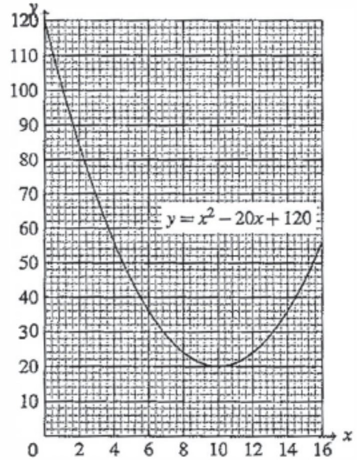
(ii) less than \$90.

(b) Miss Lee also finds that if a batch of x handbags is made, where $1 \le x \le 8$, the cost per handbag \$$H$ is given by $H = x^2 - 17x + c$ (c is a constant). When a batch of 3 handbags is made, the cost per handbag is \$144.

(i) Find c.

(ii) *[Out of syllabus. The following result is obtained: When $H = 120$, $x = 6$.]*

(iii) Miss Lee made a batch of 10 belts and a batch of 6 handbags. She managed to sell 6 belts at \$100 each and 4 handbags at \$300 each while the remaining belts and handbags sold at half of their respective cost. Find her gain or loss.



7D.9 HKCEE MA 2000 – I – 18

(Continued from **8C.11**.)

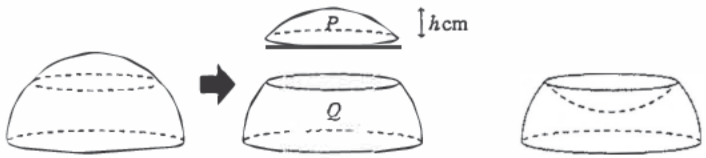


Figure (1) Figure (2)

Figure (1) shows a solid hemisphere of radius 10 cm. It is cut into two portions, P and Q, along a plane parallel to its base. The height and volume of P are h cm and V cm^3 respectively. It is known that V is the sum of two parts. One part varies directly as h^2 and the other part varies directly as h^3. $V = \frac{29}{3}\pi$ when $h = 1$ and $V = 81\pi$ when $h = 3$.

(a) Find V in terms of h and π.

(b) A solid congruent to P is carved away from the top of Q to form a container as shown in Figure (2).

(i) Find the surface area of the container (excluding the base).

(ii) It is known that the volume of the container is $\frac{1400}{3}\pi$ cm^3. Show that $h^3 - 30h^2 + 300 = 0$.

(iii) Using the graph in Figure (3) and a suitable method, find the value of h correct to 2 decimal places.

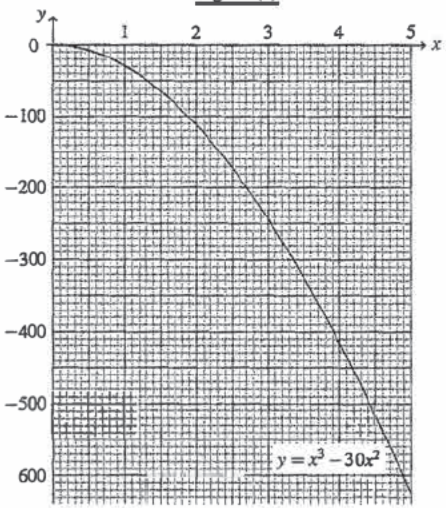


Figure (3)

7E Transformation of graphs of functions

7E.1 HKCEE MA 2010 – I – 16 (Continued from **7B.8.**)

Let $f(x) = \frac{1}{2}x \quad \frac{1}{144}x^2 \quad 6$.

(a) (i) Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

(ii) If the graph of $y = g(x)$ is obtained by translating the graph of $y = f(x)$ leftwards by 4 units and upwards by 5 units, find $g(x)$.

(iii) If the grpah of $y = h(x)$ is obtained by translating the graph of $y = 2^{f(x)}$ leftwards by 4 units and upwards by 5 units, find $h(x)$.

(b) A researcher performs an experiment to study the relationship between the number of bacteria A (u hundred million) and the temperature (s °C) under some controlled conditions. From the data of u and s recorded in Table (1), the researcher suggests using the formula $u = 2^{f(s)}$ to describe the relationship.

s	a_1	a_2	a_3	a_4	a_5	a_6	a_7
u	b_1	b_2	b_3	b_4	b_5	b_6	b_7

Table (1)

(i) According to the formula suggested by the researcher, find the temperature at which the number of the bacteria is 8 hundred million.

(ii) The researcher then performs another experiment to study the relationship between the number of bacteria B (v hundred million) and the temperature (t °C) under the same controlled conditions and the data of v and t are recorded in Table (2).

t	a_1-4	a_2-4	a_3-4	a_4-4	a_5-4	a_6-4	a_7-4
v	b_1+5	b_2+5	b_3+5	b_4+5	b_5+5	b_6+5	b_7+5

Table (2)

Using the formula suggested by the research, propose a formula to express v in terms of t.

7E.2 HKDSE MA 2015 – I – 18 (Continued from **7B.18.**)

Let $f(x) = 2x^2 - 4kx + 3k^2 + 5$, where k is a real constant.

(a) Does the graph of $y = f(x)$ cut the x axis? Explain your answer.

(b) Using the method of completing the square, express, in terms of k, the coordinates of the vertex of the graph of $y = f(x)$.

(c) In the same rectangular system, let S and T be moving points on the graph of $y = f(x)$ and the graph of $y = 2 - f(x)$ respectively. Denote the origin by O. Someone claims that when S and T are nearest to each other, the circumcentre of $\triangle OST$ lies on the x axis. Is the claim correct? Explain your answer.

7E.3 HKDSE MA 2016 – I – 18 (Continued from **7B.19.**)

Let $f(x) = \frac{-1}{3}x^2 + 12x - 121$.

(a) Using the method of completing the square, find the coordinates of the vertex of the graph of $y = f(x)$.

(b) The graph of $y = g(x)$ is obtained by translating the graph of $y = f(x)$ vertically. If the graph of $y = g(x)$ touches the x-axis, find $g(x)$.

(c) Under a transformation, $f(x)$ is changed to $\frac{-1}{3}x^2 - 12x - 121$. Describe the geometric meaning of the transformation.

7E.4 HKDSE MA 2018 – I 18 (Continued from **7B.21.**)

It is given that $f(x)$ partly varies as x^2 and partly varies as x. Suppose that $f(2) = 60$ and $f(3) = 99$.

(a) Find $f(x)$.

(b) Let Q be the vertex of the graph of $y = f(x)$ and R be the vertex of the graph of $y = 27 \quad f(x)$.

(i) Using the method of completing the square, find the coordinates of Q.

(ii) Write down the coordinates of R.

(iii) The coordinates of the point S are $(56, 0)$. Let P be the circumcentre of $\triangle QRS$. Describe the geometric relationship between P, Q and R. Explain your answer.

7E.5 HKDSE MA 2019 I – 19 (To continue as **16C.56.**)

Let $f(x) = \frac{1}{1+k}(x^2 + (6k \quad 2)x + (9k + 25))$, where k is a positive constant. Denote the point $(4, 33)$ by F.

(a) Prove that the graph of $y = f(x)$ passes through F.

(b) The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ with respect to the y-axis and then translating the resulting graph upwards by 4 units. Let U be the vertex of the graph of $y = g(x)$. Denote the origin by O.

(i) Using the method of completing the square, express the coordinates of U in terms of k.

7 Functions and Graphs

7A General functions

7A.1 HKCEE MA 1992 – I – 4

(a) (i) $x^2-2x=x(x-2)$

(ii) $x^2-6x+8=(x-2)(x-4)$

(b) $\frac{1}{x^2-2x}+\frac{1}{x^2-6x+8}=\frac{1}{x(x-2)}+\frac{1}{(x-2)(x-4)}$

$=\frac{(x-4)+(x)}{x(x-2)(x-4)}$

$=\frac{2x-4}{x(x-2)(x-4)}$

$=\frac{2(x-2)}{x(x-2)(x-4)}=\frac{2}{x(x-4)}$

7A.2 HKCEE MA 1993 – I – 2(a)

$f(3)=\frac{(3)^2+1}{(3)-1}=5$

7A.3 HKCEE MA 2006 – I – 10

(a) (i) $1=f(1)=(1-a)(1-b)(2)-3$

$\Rightarrow (a-1)(b-1)=2$

(ii) Since $a-1$ and $b-1$ are both integers and $b-1>a-1$,

$\begin{cases}a-1=1\\ b-1=2\end{cases} \Rightarrow \begin{cases}a=2\\ b=3\end{cases}$

(b) $f(x)-g(x)$

$=(x-2)(x-3)(x+1)-3-(x^3-6x^2-2x+7)$

$=2x^2+3x-4$

$\therefore \quad f(x)=g(x)$

$\Rightarrow 2x^2+3x-4=0$

$x==\frac{-3\pm\sqrt{9+32}}{4}=\frac{-3\pm\sqrt{41}}{4}$

7A.4 HKDSE MA 2016 – I – 3

$\frac{2}{4x-5}+\frac{3}{1-6x}=\frac{2(1-6x)+3(4x-5)}{(4x-5)(1-6x)}$

$=\frac{-13}{(4x-5)(1-6x)}$

7A.5 HKDSE MA 2019 – I – 2

$\frac{3}{7x-6}-\frac{2}{5x-4}=\frac{3(5x-4)-2(7x-6)}{(7x-6)(5x-4)}=\frac{x}{(7x-6)(5x-4)}$

7B Quadratic functions

7B.1 HKCEE MA 1982(1/2/3) – I – 11

(a) Since OA and OB are the roots of the equation,

(i) $OA+OB=10$

(ii) $OA\times OB=k$

(b) (i) $OM+ON=\frac{OA}{2}+\frac{OB}{2}=\frac{OA+OB}{2}=5$

(ii) $OM\times ON=\left(\frac{OA}{2}\right)\left(\frac{OB}{2}\right)=\frac{OA\times OB}{4}=\frac{k}{4}$

(c) (i) $-p=OM+ON=5 \Rightarrow p=5$

$r=OM\times ON=\frac{k}{4}$

(ii) $OM+ON=5 \Rightarrow ON=5-2=3$

$\therefore \frac{k}{4}=OM\times ON \Rightarrow k=4\times2\times3=24$

7B.2 HKCEE MA 1992 – I – 9

(a) (i) $b=2a^2-4a+3$

$\therefore$ Area of $OAPB=a(2a^2-3a+3)=2a^3-4a^2+3a$

(ii) When $a=2a^2-4a+3$,

$2a^2-5a+3=0 \Rightarrow a=1$ or $\frac{3}{2}$

(b) (i) $2a^3-4a^2+3a=\frac{3}{2}$

$4a^3-8a^2+6a=3$

$4a^3-8a^2+6a-3=0$

(ii) *[Out of syllabus]*

7B.3 HKCEE MA 1994 – I – 8

(a) $c=y$-intercept $=6$

$\therefore \alpha\beta=$ product of roots $=6$

(b) $\alpha+\beta=$ sum of roots $=-b$

(c) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-b)^2-4(6)$

$=b^2-24$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}(\alpha-\beta)(6)$

$=3(\alpha-\beta)=3\sqrt{b^2-24}$

7B.4 HKCEE MA 1999 – I – 7

$c=y$-intercept $=-6$

When $y=0$, $x^2-x-6=0 \Rightarrow x=-2$ or 3

$\therefore a=-2,\ b=3$

7B.5 HKCEE MA 2004 – I – 4

$b=y$-intercept $=-25$

Put $(a,0)$: $0=-a^2+10a-25 \Rightarrow a=5$ (repeated)

7B.6 HKCE EMA 2008 – I – 11

(a) Put $(4,9)$: $9=(4)^2+b(4)-15 \Rightarrow b=2$

Hence, $0=x^2+2x-15=(x+5)(x-3)$

$\Rightarrow$ x-intercept $=-5$ and 3

(b) $x^2+2x-15=k \Rightarrow x^2+2x-(15+k)=0$

$\because$ 2 distinct roots

$\therefore \quad \Delta>0$

$4+4(15+k)>0 \Rightarrow k>-16$

(c) When $\Delta=0$, there is only 1 intersection. i.e. $k=-16$.

$\therefore$ Required line is $y=-16$.

7B.7 HKCEE MA 2009 – I – 12

(a) (i) $x=11$

(ii) $(11,23)$

(b) (i) Put $y=5$: $5=-2(x-11)^2+23$

$(x-11)^2=9 \Rightarrow x=11\pm3=8$ or 14

$\therefore$ Distance between P and $Q=14-8=6$

(ii) Regardless of the position of S, for ΔPQS,

$PQ=6$, Corresponding height $=5$

$\therefore$ Area of $PQRS$

$=$ Area of $\Delta PQR+$ Ar eaof ΔPQS

$=\frac{1}{2}(6)(23-5)+\frac{1}{2}(6)(5)=69$

7B.8 HKCEE MA 2010 – I – 16

(a) (i) $f(x)=\frac{-1}{144}(x^2-72x)-6$

$=\frac{-1}{144}(x^2-72x+36^2-36^2)-6$

$=\frac{-1}{144}(x-36)^2+3 \Rightarrow$ Vertex $=(36,3)$

7B.9 HKCEE MA 2011 – I – 11

(a) Let $f(x)=hx^2+kx$.

$\begin{cases}28=f(-2)=4h-2k\\ -36=f(6)=36h+6k\end{cases} \Rightarrow \begin{cases}h=1\\ k=-12\end{cases}$

$\therefore f(x)=x^2-12x$

(b) (i) $f(x)=x^2-12x=(x-6)^2-36 \Rightarrow k=-36$

(ii) Put $x=10$.

$y=3(10-6)^2-36=2 \Rightarrow A=(10,2)$

$y=(10)^2-12(10)=-20 \Rightarrow D=(10,-20)$

Since the graphs are symmetric about the common axis of symmetry $x=6$,

$B=(6-(10-6),2)=(2,2)$

$C=(10-(10-6),-20)=(2,-20)$

$\therefore$ Area of $ABCD=(2-(-20))(10-2)=176$

7B.10 HKCEE AM 1988 – I – 10

(a) (i) For $f(x)$, $\begin{cases}\text{Sum of rts}=-2\\ \text{Prod of rts}=-1\end{cases}$

For $g(x)$, $\begin{cases}\text{Sum of rts}=2k\\ \text{Prod of rts}=k^2-6\end{cases}$

$PQ=$ Difference of rts of $f(x)$

$=\sqrt{(-2)^2-4(-1)}=\sqrt{8}$

$RS=$ Difference of rts of $g(x)$

$=\sqrt{(2k)^2-4(k^2-6)}=\sqrt{24}$

(ii) Mid-pt of $RS=\left(\frac{\text{Sum of rts}}{2},0\right)=(k,0)$

If this is also the mid-point of PQ, $k=\frac{-2}{2}=-1$.

(b) $\begin{cases}y=f(x)\\ y=g(x)\end{cases} \Rightarrow x^2+2x-1=-x^2+2kx-k^2+6$

$2x^2+2(1-k)x+k^2-7=0 \quad \ldots (*)$

$\Delta=4(1-k)^2-8(k^2-7)=0$

$k^2+2k-15=0 \Rightarrow k=-5$ or 3

For $k=-5$, $(*)$ becomes $2x^2+12x+18=0$

$2(x+3)^2=0$

$x=-3$

$\Rightarrow$ Intersection $=(-3,(-3)^2+2(-3)-1)=(-3,2)$

For $k=3$, $(*)$ becomes $2x^2-4x+2=0$

$2(x-1)^2=0$

$x=1$

$\Rightarrow$ Intersection $=(1,1^2+2(1)-1)=(1,2)$

7B.11 HKCEE AM 1991 – I – 9

(a) $g(x)=-2x^2-12x-23=-2(x^2+6x+9-9)-25$

$=-2(x+3)^2-5$

$\le -5<0$

(b) (i) $f(x)+kg(x)=0$

$(x^2+2x-2)+k(-2x^2-12x-23)=0$

$(1-2k)x^2+2(1-6k)x-(2+23k)=0$

Eq uarts $\Rightarrow \Delta=0$

$4(1-6k)^2+4(1-2k)(2+23k)=0$

$10k^2-7k-3=0$

$k=1$ or $\frac{-3}{10}$

$\therefore k_1=1,\ k_2=\frac{-3}{10}$

7B.12 (HKCEE AM 1993 – I – 10)

(a) Put $y=0$: $\frac{1}{k+1}[2x^2+(k+7)x+4]=0$

$2x^2+(k+7)x+4=0$

$\because$ Sum of rts $=-\frac{k+7}{2}$, Product of rts $=2$

$\therefore \quad PQ=$ Difference of rts

$1=\sqrt{\left(-\frac{k+7}{2}\right)^2-4(2)}$

$1=\frac{(k+7)^2}{4}-8$

$(k+7)^2=36$

$k=\pm6-7=-13$ or -1 (rejected)

(b) *Method 1*

From (a), PQ does not exist when

$\left(\frac{k+7}{2}\right)^2-8<0$

$(k+7)^2<32$

$-7-\sqrt{32}<k<-7+\sqrt{32}$

Method 2

$\Delta<0$

$\left(\frac{k+7}{k+1}\right)^2-4\left(\frac{2}{k+1}\right)\left(\frac{4}{k+1}\right)<0$

$(k+7)^2-32<0$

$(k+7)^2<32$

$-7-\sqrt{32}<k<-7+\sqrt{32}$

(c) (i) $\begin{cases}C(1): y=\frac{1}{2}(2x^2+8x+4)=x^2+4x+2\\ C(-2): y=-1(2x^2+5x+4)=-2x^2-5x-4\end{cases}$

$\Rightarrow 3x^2+9x+6=0$

$x=-2$ or $-1 \Rightarrow y=-2$ or -1

$\therefore$ Pts of intersecti onare $(-2,-2)$ and $(-1,-1)$.

(ii) Put $x=-2$ into $C(k)$:

RHS $=\frac{1}{k+1}[2(-2)^2+(k+7)(-2)+4]$

$=\frac{1}{k+1}(-2k-2)=-2$

$\therefore (-2,-2)$ is on $C(k)$ for any k.

Put $x=-1$ into $C(k)$:

RHS $=\frac{1}{k+1}[2(-1)^2+(k+7)(-1)+4]$

$=\frac{1}{k+1}(-k-1)=-1$

$\therefore (-1,-1)$ is on $C(k)$ for any k.

7B.13 HKCEE AM 1998 – I – 11

(a) $f(x)=x^2-kx=x^2-kx+\left(\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)^2$

$=\left(x-\frac{k}{2}\right)^2-\frac{k^2}{4}$

$\therefore$ Least value $=-\frac{k^2}{4}$. Corresponding $x=\frac{k}{2}$

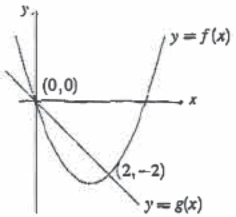
(b) $\begin{cases}y=x^2-kx\\y=-x\end{cases} \Rightarrow x^2-kx=-x$

$x(x-k+1)=0$

$x=0$ or $k-1 \Rightarrow y=0$ or $1-k$

$\therefore$ The intersections are $(0,0)$ and $(k-1,1-k)$.

(c) (i)



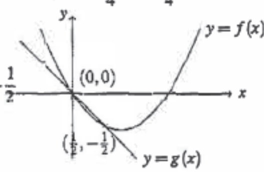
(ii) $f(x)\le g(x) \Rightarrow 0\le x\le 2$

$\therefore$ Least value of $f(x)=-\frac{(3)^2}{4}=-\frac{9}{4}$

(d) $f(x)\le g(x)$

$\Rightarrow 0\le x\le\frac{1}{2}$

$\Rightarrow$ Least $f(x)=-\frac{1}{2}$



7B.14 HKCEE AM 2000 – I – 12

(a) Δ of $f(x)=(-4m)^2+4(5m^2-6m+1)$

$=36m^2-24m+4$

$=4(3m-1)^2\ge 0$

Since $m\ne\frac{1}{3}$, $\Delta\ne 0$.

Thus, $\Delta>0$, and $f(x)$ has 2 distinct real roots.

(b) (i) $x=\frac{4m\pm\sqrt{\Delta}}{2}=\frac{4m\pm 2(3m-1)}{2}$

$\Rightarrow \beta=\frac{4m+2(3m-1)}{2}=5m-1$

$\alpha=\frac{4m-2(3m-1)}{2}=-m+1$

(ii) (1) $4<\beta=5m-1<5 \Rightarrow 5<5m<6$

$\Rightarrow 1<m<\frac{6}{5}$

(2) Sketch A:

The parabola should open upwards as the leading coefficient is positive.

Sketch B:

$1<m<\frac{6}{5} \Rightarrow -\frac{1}{5}<\alpha=-m+1<0$

The root should be larger than -1.

Sketch C:

$f(x)=x^2-4mx-(5m^2-6m+1)$

$=x^2-4mx+4m^2-9m^2+6m-1$

$=(x-2m)^2-(3m-1)^2$

$\Rightarrow$ Min value of $f(x)=-(3m-1)^2$

$1<m<\frac{6}{5} \Rightarrow -4.225<-(3m-1)^2<-4$

Thus the min value should be smaller than -1.

7B.15 HKCEE AM 2002 – 11

(a) $f(x)=x^2-2x-6=(x-1)^2-7 \Rightarrow C=(1,-7)$

$\begin{cases}y=x^2-2x-6\\y=2x+6\end{cases}$

$\Rightarrow x^2-2x-6=2x+6$

$x^2-4x-12=0 \Rightarrow x=6$ or -2

$\therefore A=(-2,2(-2)+6)=(-2,2)$

$B=(6,2(6)+6)=(6,18)$

(b) $f(x)\le g(x)$ when $-2\le x\le 6$

In this range, the horizontal line $y=k$ intersects the parabola $y=f(x)$ at one point, and thus $f(x)=k$ has only one root.

$\therefore 2<k\le 6$ or $k=-7$

7B.16 HKCEE AM 2003 – 17

Let $f(x)=-(x-a)^2+b$, where a and b are real. Point P is the vertex of the graph of $y=f(x)$.

(a) $P=(a,b)$

(b) (i) $g(x)=(x-b)^2+a$

Since $Q(b,a)$ is on the graph of $y=f(x)$,

$a=-(b-a)^2+b \Rightarrow (b-a)^2=b-a$

$\because g(a)=(a-b)^2+a$

$=(b-a)+a=b$

$\therefore (a,b)=P$ lies on $y=g(x)$.

(ii) $y=f(x)$ touches the x-axis $\Rightarrow b=0$

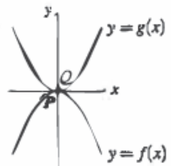
From (b)(i), $(b-a)^2=(b-a)=0$

$(b-a)(b-a-1)=0$

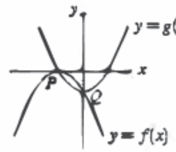
$\Rightarrow a=b$ or $a=b-1$

Thus, there are two cases:

Case 1: $a=b=0$



Case 2: $a=-1, b=0$



7B.17 HKDSE MA 2012 – I – 13

(a) $0=k(2)^3-21(2)^2+24(2)-4 \Rightarrow k=5$

(b) $P=(m,0) \Rightarrow Q=(m,15m^2-63m+72)$

$\therefore$ Area of $OPQR=m(15m^2-63m+72)$

$=15m^3-63m^2+72m$

(c) $15m^3-63m^2+72m=12$

$3(5m^3-21m^2+24m-4)=0$

$(m-2)(5m^2-11m+2)=0$ (by (a))

$(m-2)(5m-1)(m-2)=0$

$m=2,\frac{1}{5}$ or -2 (rejected as P is in Quad I)

7B.18 HKDSE MA 2015 – I – 18

(a) $\Delta=(-4k)^2-4(2)(3k^2+5)=-8k^2-40$

$\le -40<0$

$\therefore$ It does not cut the x-axis.

(b) $f(x)=2x^2-4kx+3k^2+5$

$=2(x^2-2kx+k^2-k^2)+3k^2+5$

$=2(x-k)^2+k^2+5$

$\therefore$ Vertex $=(k,k^2+5)$

7B.19 HKDSE MA 2016 – I – 18

(a) $f(x)=-\frac{1}{3}(x^2-36x)-121$

$=-\frac{1}{3}(x^2-36x+18^2-18^2)-121$

$=-\frac{1}{3}(x-18)^2-13$

$\therefore$ Vertex $=(18,-13)$

7B.20 HKDSE MA 2017 – I – 18

(a) $\begin{cases}y=2x^2-2kx+2x-3k+8\\y=19\end{cases}$

$\Rightarrow 2x^2+2(1-k)x-(3k+11)=0$

$\Delta=4(1-k)^2+8(3k+11)$

$=4(k^2-2k+1+6k+22)$

$=4(k^2+4k+23)$

$=4(k+2)^2+76\ge 76>0$

$\therefore$ There are 2 distinct intersections.

(b) (i) $\begin{cases}a+b=-\frac{2(1-k)}{2}=k-1\\ab=\frac{-(3k+11)}{2}\end{cases}$

$(a-b)^2=(a+b)^2-4ab$

$=(k-1)^2+2(3k+11)=k^2+4k+23$

(ii) $(a-b)^2=(k+2)^2+19$

Minimum value of $(a-b)^2=19$

$\Rightarrow$ Minimum distance of $AB=\sqrt{19}>4$

$\therefore$ NO

7B.21 HKDSE MA 2018 – I – 18

(a) Let $f(x)=hx^2+kx$.

$\begin{cases}60=f(2)=4h+2k\\99=f(3)=9h+3k\end{cases} \Rightarrow \begin{cases}h=3\\k=24\end{cases}$

$\therefore f(x)=3x^2+24x$

(b) (i) $f(x)=3(x^2+8x)=3(x^2+8x+16-16)$

$=3(x+4)^2-48$

$\therefore Q=(-4,-48)$

7B.22 HKDSE MA 2020 – I – 7

7a Since the equation $p(x)=0$, i.e. $4x^2+12x+c=0$, has equal roots,

$\Delta=0$

$12^2-4(4)(c)=0$

$c=9$

b Put $y=0$,

$0=p(x)-169$

$4x^2+12x+9-169=0$

$x^2+3x-40=0$

$(x+8)(x-5)=0$

$x=-8$ or 5

Therefore, the x-intercepts of the graph of $y=p(x)-169$ are -8 and 5

7C Extreme values of quadratic functions

7C.1 HKCEE MA 1985(A/B) – I – 13

(a) $DE^2=BD^2+BE^2-2\cdot BD\cdot BE\cos\angle B$

$=(2-x)^2+x^2-2(2-x)(x)\cos 60^\circ$

$=3x^2-6x+4$

(b) Area of $\triangle DEF=\frac{1}{2}DE\cdot DE\sin 60^\circ$

$=\frac{1}{2}(3x^2-6x+4)\cdot\frac{\sqrt{3}}{2}$

$=\frac{\sqrt{3}}{4}(3x^2-6x+4)$

$=\frac{3\sqrt{3}}{4}\left(x^2-2x+\frac{4}{3}\right)$

$=\frac{3\sqrt{3}}{4}\left(x^2-2x+1+\frac{1}{3}\right)$

$=\frac{3\sqrt{3}}{4}(x-1)^2+\frac{\sqrt{3}}{4}$

$\therefore$ Minimum area is attained when $x=1$.

7C.2 HKCEE MA 1982(1/2) – I – 12

(a) Let $P=ax+bx^2$.

$\begin{cases}80000=20a+400b \Rightarrow a+20b=4000\\87500=35a+1225b \Rightarrow a+35b=2500\end{cases}$

$\Rightarrow \begin{cases}a=6000\\b=-100\end{cases} \Rightarrow P=6000x-100x^2$

Hence, when $x=15$, $P=5000(15)-100(15)^2=67500$.

(b) $P=-100(x^2-60x)=-100(x^2-60x+30^2-30^2)$

$=90000-(x-30)^2$

i.e. $a=90000$, $b=1$, $c=30$

(c) When P is maximum, $x=30$.

7C.3 HKCEE MA 1988 – I – 10

(a) Let $y=ax+bx^2$

$\begin{cases}-5=a+b\\-8=2a+4b\end{cases} \Rightarrow \begin{cases}a=-6\\b=1\end{cases} \Rightarrow y=x^2-6x$

Hence, when $x=6$, $y=(6)^2-6(6)=0$

(b) $y=x^2-6x+9-9=(x-3)^2-9$

$\therefore$ Least possible value of $y=-9$

7C.4 HKCEE MA 2011 – I – 12

(a) $\angle C=180^\circ-\angle B=90^\circ$ (int. $\angle$s, $AB//DC$)

$\angle DPC=180^\circ-\angle APD-\angle APB$ (adj. $\angle$s on st. line)

$=90^\circ-\angle APB$

$\angle PAB=180^\circ-\angle B-\angle APB$ ($\angle$ sum of $\triangle$)

$=90^\circ-\angle APB=\angle DPC$

In $\triangle ABP$ and $\triangle PCD$,

$\angle B=\angle C=90^\circ$ (proved)

$\angle DPC=\angle PAB$ (proved)

$\angle PDC=\angle APB$ ($\angle$ sum of $\triangle$)

$\therefore \triangle ABP\sim\triangle PCD$ (AAA)

(b) $\frac{AB}{BP}=\frac{PC}{CD}$ (corr. sides, $\sim\triangle$s)

$\frac{3}{x}=\frac{11-x}{k}$

$3k=11x-x^2 \Rightarrow x^2-11x+3k=0$

(c) $\Delta\ge 0 \Rightarrow (-11)^2-4(3k)\ge 0 \Rightarrow k\le\frac{121}{12}$

Hence, the greatest integral value of k is 10.

7B.23 HKDSE MA 2020 I – 17

17a

$g(x)=x^2-2kx+2k^2+4$

$=x^2-2kx+\left(\frac{2k}{2}\right)^2+2k^2+4-\left(\frac{2k}{2}\right)^2$

$=(x-k)^2+k^2+4$

Therefore, the coordinates of the vertex of the graph of $y=f(x)$ are (k,k^2+4).

b

Since the graph of $y=g(x+2)$ can be obtained by translating be graph of $y=g(x)$ leftwards by 2 units, we know that $D=(k-2,k^2+4)$.

Since the graph of $y=-g(x-2)$ can be obtained by translating the graph of $y=g(x)$ rightwards by 2 units followed by reflecting the resulting graph along the x-axis, we know that $E=(k+2,-(k^2+4))=(k+2,-k^2-4)$.

Let M be the mid-point of DE and O be the circumcentre of ΔDEF.

$M=\left(\frac{(k-2)+(k+2)}{2},\frac{(k^2+4)+(-k^2-4)}{2}\right)$

$=(k,0)$

Suppose there exists such a point F.

$OM \perp DE$ (circumcentre of ΔDEF)

The slope of OM × The slope of $DE=-1$

$\frac{0-3}{k-0}\times\frac{(k^2+4)-(-k^2-4)}{(k-2)-(k+2)}=-1$

$-6(k^2+4)=4k$

$3k^2+2k+12=0$

$\Delta=2^2-4(3)(12)$

$=-140$

<0

Hence, there is no real solution to k so contradiction arises.

Therefore, there is no such a point F.

7C.5 HKCEE AM 1986 – I – 3

$f(x)=-kx^2+18x+4k$

$=-k\left[x^2-\frac{18}{k}x+\left(\frac{9}{k}\right)^2-\left(\frac{9}{k}\right)^2\right]+4k$

$=-k\left(x-\frac{9}{k}\right)^2+\frac{81}{k}+4k$

$\therefore \frac{81}{k}+4k=45$

$4k^2-45k+81=0 \Rightarrow k=\frac{9}{4}$ or 9

7C.6 HKCEE AM 1996 – I – 4

(a) $x^2-6x+11=(x-3)^2+2$

$\therefore a=-3,\ b=2$

(b) $x^2-6x+11\geq 2 \Rightarrow \frac{1}{x^2-6x+11}\leq\frac{1}{2}$

$\therefore 0<\frac{1}{x^2-6x+11}\leq\frac{1}{2}$

7C.7 HKDSE MA 2013 – I – 17

(a) $f(x)=-x^2+36x=-(x^2-36x+18^2-18^2)$

$=-(x-18)^2+324$

$\therefore$ Vertex $=(18,324)$

(b) (i) $A=x\left(\frac{108-3x}{2}\right)=\frac{3}{2}(36x-x^2)$

(ii) Max value of $A=\frac{3}{2}(324)$ (by (a))

$=486<500$

$\therefore$ NO.

7D Solving equations using graphs of functions

7D.1 HKCEE MA 1980(3) – I – 16

(a) $30=25x-x^3 \Rightarrow \begin{cases}y=25x-x^3\\y=30\end{cases}$

Add $y=30 \Rightarrow x=1.3$ or 4.2

(b) (i) $AC^2=b^2+b^2=2b^2$

$5^2=h^2+\left(\frac{AC}{2}\right)^2$

$25=h^2+\frac{1}{2}b^2 \Rightarrow b=\sqrt{50-2h^2}$

$V=\frac{1}{3}b^2h=\frac{1}{3}(50-2h^2)h$

$=\frac{2}{3}(25h-h^3)$

(ii) $20=\frac{2}{3}(25h-h^3) \Rightarrow 30=25h-h^3$

From (a), $h=1.3$ or 4.2.

7D.2 HKCEE MA 1981(1) – I – 11

(a) One side $=x$ cm

The other side $=\frac{20-2x}{2}=10-x$ (cm)

$\therefore y=x(10-x)=10x-x^2$

(b) (i) $y=18.4$

(ii) Add $y=12 \Rightarrow x=1.4$ or 8.6

(iii) Greatest area $=y$-coordinate of vertex $=25$

7D.3 HKCEE MA 1983(A) – I – 14

(a) $V=k(7-2k)^2=4k^3-28k^2+49k$

(b) $4x^3-28x^2+49x=20 \Rightarrow \begin{cases}y=4x^3-28x^2+49x\\y=20\end{cases}$

Add $y=20 \Rightarrow x=0.6$, 1.9 or 4.5

(c) $k=0.6$ or 1.9 or 4.5 (rejected)

7D.4 HKCEE MA 1985(A) – I – 12

(a) (i) $x^3+x-1=0 \Rightarrow \begin{cases}y=x^3+x\\y=1\end{cases}$

Add $y=1 \Rightarrow x=0.7$

(b) (i) $(x+1)^4-(x-1)^4$

$=[(x+1)^2+(x-1)^2][(x+1)^2-(x-1)^2]$

$(2x^2+2)(4x)=8x^3+8x$

(ii) $8x^3+8x=8 \Rightarrow x^3+x-1=0$

By (a)(ii), $x=0.69$.

7D.5 HKCEE MA 1985(B) – I – 12

(a) Since ΔABC and thus ΔBPQ are right-angled isosceles, $QR=(16-2x)$ cm.

$\therefore$ Area of $PQRS=x(16-2x)=2(8x-x^2)$ (cm^2)

(b) (i) The greatest area is attained when $x=4$.

(ii) $28=2(8x-x^2)$

$14=8x-x^2 \Rightarrow \begin{cases}y=8x-x^2\\y=14\end{cases}$

Add $y=14 \Rightarrow x=2.6$ or 5.4.

7D.6 HKCEE MA 1986(B) – I – 14

(a) $c = y\text{-intercept} = 6$

Roots $=$ 2 and 3 $\Rightarrow \begin{cases} \frac{c}{a} = (-2)(3) \Rightarrow a = 1 \\ -\frac{b}{a} = (-2)+(3) \Rightarrow b = 1 \end{cases}$

(b) (i) $(x+2)(x-3) = -1 \Rightarrow \begin{cases} y = x^2+x+6 \\ y = -1 \end{cases}$

Add $y = 1 \Rightarrow x =$ 2.2 or 3.2

7D.7 HKCEE MA 1987(A) – I – 14

(a) (i) $x^3 - 6x^2 + 9x - 1 = 0 \Rightarrow \begin{cases} y = x^3 - 6x^2 + 9x \\ y = 1 \end{cases}$

Add $y = 1 \Rightarrow x = 0.1, 2.3$ or 3.5

(c) $\begin{cases} y = x^3 \quad 6x^2 + 9x \\ y = k \end{cases}$

To have 3 intersections, $0 < k < 4$.

7D.8 HKCEE MA 1997 – I – 13

(a) (i) 10

(ii) $1.8 < x \leq 16 \Rightarrow 2 \leq x \leq 16$

(b) (i) Put $x = 3$ and $H = 144$: $144 = 3^2 - 51 + c$

$c = 186$

(iii) Total cost $= 10 \times \$20 + 6 \times 120 = \$520$

Total proceeds

$= 6 \times \$100 + 4 \times \$300 + 4 \times \$10 + 2 \times \$60$

$= \$1960$

$\therefore$ Gain $= 1960 - 520 = (\$)1440$

7D.9 HKCEE MA 2000 – I – 18

(a) Let $V = ah^2 + bh^3$.

$\begin{cases} \frac{29\pi}{3} = a+b \\ 81\pi = 9a + 27b \end{cases} \Rightarrow \begin{cases} a = 10\pi \\ b = -\frac{\pi}{3} \end{cases}$

$\therefore V = 10h^2 - \frac{\pi}{3}h^3$

(b) (i) Surface area = Surface area of original hemisphere

$= 2\pi(10)^2 = 200\pi \text{ (cm}^2\text{)}$

(ii) $\frac{1}{2} \cdot \frac{4}{3}\pi(10)^3 - 2\left(10h^2 - \frac{\pi}{3}h^3\right) = \frac{1400}{3}\pi$

$\frac{2000}{3}\pi - 20h^2 + \frac{2\pi}{3}h^3 = \frac{1400}{3}\pi$

$h^3 - 30h^2 + 300 = 0$

(iii) $\begin{cases} y = x^3 - 30x^2 \\ y = -300 \end{cases}$

Add $y = -300$ to the graph $\Rightarrow h = 3.35$

7E Transformation of graphs of functions

7E.1 HKCEE MA 2010 – I – 16

(a) (i) $f(x) = \frac{-1}{144}(x^2 - 72x) - 6$

$= \frac{-1}{144}(x^2 - 72x + 36^2 - 36^2) - 6$

$= \frac{-1}{144}(x-36)^2 + 3$

$\therefore$ Vertex $= (36, 3)$

(ii) $g(x) = f(x+4) + 5 = \frac{-1}{144}(x-32)^2 + 8$

(iii) $h(x) = 2^{f(x+4)} + 5 = 2^{\frac{-1}{144}(x-32)^2+3} + 5$

(b) (i) When $u = 8$, $8 = 2^{f(s)}$

$3 = f(s) = \frac{-1}{144}(s-36)^2 + 3$

$s = 36$

$\therefore$ The temperature is 36°C.

(ii) From the table, $\begin{cases} t = s - 4 \\ v = u + 5 \end{cases}$.

Hence, $u = 2^{f(s)}$ becomes: $v - 5 = 2^{f(t+4)}$

$\Rightarrow v = 2^{f(t+4)} + 5 = 2^{\frac{-1}{144}(t-32)^2+3} + 5$

7E.2 HKDSE MA 2015 – I – 18

(a) $\Delta = (-4k)^2 - 4(2)(3k^2+5) = -8k^2 - 40$

$\leq -40 < 0$

$\because$ It does not cut the x-axis.

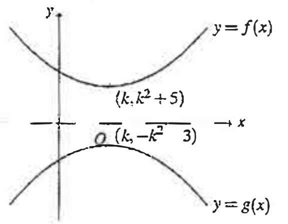
(b) $f(x) = 2x^2 \quad 4kx + 3k^2 + 5$

$= 2(x^2 - 2kx + k^2 - k^2) + 3k^2 + 5$

$= 2(x-k)^2 + k^2 + 5$

$\therefore$ Vertex $= (k, k^2+5)$

(c)



S and T are nearest to each other when they are the vertices of the two parabolas respectively. Since $OS \neq OT$, $\triangle OST$ is not isosceles, and thus the x-axis is not the $\perp$ bisector of ST. NOT correct.

7E.3 HKDSE MA 2016 – I – 18

(a) $f(x) = -\frac{1}{3}(x^2 - 36x) - 121$

$= -\frac{1}{3}(x^2 - 36x + 18^2 - 18^2) - 121$

$= \frac{1}{3}(x \quad 18)^2 \quad 13$

$\therefore$ Vertex $= (18, -13)$

(b) $g(x) = f(x) + 13 = -\frac{1}{3}(x-18)^2$

(c) $-\frac{1}{3}x^2 - 12x - 121 = f(-x)$

Hence, the transformation is a reflection in the y axis

7E.4 HKDSE MA 2018 – I – 18

(a) Let $f(x) = hx^2 + kx$.

$\begin{cases} 60 = f(2) = 4h + 2k \\ 99 = f(3) = 9h + 3k \end{cases} \Rightarrow \begin{cases} h = 3 \\ k = 24 \end{cases}$

$\therefore f(x) = 3x^2 + 24x$

(b) (i) $f(x) = 3(x^2 + 8x) = 3(x^2 + 8x + 16 - 16)$

$= 3(x+4)^2 - 48$

$\therefore Q = (-4, -48)$

(ii) $R = (-4, 75)$

(iii) $QR = 75 - (-48) = 123$

$SQ = \sqrt{60^2 + 48^2} = \sqrt{5904}$

$RS = \sqrt{60^2 + 75^2} = \sqrt{9225}$

Hence, $QR^2 = SQ^2 + RS^2$. $\triangle QRS$ is right-$\angle$ed at S. (converse of Pyth. thm)

$\therefore P$ is the mid-point of QR.

7E.5 HKDSE MA 2019 – I – 19

(a) $f(4) = \frac{1}{1+k}\left((4)^2 + (6k-2)(4) + (9k+25)\right)$

$= \frac{1}{1+k}(33 + 33k) = 33$

Hence, the graph passes through F.

(b) (i) $g(x) = f(-x) + 4$

$= \frac{1}{1+k}\left((-x)^2 + (6k-2)(-x) + (9k+25)\right) + 4$

$= \frac{1}{1+k}\Big(x^2 - (6k-2)x + (3k-1)^2$

$-(3k-1)^2 + (9k+25)\Big) + 4$

$= \frac{1}{1+k}\left((x-3k+1)^2 - 9k^2 + 3k + 24\right) + 4$

$= \frac{1}{1+k}\left((x-3k+1)^2 - 3(1+k)(3k-8)\right) + 4$

$= \frac{1}{1+k}(x-3k+1)^2 - 3(3k-8) + 4$

$= \frac{1}{1+k}(x-3k+1)^2 + 28 - 9k$

$\therefore U = (3k-1, 28-9k)$